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The Clarkson-Polyak modulus of convexity and its applications

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J. A. Clarkson, 1936

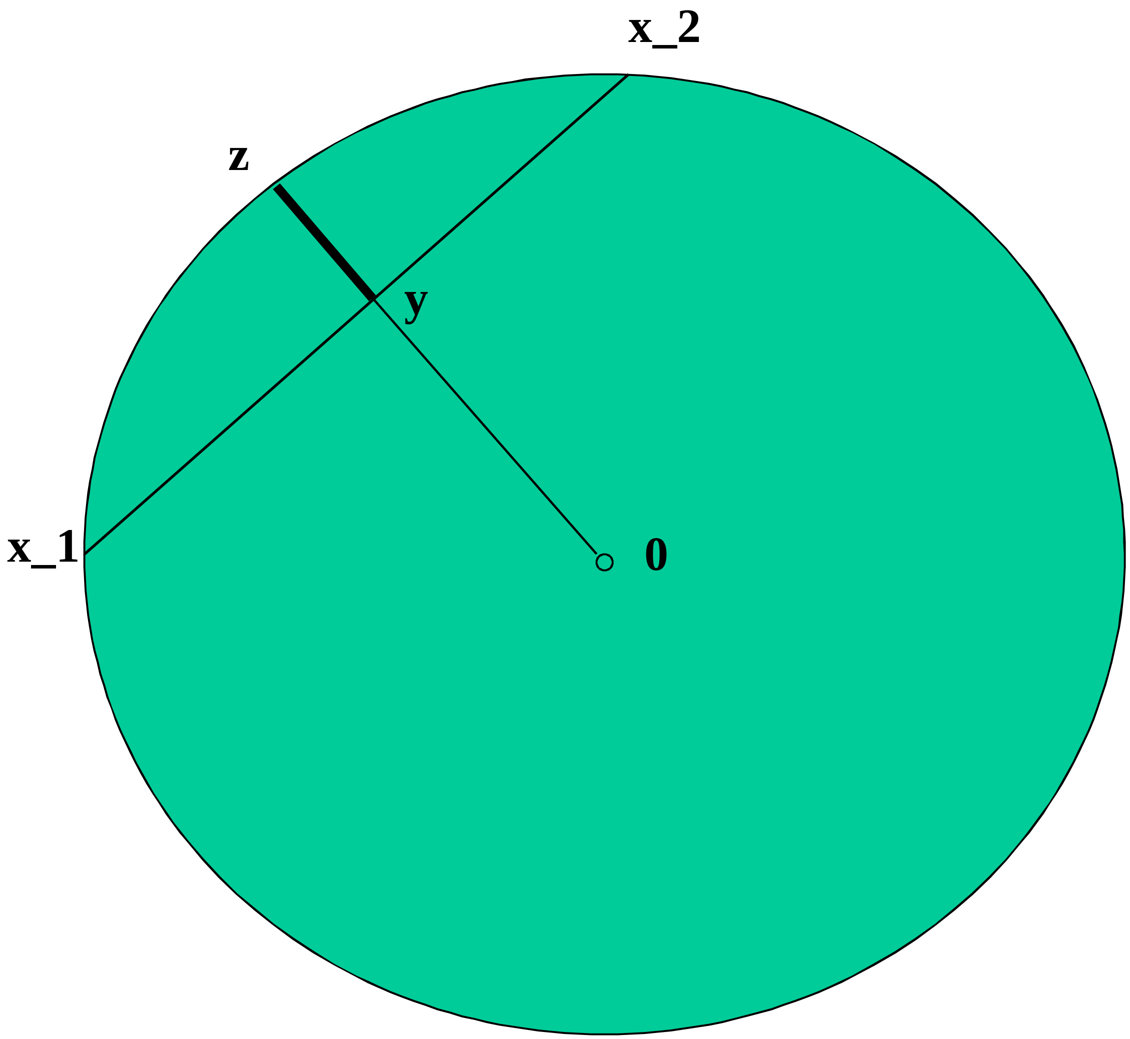
Let X be a real Banach space and $B_r(x)$ be the closed ball with center $x \in X$ and radius $r > 0$.

In 1936 J. A. Clarkson introduced the modulus of convexity for the space X by the formula

$$\delta_X : [0,2] \to [0,1],$$

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x_1 + x_2}{2} \right\| : \forall x_1, x_2 \in B_1(0),\ \|x_1 - x_2\| = \varepsilon \right\}$$

A Banach space X is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all permissible $\varepsilon > 0$. This concept led to the development of the theory of uniformly convex spaces.



y=(x_1+x_2)/2.

