Optimization problems in traffic control

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Outline of the talk

Ramp metering for congested freeways

> Model - the ACTM
> Problem formulation
> Solution via relaxation

Signal coordination on streets

> The bandwidth function
> Two-path problem formulation
> Solution via relaxation
> Extensions

- Gaussian arrival functions
- von Mises arrival functions
- Networks

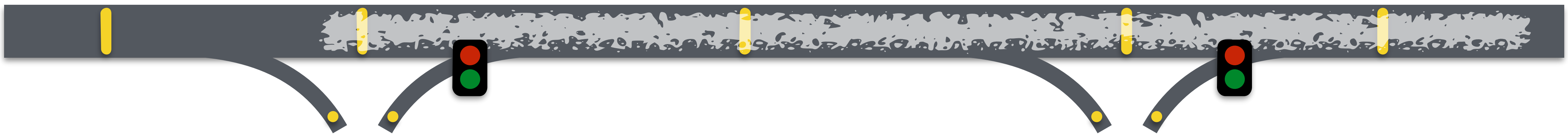
Part I

Ramp metering for congested freeways

Ramp metering

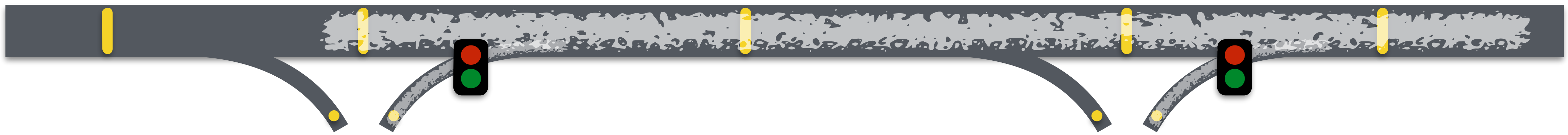
- A linear freeway with:
  - > traffic sensors (sufficient to estimate the state),
    - mainline
    - on-ramps
    - off-ramps
  - > metering lights.

Ramp metering



- A linear freeway with:
  - traffic sensors (sufficient to estimate the state),
    - mainline
    - on-ramps
    - off-ramps
  - metering lights.
- congestion!

Ramp metering



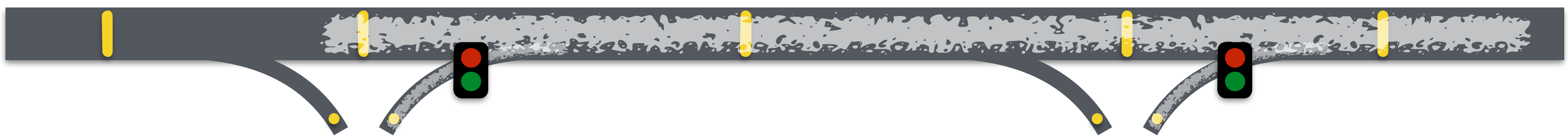
Question:

Can the total congestion (mainline + ramps) be diminished by metering the ramps?

Short answer:

Yes.

Ramp metering

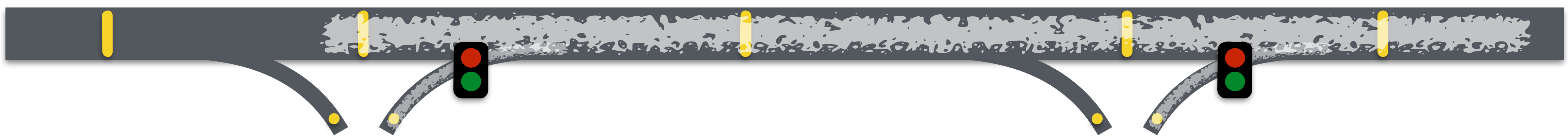


Long answer:

Mainline congestion inefficiencies,

1. Offramp blockage,
2. Finite acceleration,
3. Weaving.

Ramp metering



Long answer:

Mainline congestion inefficiencies,

1. Offramp blockage,
2. ~~Finite acceleration,~~
3. ~~Weaving.~~

First order theory

Freeway model: ACTM

ACTM: Asymmetric Cell-Transmission Model [Gomes & Horowitz 2006]

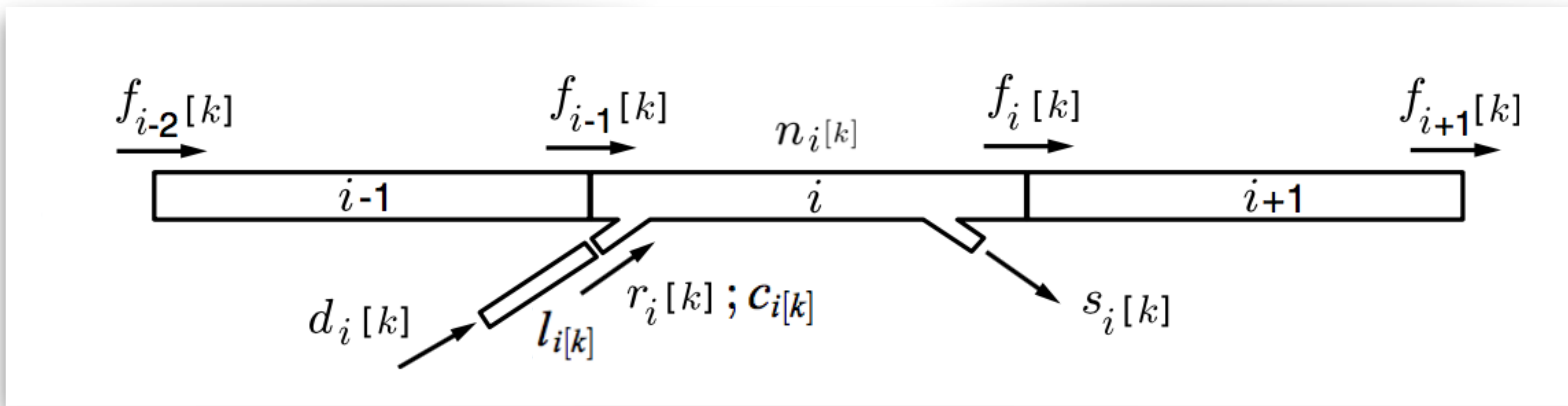
↑ **[Freeways only]**

CTM: Cell-Transmission Model [Daganzo 1995]

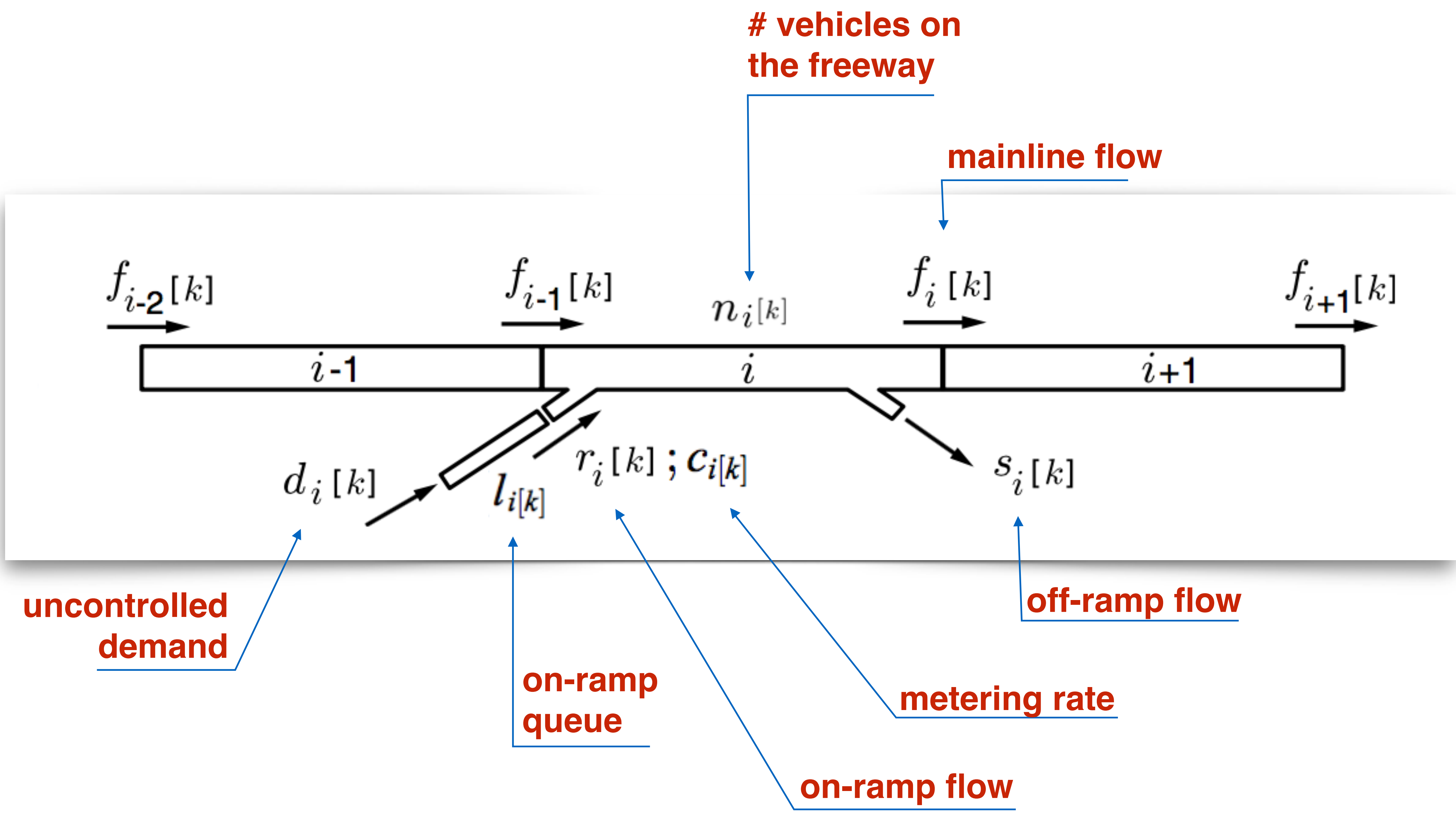
↑ **[Godunov discretization]**

LWR: Continuous hydrodynamic theory [Lighthill & Whithham 1958]

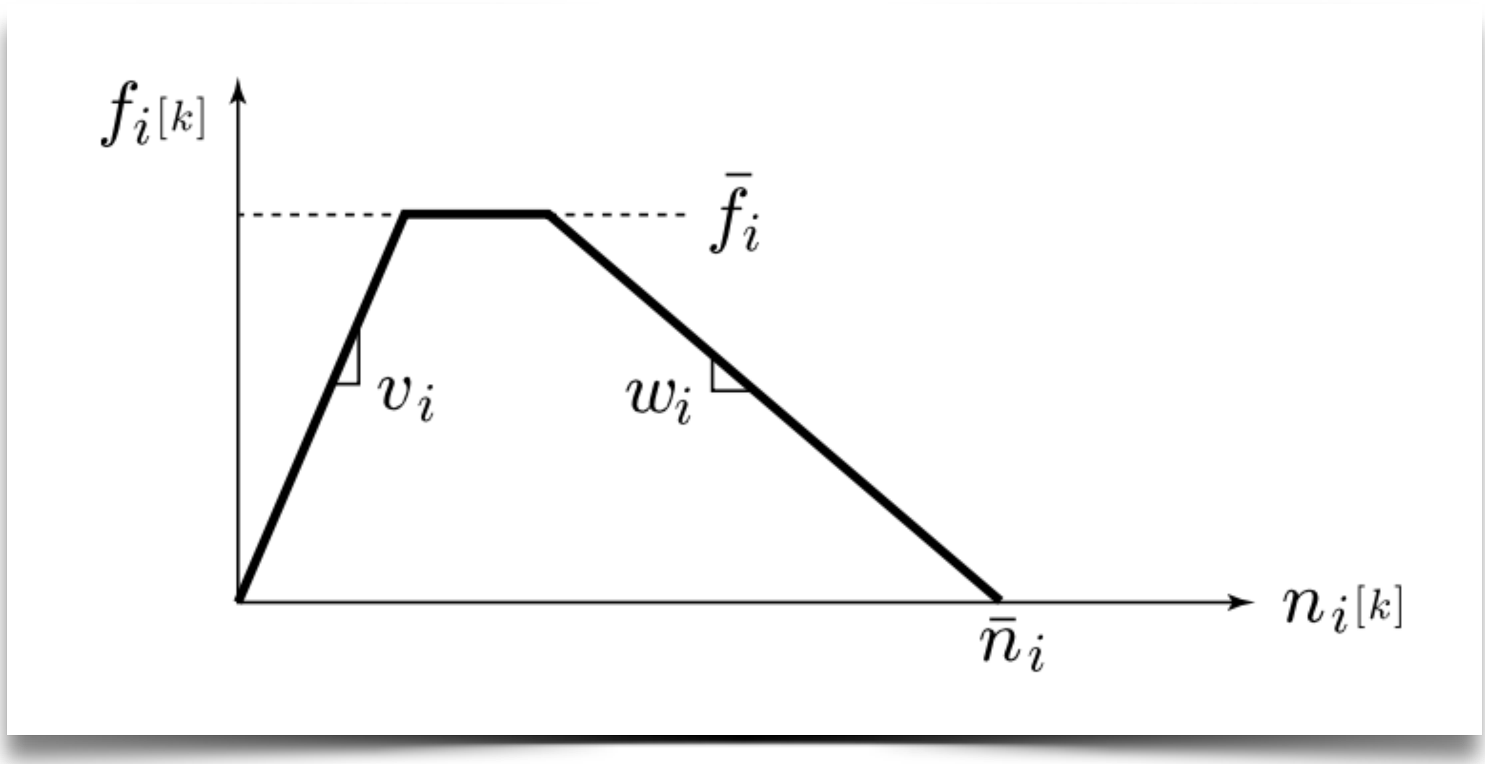
Freeway model: ACTM



Freeway model: ACTM



Flow/Density relation:



