Weierstrass Institute for
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Leibniz Association

Geometry of barriers for 3-dimensional cones

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Conic programs

Definition: A regular convex cone $K \subset \mathbb{R}^n$ is a closed convex cone with nonempty interior and containing no lines.

Definition: A conic program over a regular convex cone $K \subset \mathbb{R}^n$ is an optimization problem of the form

$$\min_{x \in K} \langle c, x \rangle : \quad Ax = b.$$

every convex optimization problem can be written as a conic program let $K \subset \mathbb{R}^n$ be a regular convex cone

a logarithmically homogeneous self-concordant barrier on K is a smooth function $F : K^o \to \mathbb{R}$ satisfying

- $F(\lambda x) = -\nu \log \lambda + F(x)$ for all $x \in K^o, \lambda > 0$
- $F(x) \to +\infty$ as $x \to \partial K$
- Hessian $F'' \succ 0$ defines a Riemannian metric g on K^o
- self-concordance: $|F'''(x)[h,h,h]| \leq 2(F''(x)[h,h])^{3/2}$ for all $h \in T_x K^o$

ν is called self-concordance parameter

path-following method to solve conic programs over K:

approximately solve unconstrained penalized convex problem

$$\min_x \left(F(x) + \tau\langle c, x\rangle\right) : \quad Ax = b$$

and increase τ, alternating these steps
for $\tau \to \infty$ the solution converges to optimal solution of original program

the smaller the self-concordance parameter, the faster the convergence

Question: Given a cone $K \subset \mathbb{R}^n$, what is the minimal self-concordance parameter ν_{opt} that a barrier on K can have?

- [Nesterov, Nemirovski 1993] universal barrier, $\nu \leq O(n)$
- [H. 2014; Fox 2015] canonical barrier, $\nu \leq n$
- [Bubeck, Eldan, in preparation] entropic barrier, $\nu \leq n + O(\sqrt{n})$
- [Güler, Tuncel 1998] for homogeneous cones ν_{opt} equals the Siegel rank
- if K has an extreme ray which is isomorphic to an extreme ray of $\mathbb{R}^n_+$, then $\nu_{opt} = n$ (consequence of [H. 2013; H. 2014])

let F be a barrier on K with self-concordance parameter ν, and set

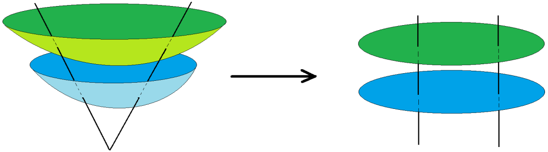
$$\mu(F) = \sup_{x \in K^o, h \in T_x K^o} (F''(x)[h,h])^{-3/2} |F'''(x)[h,h,h]| \leq 2$$

then $\tilde{F} = \frac{\mu^2}{4} F$ is a barrier on K with parameter $\tilde{\nu} = \frac{\mu^2}{4}\nu \leq \nu$

Definition: Let $K \subset \mathbb{R}^n$ be a regular convex cone and F a self-concordant barrier on K with parameter ν. We call F minimal if $\mu(F) = 2$, and optimal if for every other barrier $\tilde{F}$ on K with parameter $\tilde{\nu}$, we have $\nu \leq \tilde{\nu}$.

let F be a logarithmically homogeneous barrier on $K \subset \mathbb{R}^n$

level surfaces of F are centro-affine and homothetic



interior K^o is diffeomorphic to a direct product of a level surface and a radial ray

Theorem: [Loftin 2002] Under the above diffeomorphism the Riemannian metric defined on K^o by the Hessian F'' splits into a direct product $g = \nu h \oplus s$, where h is the centro-affine metric of the level surface and s the trivial 1-dimensional metric on the ray.

the restriction of $\nu^{-1} F'''$ to the horizontal factor is the cubic form C of the level surface

given h and C the level surfaces of F and the cone K can be recovered up to linear isomorphism

for three-dimensional cones K the level surfaces M of a barrier F are two-dimensional
hence M is a complete non-compact simply connected Riemann surface

Uniformization theorem: Every simply connected Riemann surface is conformally equivalent to either the unit disc D, or the complex plane $\mathbb{C}$, or the Riemann sphere S, equipped with either the hyperbolic metric, or the flat (parabolic) metric, or the spherical (elliptic) metric, respectively.