B. T. Polyak, 1966

B. T. Polyak introduced the concept of modulus of convexity for an arbitrary convex closed bounded subset $A \subset X$. The modulus of convexity is the function

$$\delta_A : \ (0, \operatorname{diam} A) \to [0, +\infty)$$

and it is defined as

$$\delta_A(\varepsilon) = \sup\{\delta \geq 0 \ : B_\delta\left(\frac{x_1+x_2}{2}\right) \subset A, \\ \forall x_1, x_2 \in A : \ \|x_1 - x_2\| = \varepsilon\}$$

It is easy to see that in the case $A = B_1(0)$ we have $\delta_A(\varepsilon) = \delta_X(\varepsilon)$, for all $\varepsilon \in [0, 2]$.

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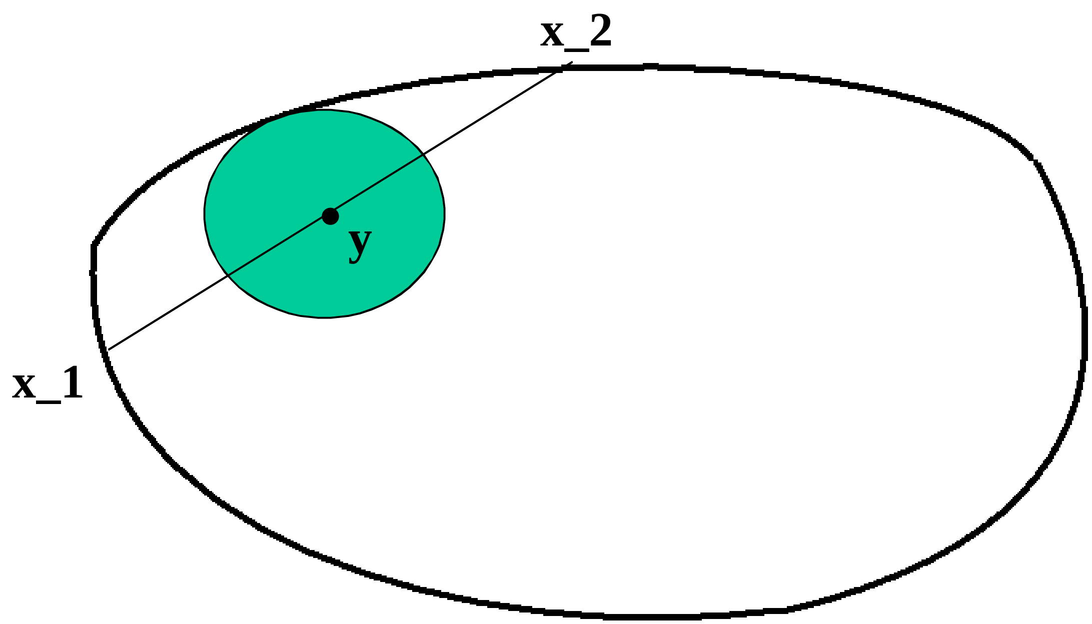


The concept of B. T. Polyak appeared to be very useful in the field of optimization, for estimate of the rate of convergence for minimization sequences.

$$\min_{x\in A} f(x),$$

where the set A is uniformly convex. Then there exists $C > 0$ with

$$\|x_k - x_*\| \leq C \cdot \delta_A^{-1}\left(|f(x_k) - f(x_*)|\right).$$

In approximation theory, mainly for the question of stability of functionals.

$$\min_{x \in A} f(x), \tag{1}$$
$$\min_{x \in B} f(x), \tag{2}$$

and the set A is uniformly convex with the modulus δ_A. Let a and b be solutions of the problems (1) and (2), respectively. Then there exists $C > 0$ with

$$\|a - b\| \leq C \cdot \delta_A^{-1}\left(h(A, B)\right).$$

1 Strongly convex set of radius R. Dual description.

A nonempty set $A \subset X$ is strongly convex of radius $R > 0$ if

$$A = \bigcap_{x \in B} B_R(x), \qquad B \subset X.$$

Let X be a real Hilbert space. Then

$$\delta_A(\varepsilon) \geq R\delta_X\left(\frac{\varepsilon}{R}\right) =$$

$$= R\left(1 - \sqrt{1 - \frac{\varepsilon^2}{4R^2}}\right) \sim \frac{\varepsilon^2}{8R}, \varepsilon \to 0.$$