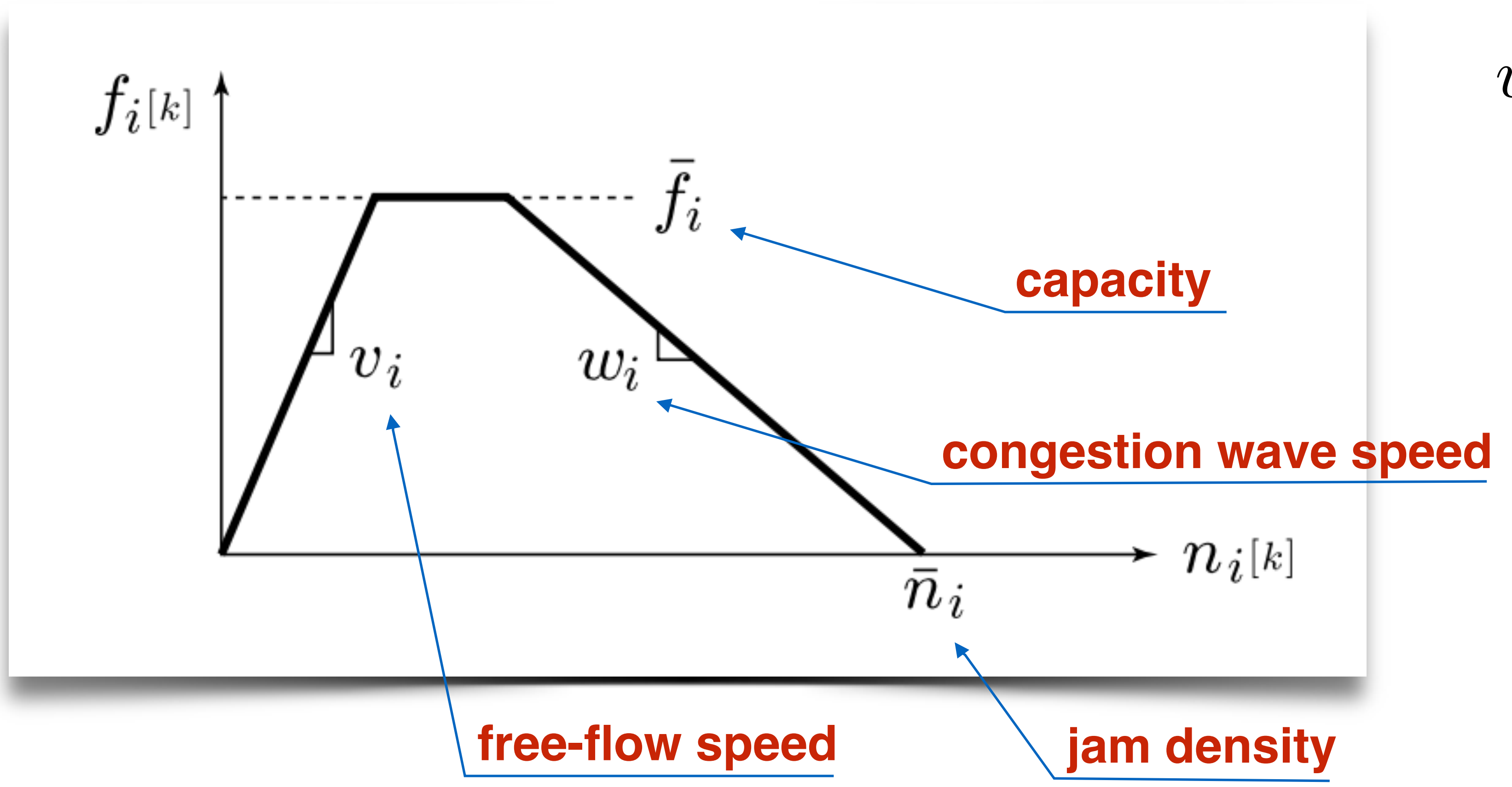
$$w_i \in (0, 1]$$
$$v_i \in (0, 1]$$

Off-ramp flows:

$$s_{i[k]} = \beta_{i[k]}(s_{i[k]} + f_{i[k]})$$

$$\beta_i[k] \in [0, 1]$$

Flow/Density relation:



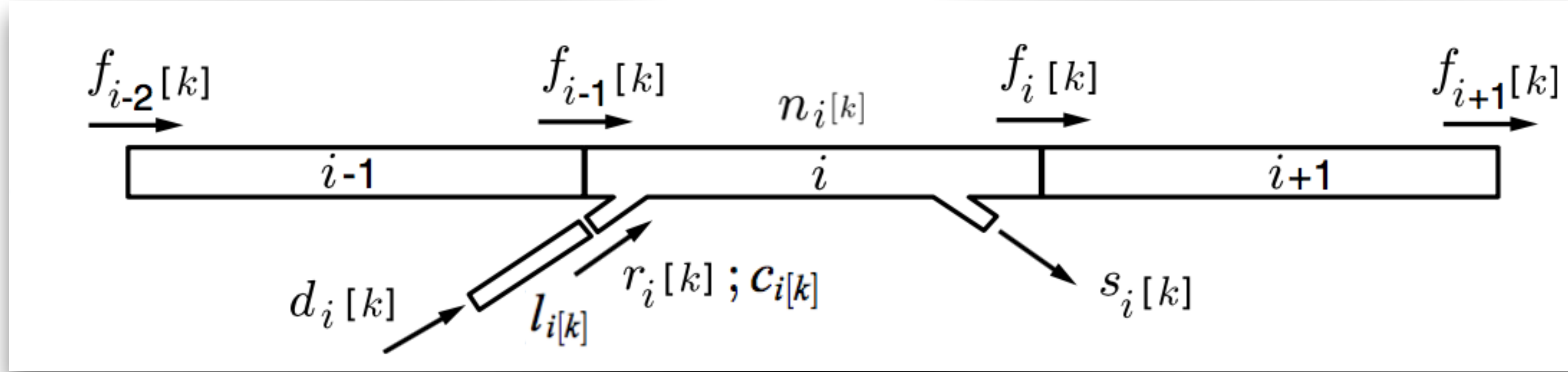
$$w_i \in (0, 1]$$
$$v_i \in (0, 1]$$

Off-ramp flows:

$$s_{i[k]} = \beta_{i[k]}(s_{i[k]} + f_{i[k]})$$

split ratio

$$\beta_i[k] \in [0, 1]$$



On-ramp flows:

$$r_i[k] = \min\big(\; l_i[k] + d_i[k]\;,\; \xi_i(\bar{n}_i - n_i[k])\;,\; c_i[k]\;\big)$$

Mainline flows:

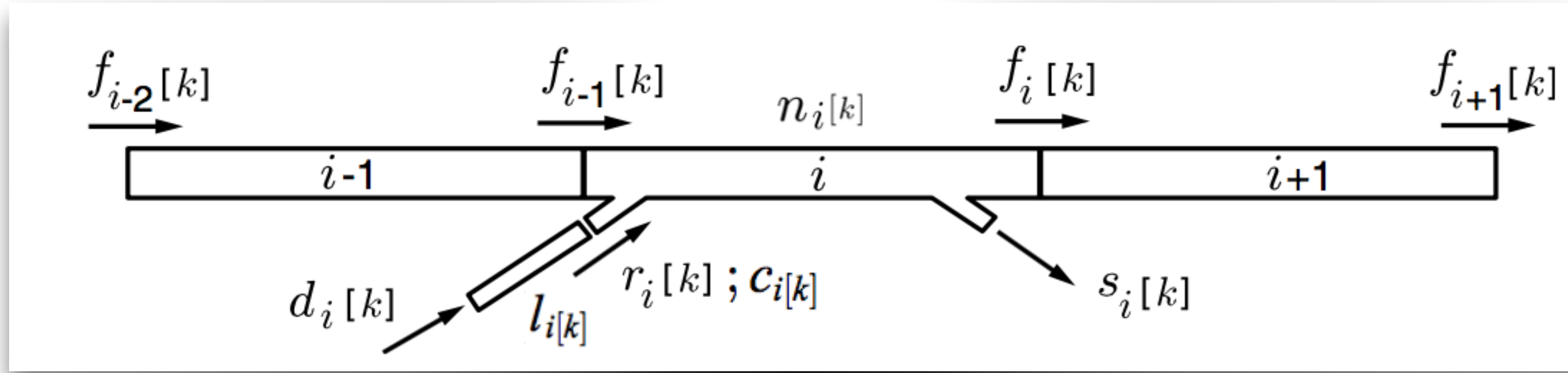
$$f_i[k] = \min\Big(\;(1-\beta_i[k])\,v_i\,n_i[k]\;,\; w_{i+1}(\bar{n}_{i+1} - n_{i+1}[k])\;,\; \bar{f}_i\;\Big)$$

On-ramp conservation:

$$l_i[k+1] = l_i[k] + d_i[k] - r_i[k]$$

Mainline conservation:

$$n_i[k+1] = n_i[k] + f_{i-1}[k] - f_i[k] - s_i[k]$$



On-ramp flows:

$$r_i[k] = \min\left(\, l_i[k] + d_i[k] \;,\; \xi_i(\bar{n}_i - n_i[k]) \;,\; c_i[k] \,\right)$$

on-ramp allocation

ramp metering rate

$\xi_i \in (0, 1]$

Mainline flows:

$$f_i[k] = \min\Big(\, (1-\beta_i[k])\, v_i\, n_i[k] \;,\; w_{i+1}(\bar{n}_{i+1} - n_{i+1}[k]) \;,\; \bar{f}_i \,\Big)$$

free flow

congestion

capacity

On-ramp conservation:

$$l_i[k+1] = l_i[k] + d_i[k] - r_i[k]$$

Mainline conservation:

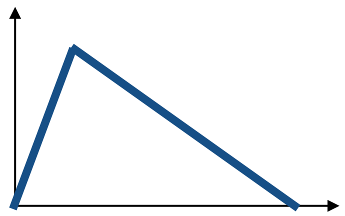
$$n_i[k+1] = n_i[k] + f_{i-1}[k] - f_i[k] - s_i[k]$$

Total travel time / Total travel distance

$$\mathrm{TTT} = \sum_k \sum_i (n_i[k] + l_i[k])$$

$$\mathrm{TTD} = \sum_k \sum_i (f_i[k] + r_i[k])$$

$$\text{Problem } \mathcal{N} \qquad \begin{array}{rl} \text{minimize} & \mathrm{TTT} - \eta\, \mathrm{TTD} \\ \text{subject to} & \text{Mainline and on-ramp flow,} \\ & \text{Mainline and on-ramp conservation,} \\ & l_i[k] \leq \bar{l}_i \\ & c_i[k] \in [\, \underline{c}_i \,,\, \bar{c}_i \,] \end{array}$$