due to Klein, Riemann, Schwarz, Koebe, Poincaré, Hilbert, Weyl, Radó ... 1880–1920

- there exists an oriented atlas of charts on M such that $h = e^{2\phi}(dx_1^2 + dx_2^2)$
- each chart parameterized by one complex parameter $z = x_1 + ix_2$, $h = e^{2\phi}|dz|^2$
- transition maps holomorphic (conformal $+$ oriented $=$ holomorphic)
- global chart with values in $D, \mathbb{C}$, or S exists and is unique up to automorphisms

- D: $h = e^{2\tilde{\phi}} \frac{4|dz|^2}{(1-|z|^2)^2}$ with $\tilde{\phi}$ uniquely defined scalar field on M
- $\mathbb{C}$: $h = e^{2\tilde{\phi}}|dz|^2$ with $\tilde{\phi}$ scalar field defined up to additive constant
- non-compactness of M rules out elliptic case S

Decomposition of the cubic form

consider a conformal chart on M such that $h = e^{2\phi}(dx_1^2 + dx_2^2)$

the cubic form C can be decomposed as

$$C = \left[\begin{pmatrix} \frac{3}{4}e^{2\phi}T_1 + U_1 & \frac{1}{4}e^{2\phi}T_2 - U_2 \\ \frac{1}{4}e^{2\phi}T_2 - U_2 & \frac{1}{4}e^{2\phi}T_1 - U_1 \end{pmatrix}, \begin{pmatrix} \frac{1}{4}e^{2\phi}T_2 - U_2 & \frac{1}{4}e^{2\phi}T_1 - U_1 \\ \frac{1}{4}e^{2\phi}T_1 - U_1 & \frac{3}{4}e^{2\phi}T_2 + U_2 \end{pmatrix} \right]$$

T is the Tchebycheff form and represents the trace part of C; define $E = \frac{1}{4}(T_1 - iT_2)$
$U = U_1 + iU_2$ is a cubic differential representing the trace-free part of C, $U(w) = U(z)(\frac{dz}{dw})^3$

compatibility requirements on ϕ, C [Liu, Wang 1997]:
T is closed with (real) potential t, and with $E = \frac{1}{2}\frac{\partial t}{\partial z}$

$$\frac{\partial U}{\partial \bar{z}} = e^{4\phi}\frac{\partial}{\partial z}(e^{-2\phi}E),$$

$$|U|^2 = 2e^{6\phi} + e^{4\phi}|E|^2 - 8e^{4\phi}\frac{\partial^2 \phi}{\partial z \partial \bar{z}}$$

here $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$

the barrier F is canonical if and only if the Tchebycheff form T vanishes: $E = 0$ and U holomorphic

Theorem: (follows from [Simon, Wang 1993])

- Let $K \subset \mathbb{R}^3$ be a regular convex cone. Then the canonical barrier on K defines a unique complete canonical Riemann metric $h = e^{2\phi}|dz|^2$ on the Riemann surface M of the rays in K^o and an associated holomorphic cubic differential U satisfying the relation

$$|U|^2 = 2e^{6\phi} - 2e^{4\phi}\Delta\phi = 2e^{6\phi}(1 + \mathbf{K}),$$

  where Δ is the ordinary Laplacian and $\mathbf{K}$ the Gaussian curvature.
- Every simply connected non-compact Riemann surface with complete metric $h = e^{2\phi}|dz|^2$ and holomorphic cubic differential U satisfying above relation defines a regular convex cone $K \subset \mathbb{R}^3$ with its canonical barrier.

Remarks:

- level surfaces of F can be recovered from (h, U) by solving a Cauchy initial value problem of a PDE
- [Simon, Wang 1993] gives a necessary and sufficient integrability condition on ϕ
- for given ϕ, U is determined up to a constant factor $e^{i\varphi}$
- for given U, there exists at most one solution ϕ (maximum principle)
- symmetry group of K = symmetry group of (h, U) times homothety subgroup

[Dumas, Wolf 2014 (preprint)] polynomials U of degree k correspond to polyhedral cones K with $k+3$ extreme rays