Theorem 1 Let X be a real Hilbert space. Suppose that a nonempty closed convex subset $A \subset X$ is uniformly convex with the modulus of convexity of the second order at zero: there exists $C > 0$ such that

$$\delta_A(\varepsilon) = C\varepsilon^2 + o(\varepsilon^2), \quad \varepsilon \to +0.$$

Then there exists a subset $B \subset X$ such that

$$A = \bigcap_{x \in B} \left(x + \frac{1}{8C} B_1(0) \right),$$

and $\frac{1}{8C}$ is sharp in the sense that for any $r < \frac{1}{8C}$ and any subset $C \subset X$,

$$A \neq \bigcap_{x \in C} B_r(x).$$

2 Approximations of strictly convex compacta in $\mathbb{R}^n$.

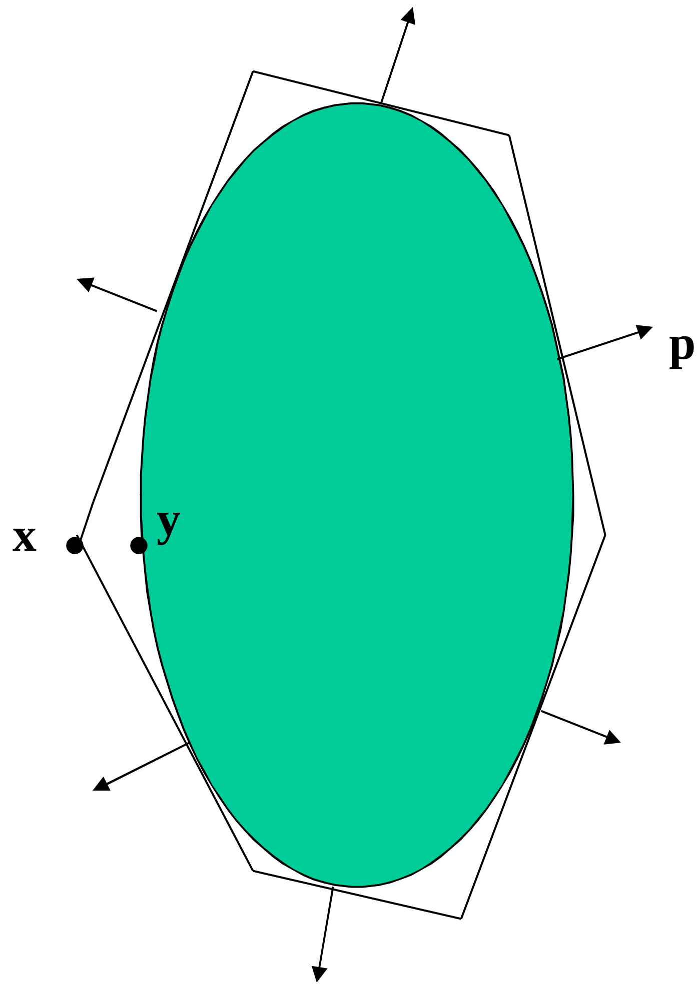
The *supporting function* of the subset $A \subset \mathbb{R}^n$ is defined as follows

$$s(p, A) = \sup_{x \in A} (p, x), \qquad \forall p \in \mathbb{R}^n.$$

We shall consider external polyhedral approximation of a convex compact $A \subset \mathbb{R}^n$ on the grid $\mathbb{G}$

$$\hat{A} = \{x \in \mathbb{R}^n \mid (p, x) \leq s(p, A), \quad \forall p \in \mathbb{G}\}.$$

$$h(A, \hat{A}) = \sup_{\|p\|=1} (s(p, \hat{A}) - s(p, A)).$$


p
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Theorem 2 Let $A \subset \mathbb{R}^n$ be a convex compact set with the modulus of convexity $\delta_A(\varepsilon)$, $\varepsilon \in [0, \operatorname{diam} A]$. Let $\mathbb{G}$ be a grid with the step $\Delta \in (0, \frac{1}{2})$ and $\frac{\delta_A(\operatorname{diam} A)}{\operatorname{diam} A} > \frac{\Delta}{4-\Delta^2}$. Then

$$h(A, \hat{A}) \leq \frac{8}{7}\varepsilon(\Delta)\Delta,$$

where $\varepsilon(\Delta)$ is a solution of the equation $\frac{\delta_A(\varepsilon)}{\varepsilon} = \frac{\Delta}{4-\Delta^2}$.

Thank you!