Total travel time / Total travel distance

$$\mathrm{TTT} = \sum_k \sum_i \left(n_i[k] + l_i[k]\right)$$

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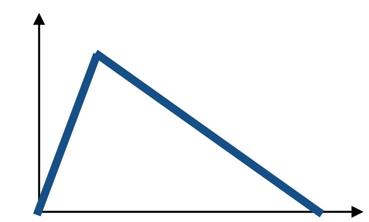
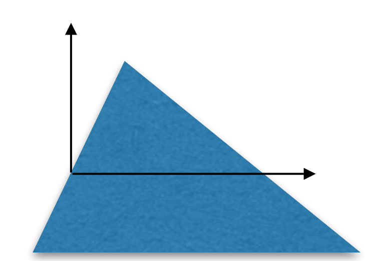
Problem $\mathcal{N}$

$$\begin{aligned} \text{minimize} \quad & \mathrm{TTT} - \eta\, \mathrm{TTD} \\ \text{subject to} \quad & \text{Mainline and on-ramp flow,} \\ & \text{Mainline and on-ramp conservation,} \\ & l_i[k] \le \bar{l}_i \\ & c_i[k] \in [\, \underline{c}_i \,,\, \bar{c}_i \,] \end{aligned}$$



Problem $\mathcal{L}$

$$\begin{aligned} \text{minimize} \quad & \mathrm{TTT} - \eta\, \mathrm{TTD} \\ \text{subject to} \quad & \boxed{\textit{Relaxed} \text{ mainline and on-ramp flow,}} \\ & \text{Mainline and on-ramp conservation,} \\ & l_i[k] \le \bar{l}_i \\ & c_i[k] \in [\, \underline{c}_i \,,\, \bar{c}_i \,] \end{aligned}$$



Theorem

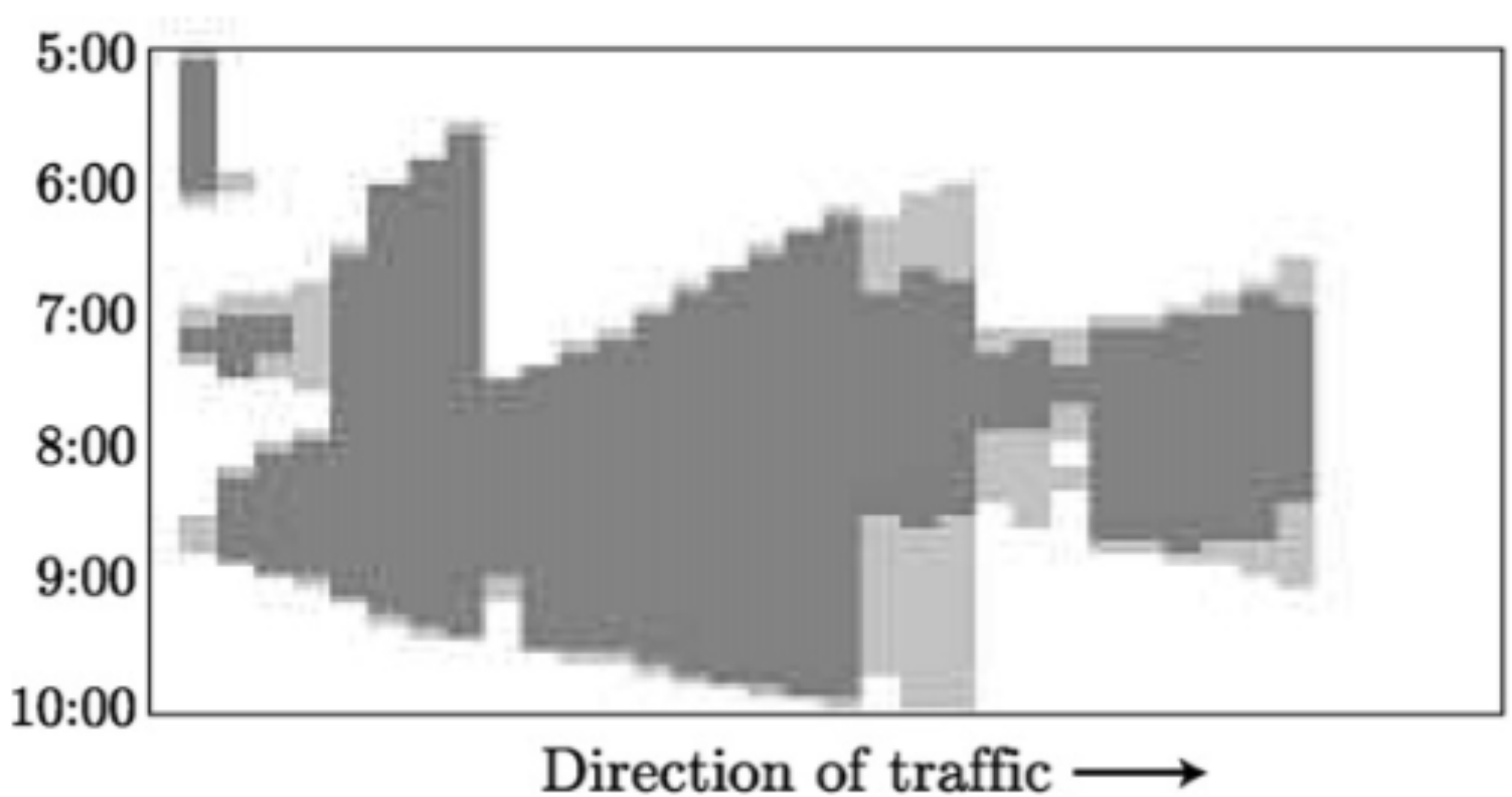
A solution to $\mathcal{L}$ also solves $\mathcal{N}$ whenever,

- $v_i < 1$,
- $w_i < 1$,
- $\beta_i[k]$'s are constant in time,
- $\underline{c}_i = 0$,
- the final condition of the freeway is *empty*,
- $r_i[k] < \xi_i(\bar{n}_i - n_i[k])$

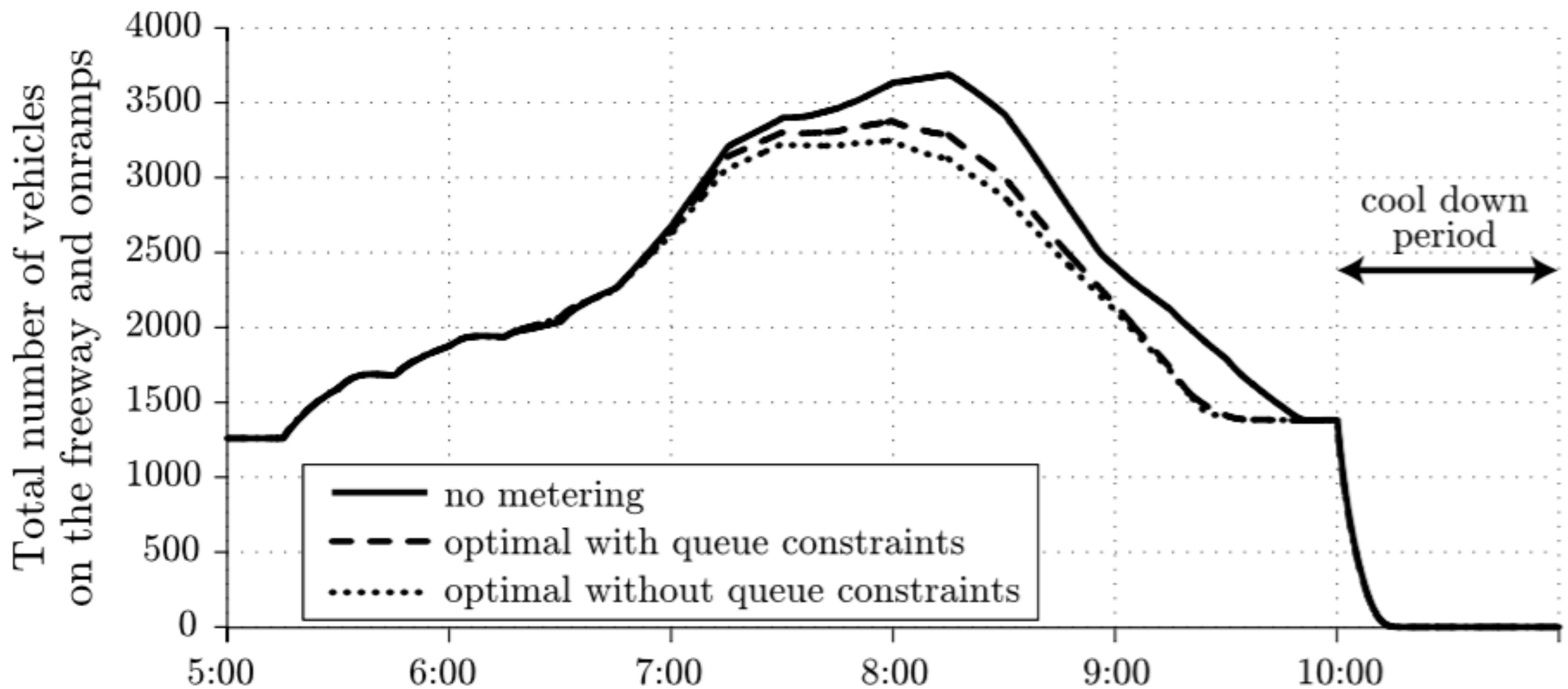
congestion does not propagate onto the onramps

Numerical experiment

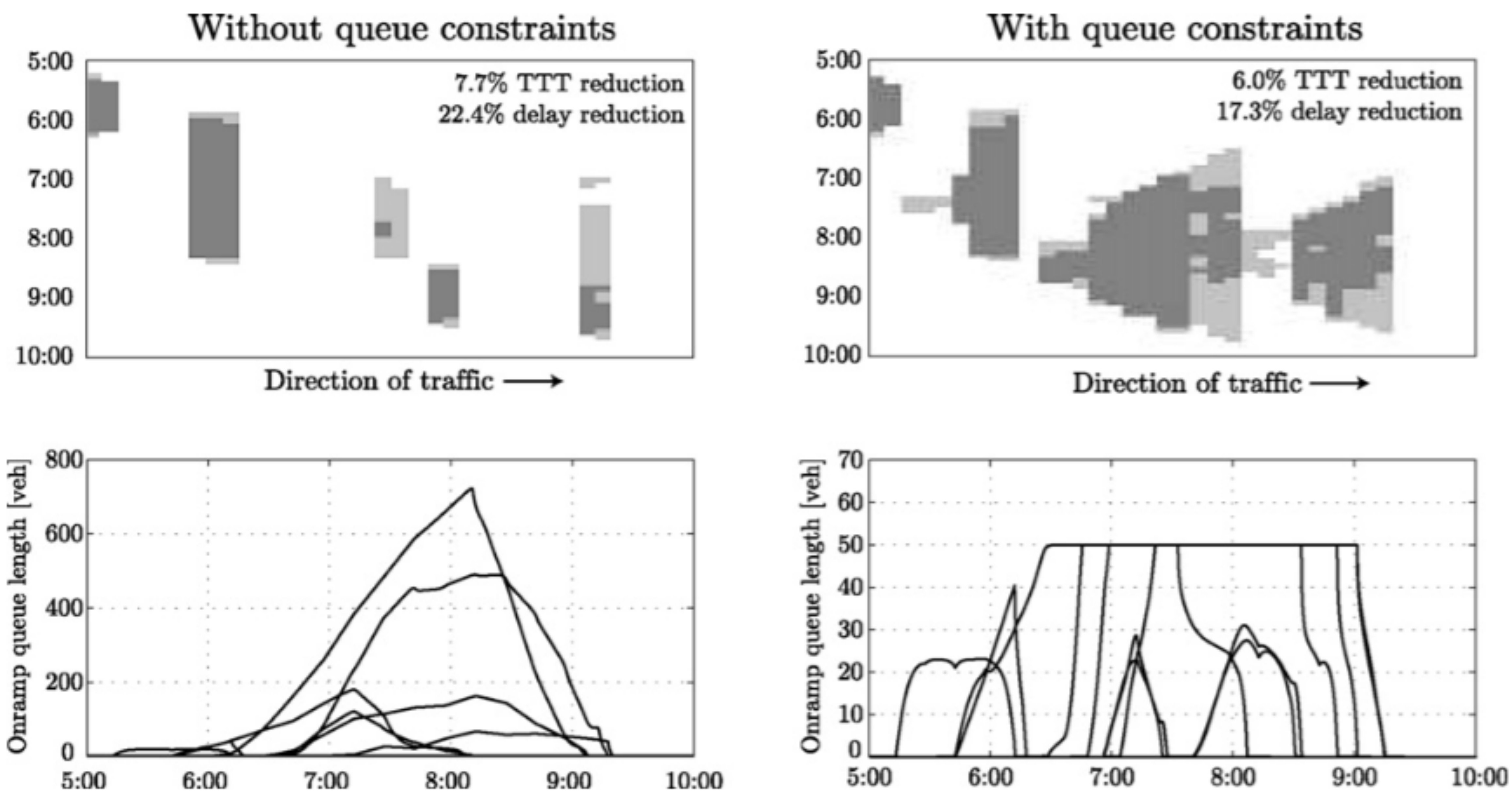
30 miles of I-210 in Pasadena, California, USA