$U = z^k$ corresponds to the cone over the regular $(k+3)$-gon

Riemann surface conformally equivalent to $\mathbb{C}$

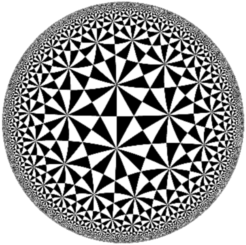
[Wang 1997; Loftin 2001; Labourie 2007] holomorphic functions on compact Riemann surface of genus $g \geq 2$ form finite-dimensional space

each such function U determines a unique metric h on the surface and its universal cover

the corresponding cone K has an automorphism group with cocompact action on the level surfaces on F

∂K is C^1, but in general nowhere C^2

Riemann surface conformally equivalent to $\mathbb{D}$



[Benoist, Hulin 2014] the following are equivalent:

- $k = \sup_M \mathbf{K} < 0$
- M is Gromov hyperbolic (geodesic triangles have bounded width)
- $\mathbb{R}^3_+$ is not in the closure of the orbit of K under $SL(3,\mathbb{R})$
- M is conformally equivalent to $\mathbb{D}$ and U is bounded in the hyperbolic metric
- ∂K is C^1 and quasi-symmetric

Lemma: [H. 2011] Let F be a minimal barrier on $K \subset \mathbb{R}^n$ and M a level surface of F. Then the ∞-norm of the cubic form on M,

$$\gamma = \sup |C(x)[h,h,h]| \;:\qquad x \in M,\; h \in T_xM,\; ||h|| = 1$$

relates to the barrier parameter ν of F by

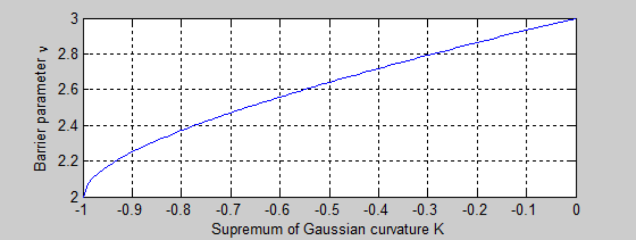
$$\gamma = \frac{2(\nu-2)}{\sqrt{\nu-1}}, \qquad \nu = \frac{\gamma^2 + 16 + \gamma\sqrt{\gamma^2+16}}{8}.$$

Lemma: [Simon, Wang 1993] Let (h, U) be a compatible pair of a metric and a holomorphic cubic differential. Then $|U|^2 = 2(K+1)e^{6\phi}$, where K is the Gaussian curvature, $-1 \leq K \leq 0$.

Corollary: Let $K \subset \mathbb{R}^3$ be a regular convex cone, F the canonical barrier on it, and (h, U) the metric and holomorphic cubic differential defined by F. Then

$$\nu = \frac{k + 9 + \sqrt{(k+1)(k+9)}}{4},$$

where $k = \sup_M \mathbf{K}$ is the supremum of the Gaussian curvature.



extreme cases:

- $\mathbf{K} \equiv 0$: flat metric, $K = \mathbb{R}^3_+$
- $\mathbf{K} \equiv -1$: hyperbolic metric, $K = L_3$

generalize to arbitrary dimension

Lemma: (follows from [H., 2013; H., 2014]) Let $K \subset \mathbb{R}^n$ be a regular convex cone such that $\mathbb{R}^n_+$ is in the closure of the orbit of K under $SL(n, \mathbb{R})$. Then $\nu_{opt}(K) = n$.

Corollary: Let $K \subset \mathbb{R}^3$ be a regular convex cone. Then the following are equivalent:

- $\nu_{opt}(K) < 3$
- $\nu_{can}(K) < 3$
- $k = \sup_M \mathbf{K} < 0$
- M is Gromov hyperbolic
- $\mathbb{R}^3_+$ is not in the closure of the orbit of K under $SL(3, \mathbb{R})$
- M is conformally equivalent to $\mathbb{D}$ and U is bounded in the hyperbolic metric
- ∂K is C^1 and quasi-symmetric

There is a 1-to-1 correspondence between such cones and bounded holomorphic cubic differentials U on $\mathbb{D}$.

Open questions:

Which cones allow barriers such that the corresponding Riemann surface is conformally equivalent to $\mathbb{C}$?
Is there an easy way to compute ν_{opt} (there are cones such that $\nu_{opt} < \nu_{can}$)?

Thank you!