Numerical experiment

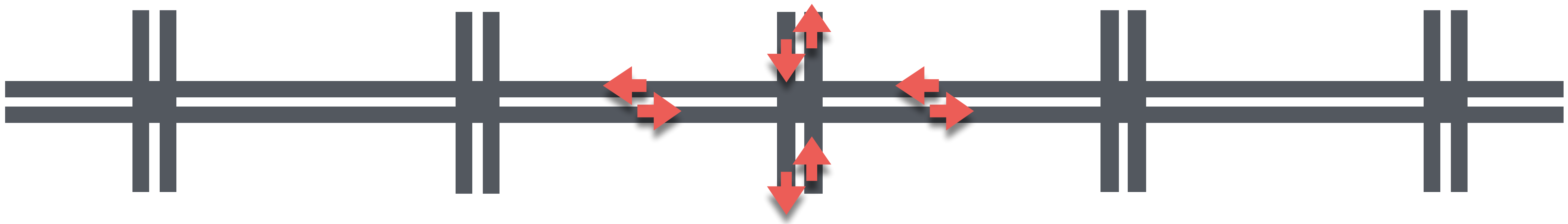


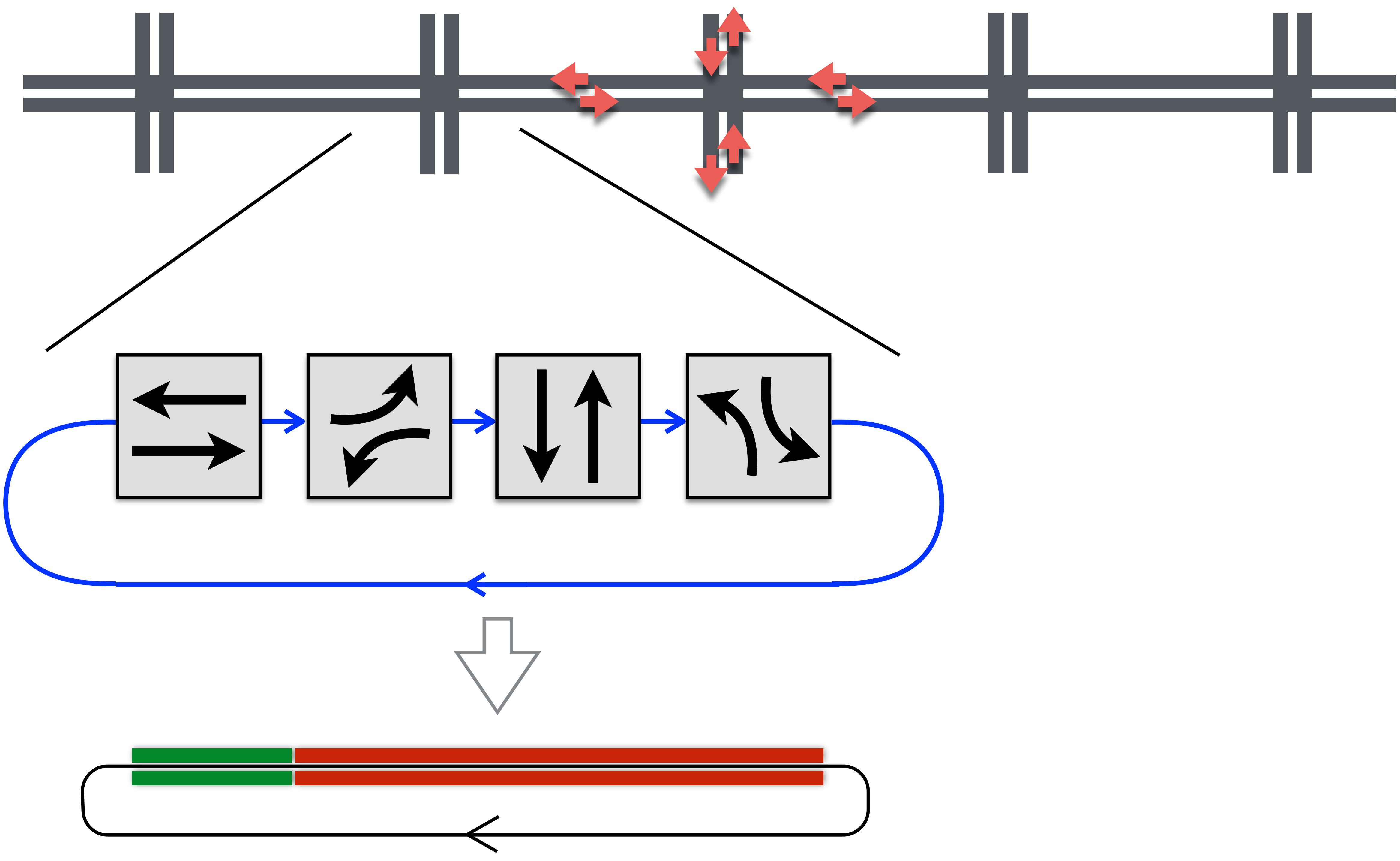
Numerical experiment

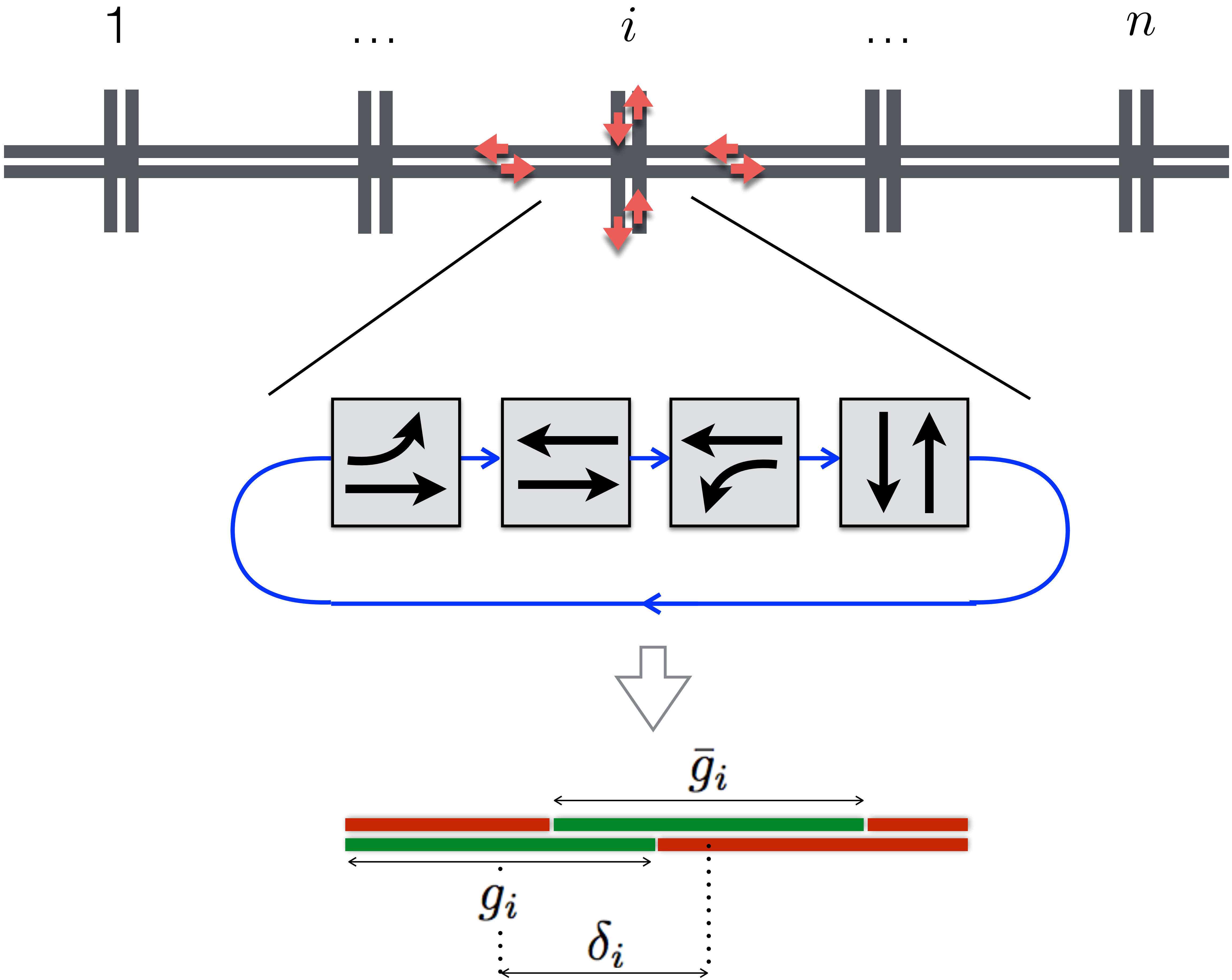


Part II

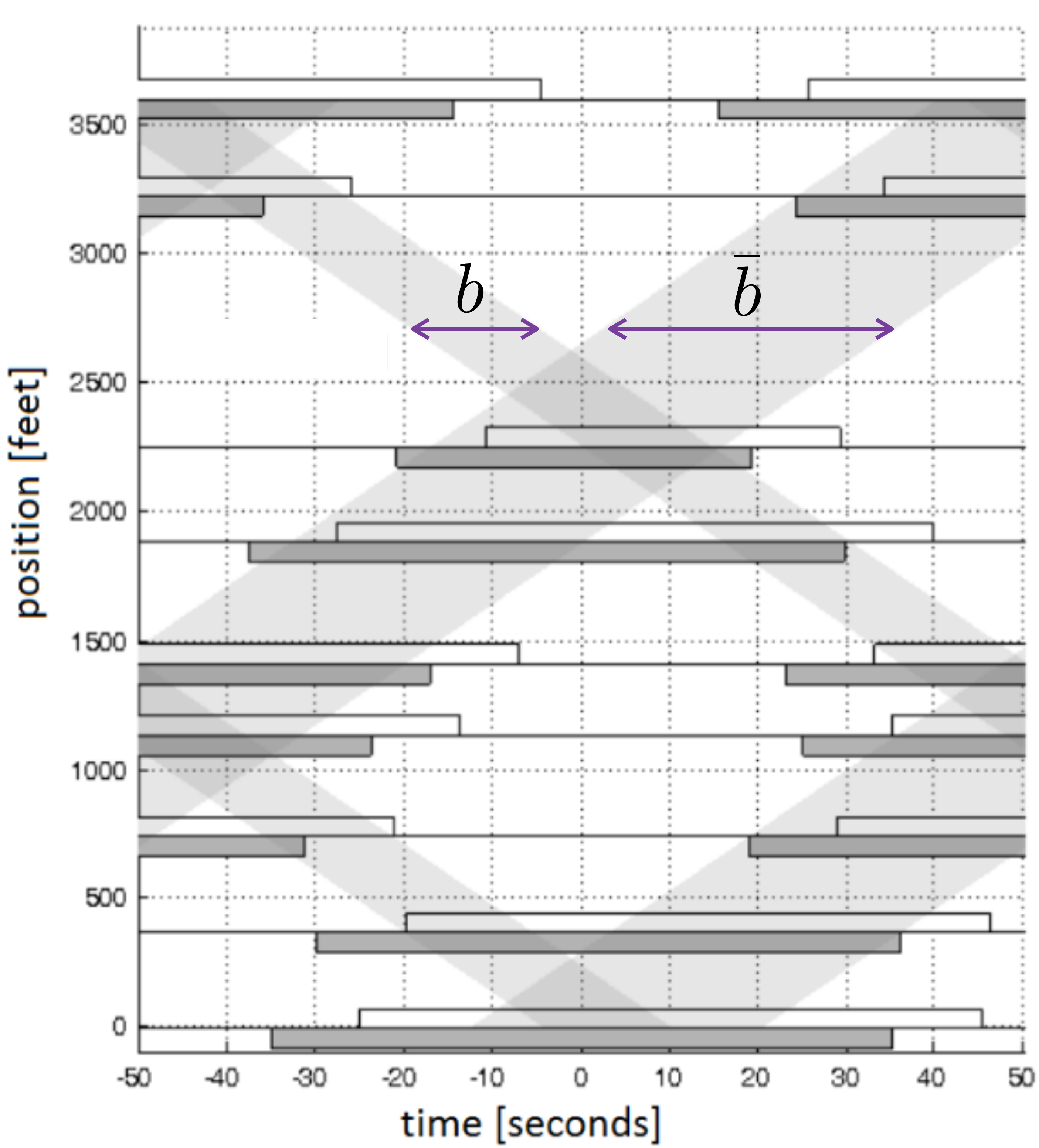
Signal coordination on streets

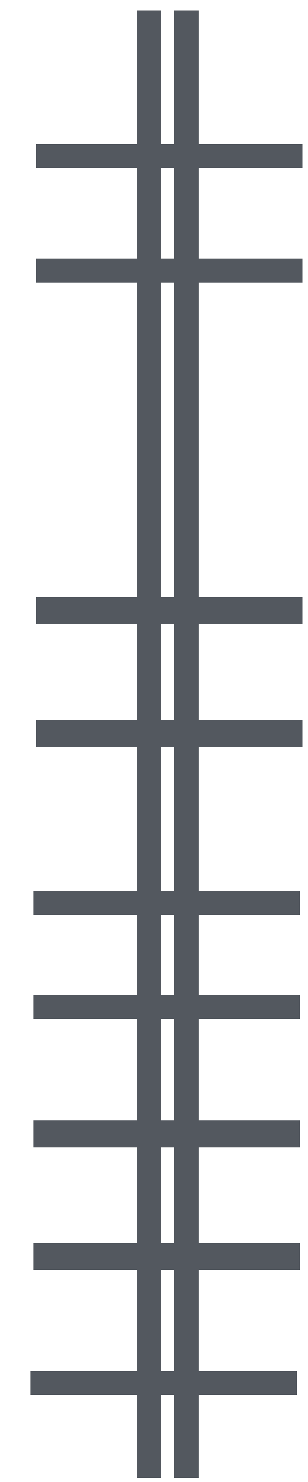


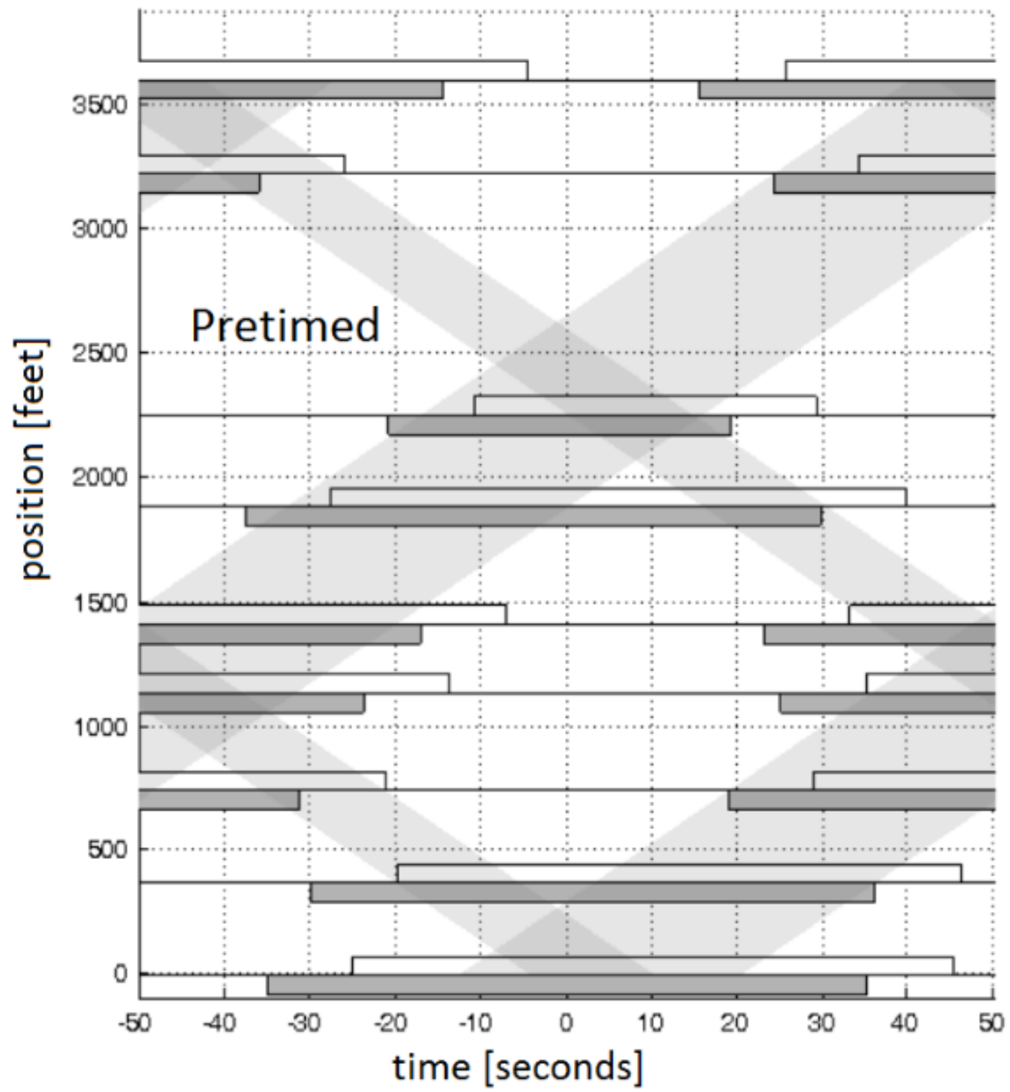




1
...
i
...
n
$\bar{g}_i$
g_i
δ_i


position [feet]
3500
3000
2500
2000
1500
1000
500
0
-50
-40
-30
-20
-10
0
10
20
30
40
50
time [seconds]
b
$\bar{b}$





Pretimed
position [feet]
time [seconds]
3500
3000
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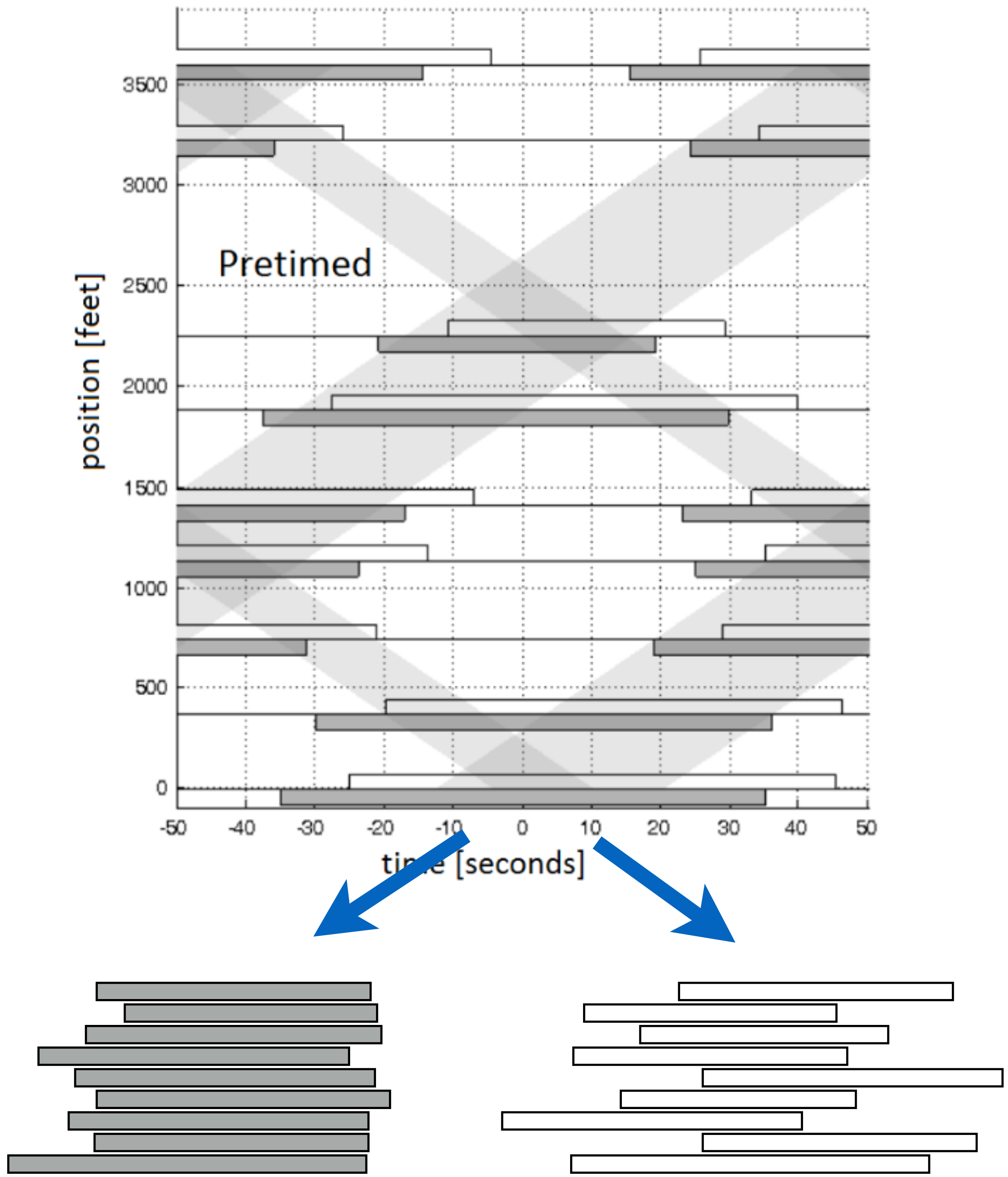
-20
-10
0
10
20
30
40
50



Pretimed
position [feet]
3500
3000
2500
2000
1500
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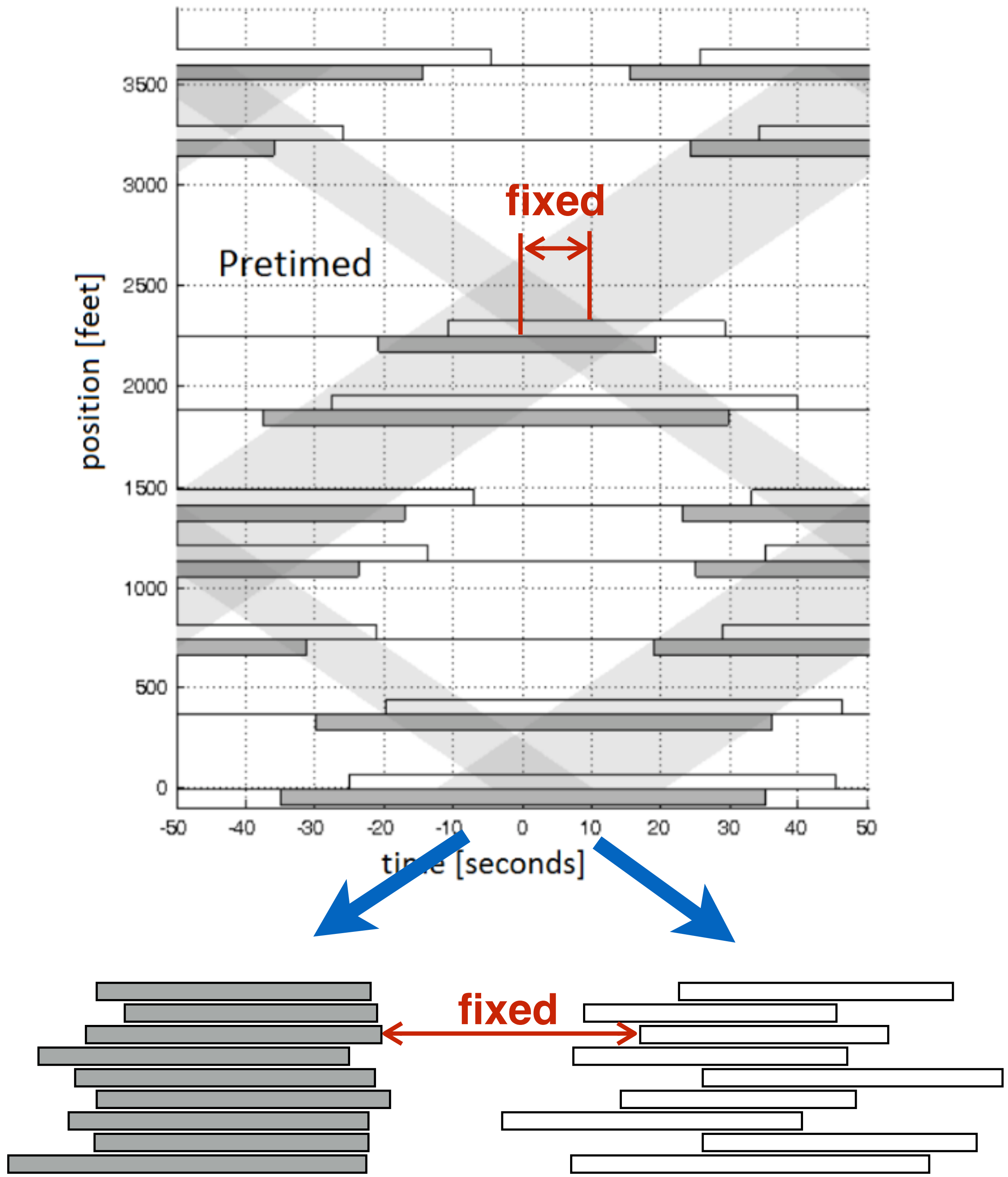
0
-50
-40
-30
-20
-10
0
10
20
30
40
50
time [seconds]



fixed
Pretimed
position [feet]
3500
3000
2500
2000
1500
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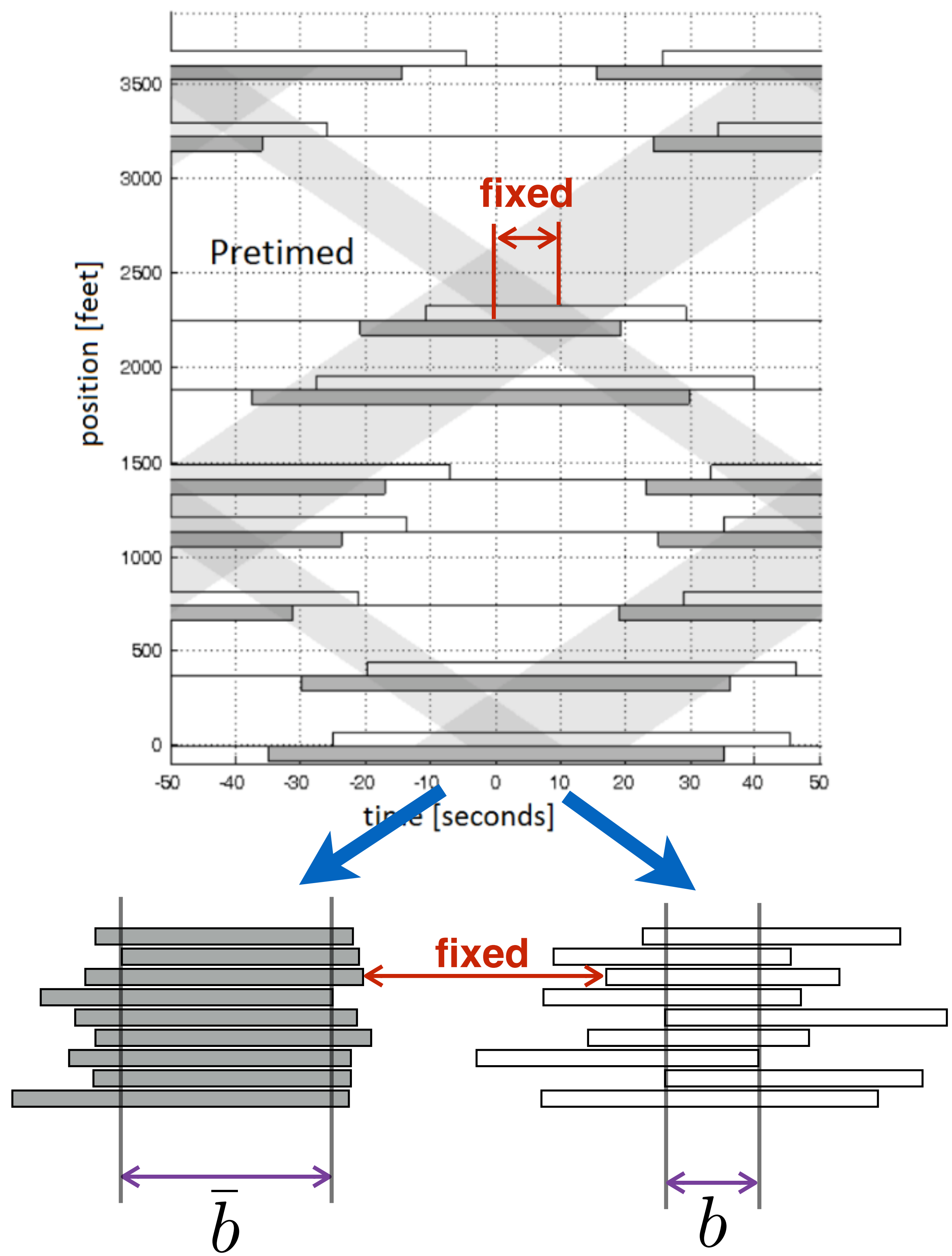
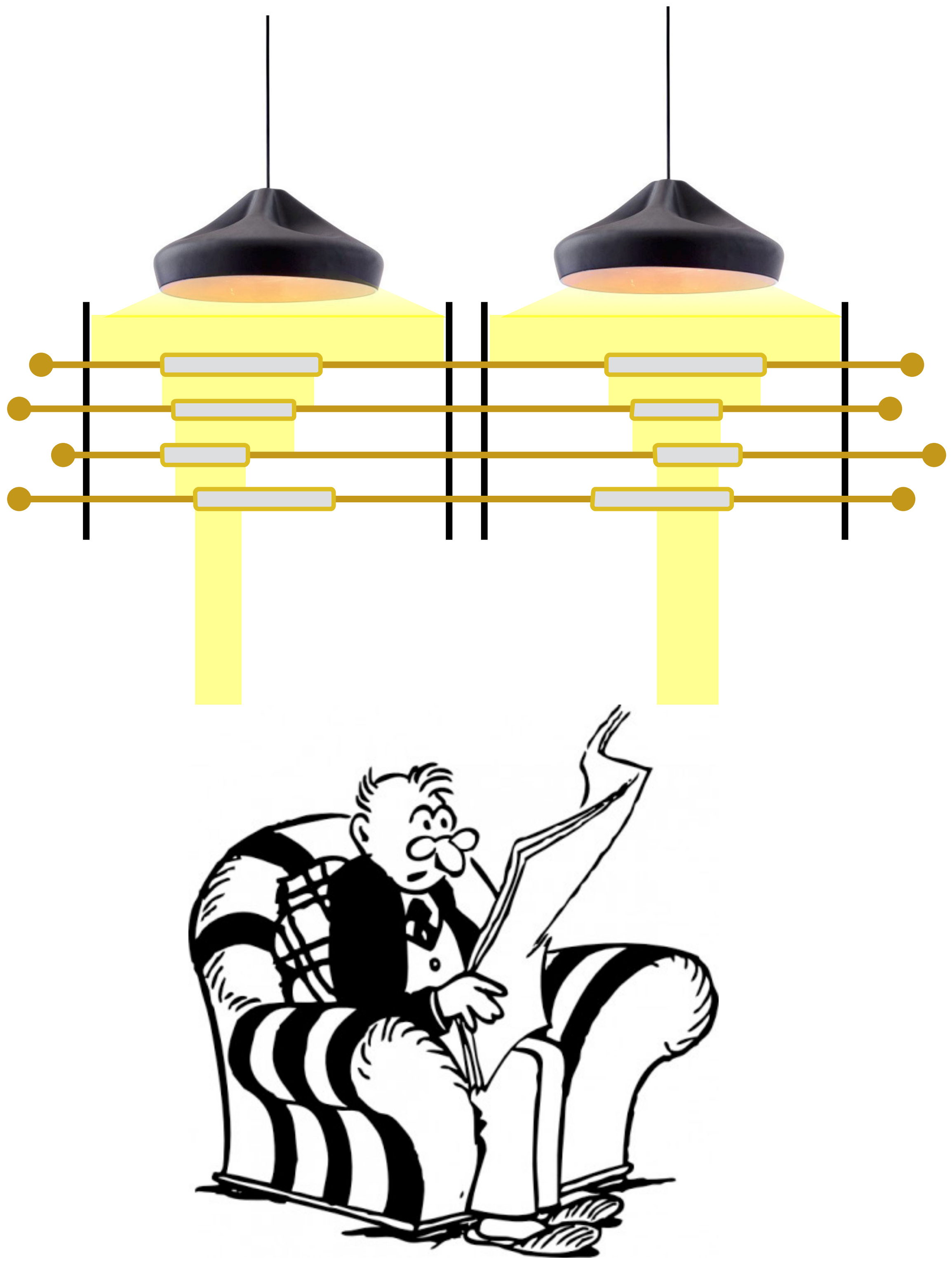
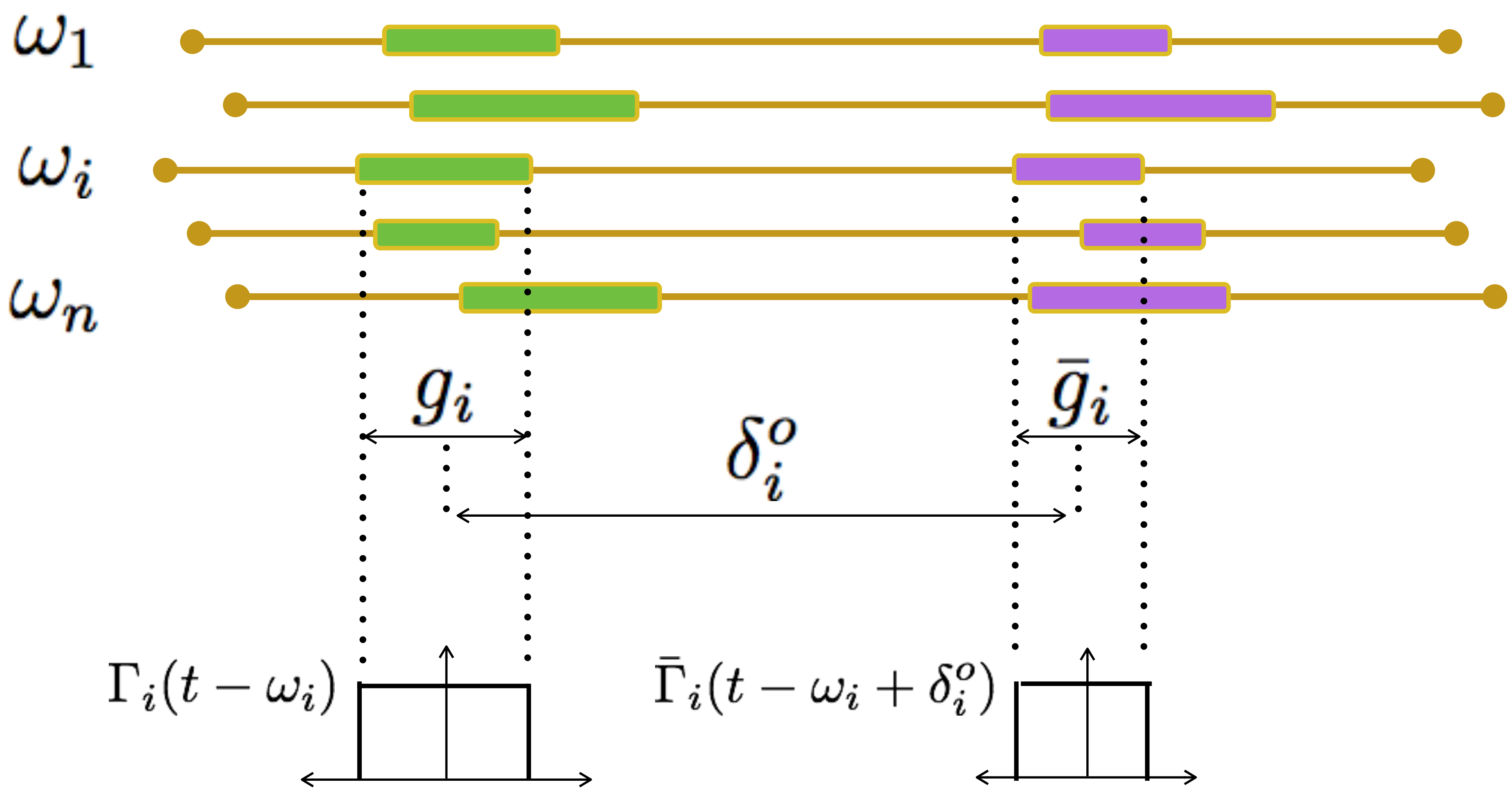
0
-50
-40
-30
-20
-10
0
10
20
30
40
50
time [seconds]
fixed



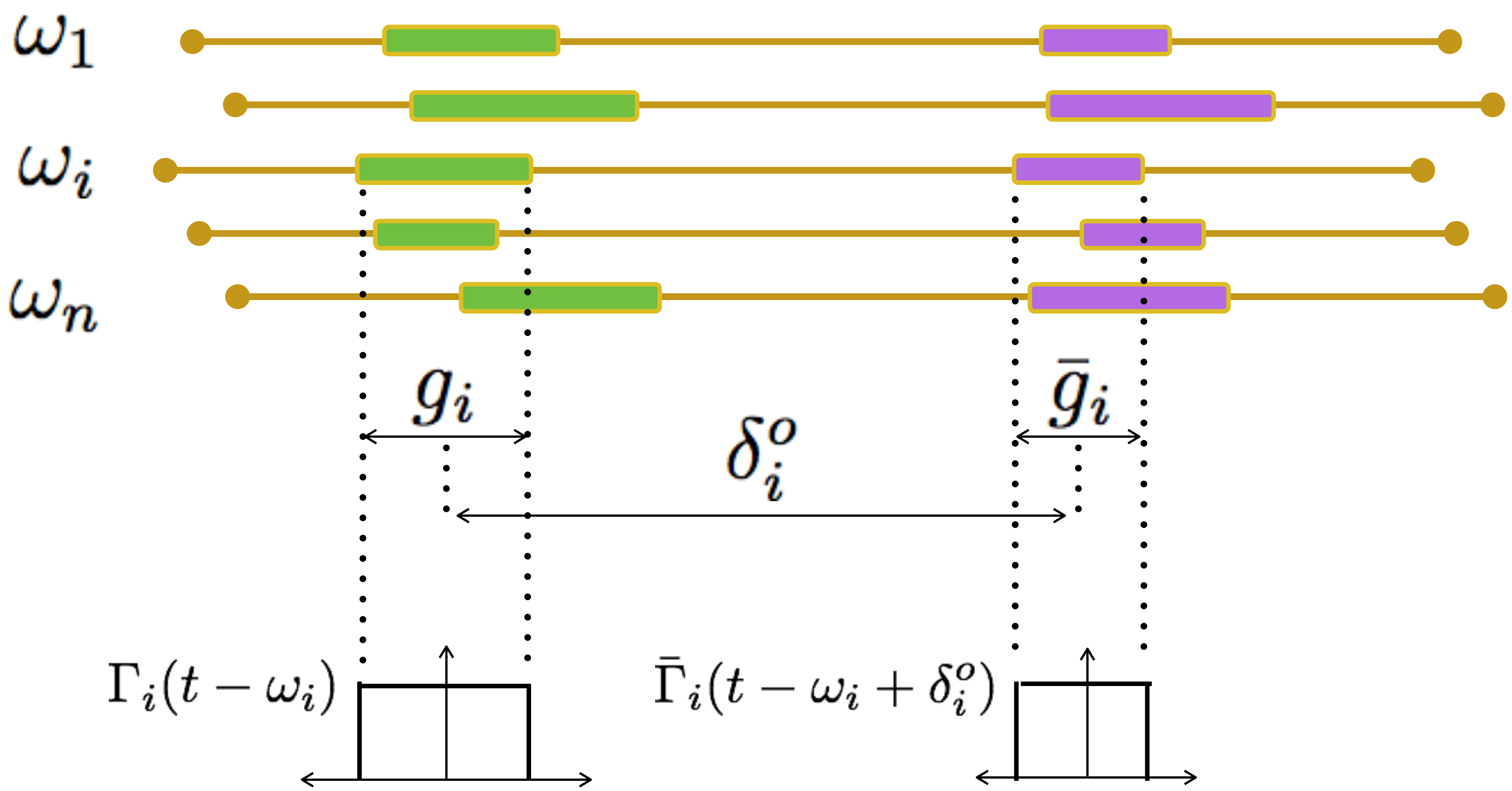
Pretimed
fixed
position [feet]
time [seconds]
3500
3000
2500
2000
1500
1000
500
0
-50
-40
-30
-20
-10
0
10
20
30
40
50
fixed
$\bar{b}$
b

Analogy





$$b(\omega) \;=\; \int \prod_{i=1}^{n} \Gamma_i(t-\omega_i) \;=\; \max\left(0\,,\; \min_{i,j}(\omega_i - \omega_j + g_{i,j})\right)$$



by definition

apply square Gamma

$$b(\omega) \;=\; \int \prod_{i=1}^{n} \Gamma_i(t-\omega_i) \;=\; \max\left(0\ ,\ \min_{i,j}(\omega_i-\omega_j+g_{i,j})\right)$$

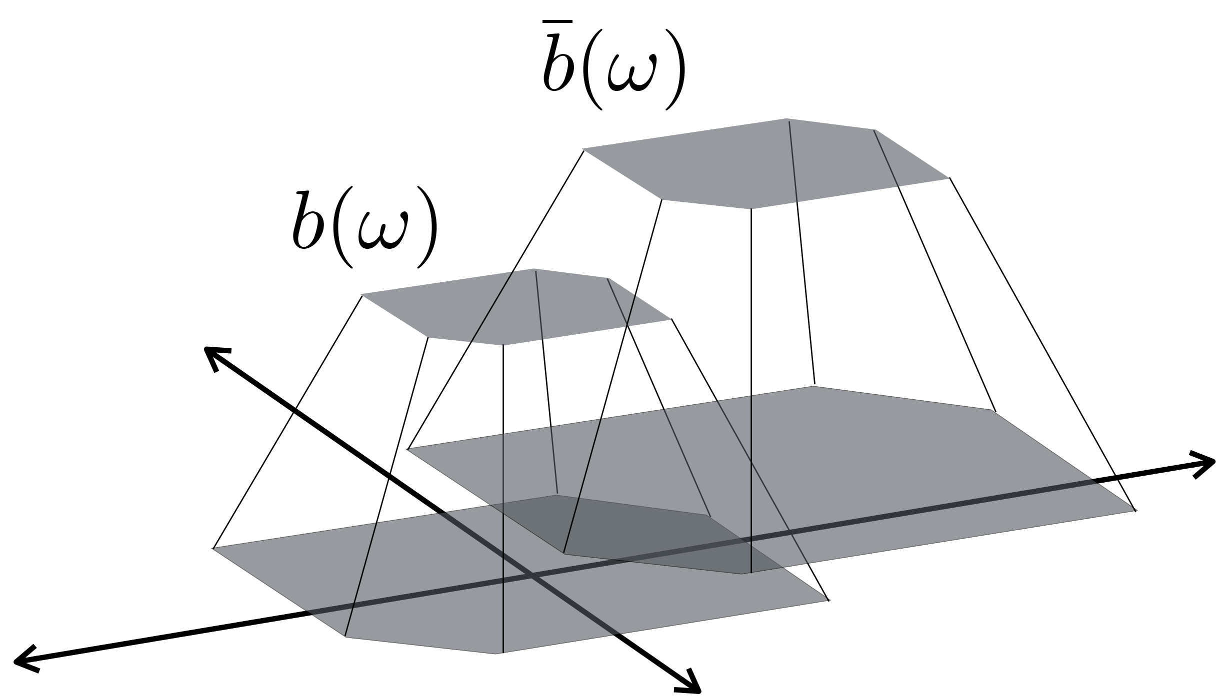
$$\omega = \{\omega_1, \ldots, \omega_n\} \qquad\qquad g_{i,j} \triangleq (g_i+g_j)/2$$

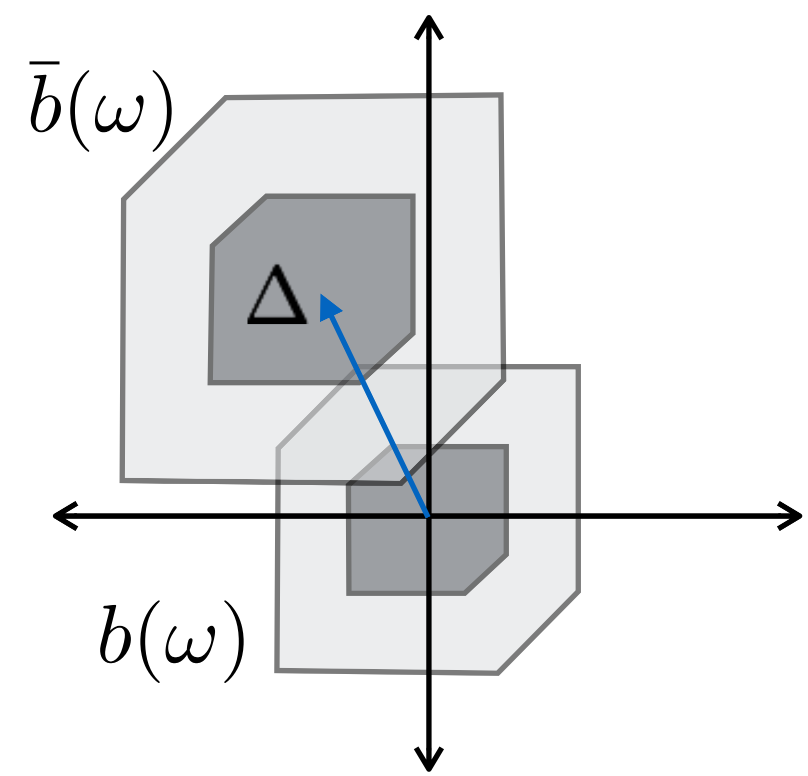
$$b(\omega) = \max\left(0, \min_{i,j}\left(\omega_i - \omega_j + g_{i,j}\right)\right)$$

$$\bar{b}(\omega) = \max\left(0, \min_{i,j}\left((\omega_i + \delta_i^o) - (\omega_j + \delta_j^o) + \bar{g}_{i,j}\right)\right)$$

$$\Delta = \{\delta_1^o, \ldots, \delta_n^o\}$$

$$\omega = \{\omega_1, \ldots, \omega_n\}$$



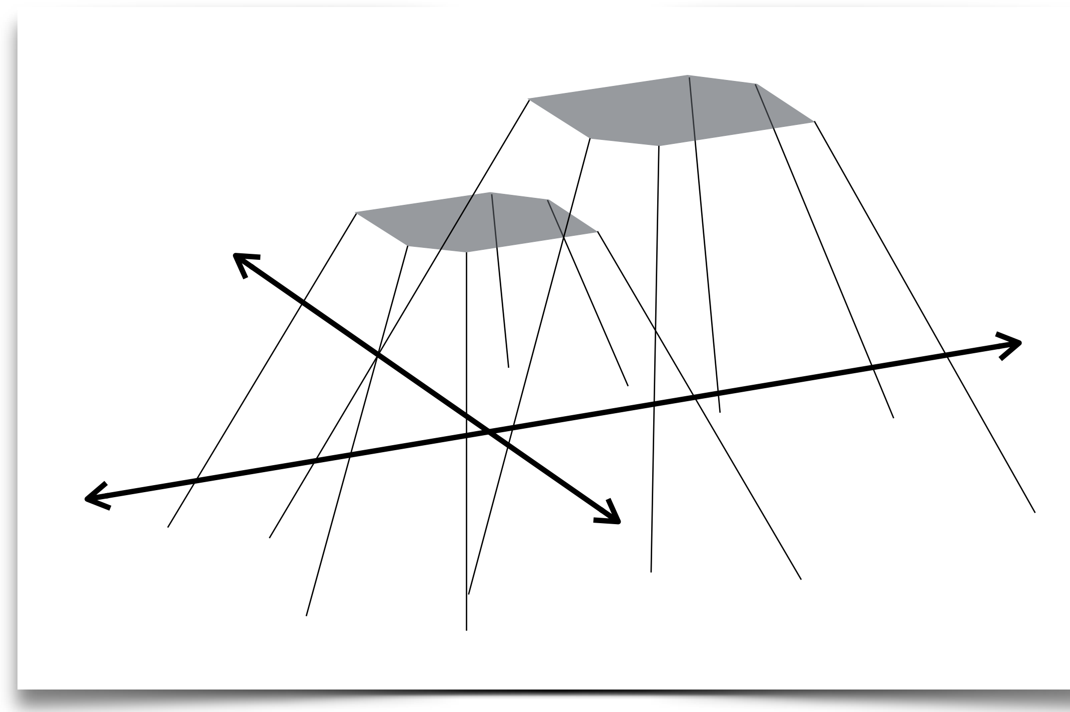


Problem $\mathcal{N}$: $\quad$ maximize $b(\omega) + \bar{b}(\omega)$

Relaxation:

$$b_r(\omega) = \min_{i,j} \left(\omega_i - \omega_j + g_{i,j}\right)$$
$$\bar{b}_r(\omega) = \min_{i,j} \big((\omega_i + \delta_i^o) - (\omega_j + \delta_j^o) + \bar{g}_{i,j}\big)$$

- $b_r(\omega)$ and $\bar{b}_r(\omega)$ are concave.
- Hence we can pose a *linear* program.

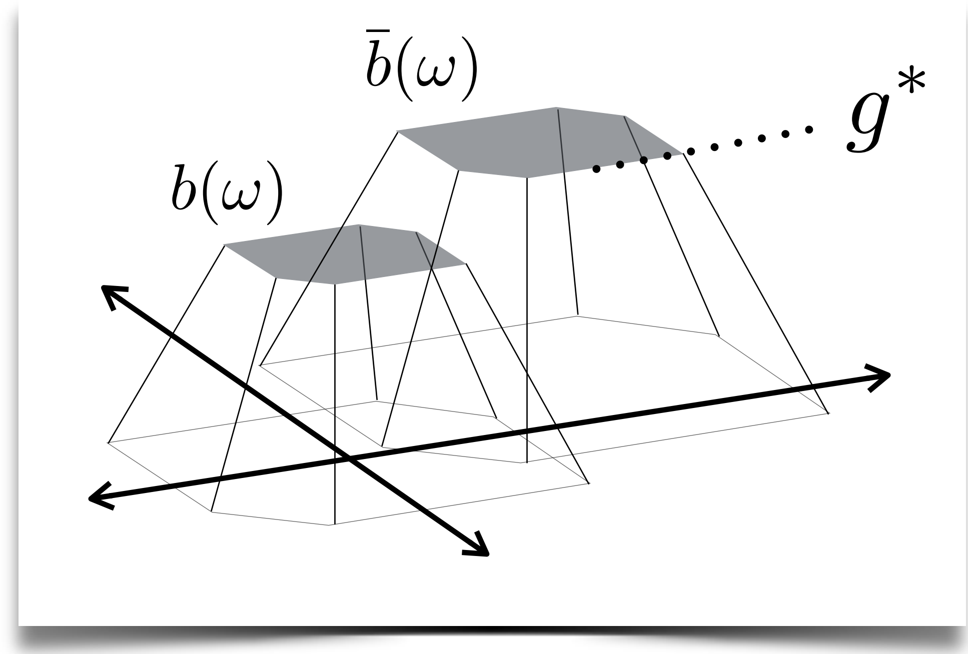


Problem $\mathcal{L}$:

$$\begin{aligned}
\text{maximize} \quad & b_r + \bar{b}_r & \\
\text{subject to} \quad & b_r \leq \omega_i - \omega_j + g_{i,j} & \forall i, j \\
& \bar{b}_r \leq \omega_i + \delta_i^o - \omega_j - \delta_j^o + \bar{g}_{i,j} & \forall i, j
\end{aligned}$$

Definitions:

- $$g^* \triangleq \max\left(\max_{\Omega}(b(\omega)) \ , \ \max_{\Omega}(\bar{b}(\omega)) \right)$$
  $$= \max\left(\quad \min_{i}(g_i) \quad , \quad \min_{i}(\bar{g}_i) \quad \right)$$



- $\omega^*_{\mathcal{L}}$ ... solution to $\mathcal{L}$
- $\omega^*_{\mathcal{N}}$ ... solution to $\mathcal{N}$

Definitions:

- $$g^* \triangleq \max\left(\max_{\Omega}(b(\omega)) \ , \ \max_{\Omega}(\bar{b}(\omega)) \right)$$
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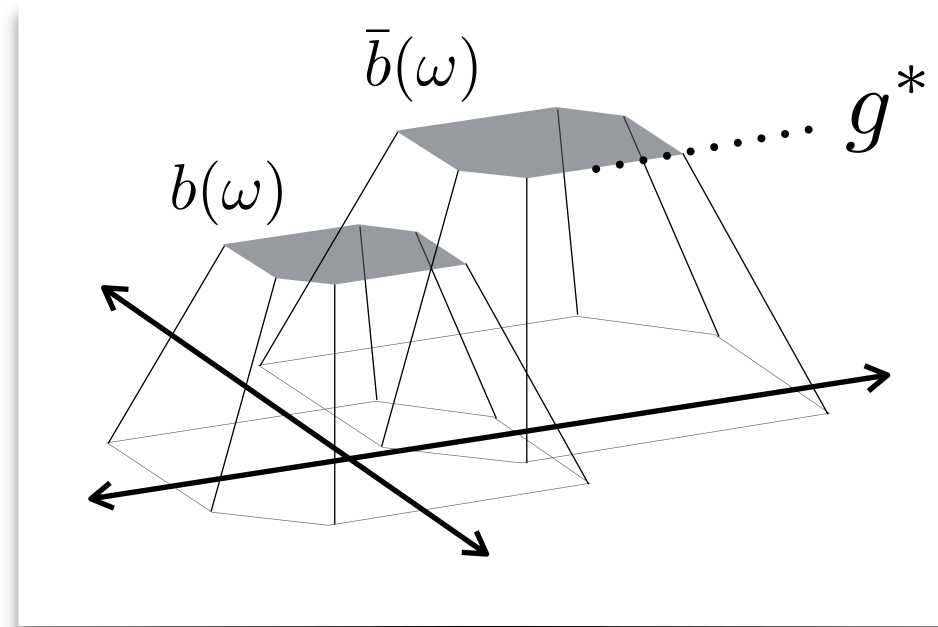
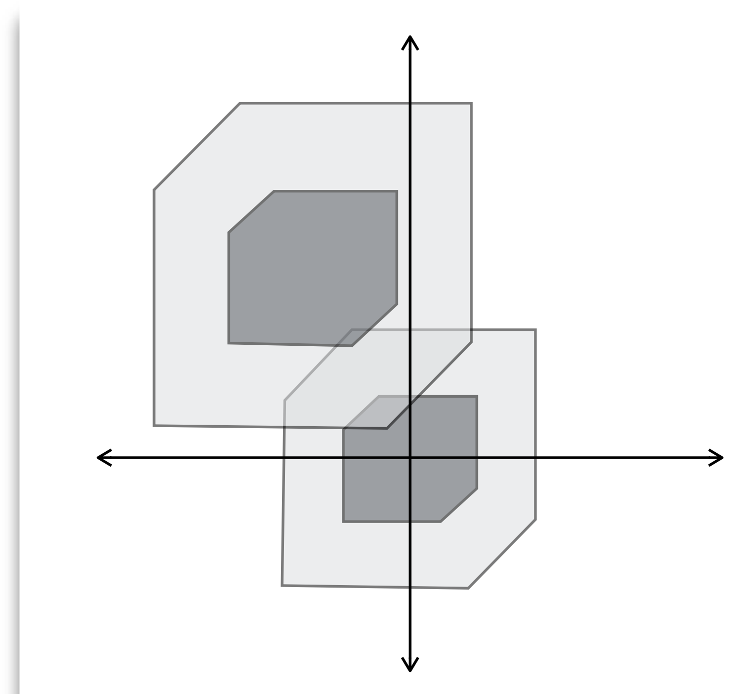


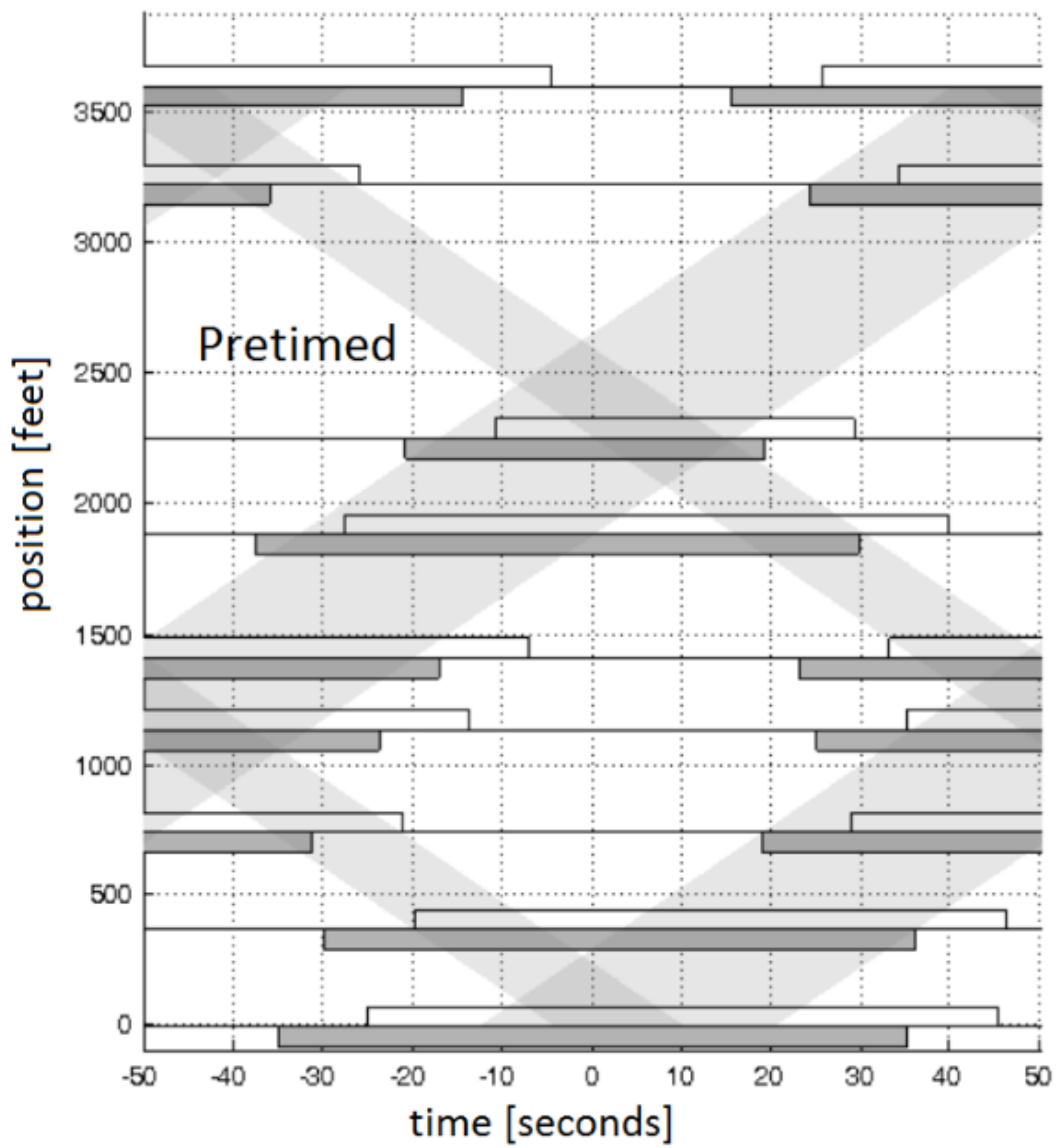
- $\omega^*_{\mathcal{L}}$ ... solution to $\mathcal{L}$
- $\omega^*_{\mathcal{N}}$ ... solution to $\mathcal{N}$

Theorem:

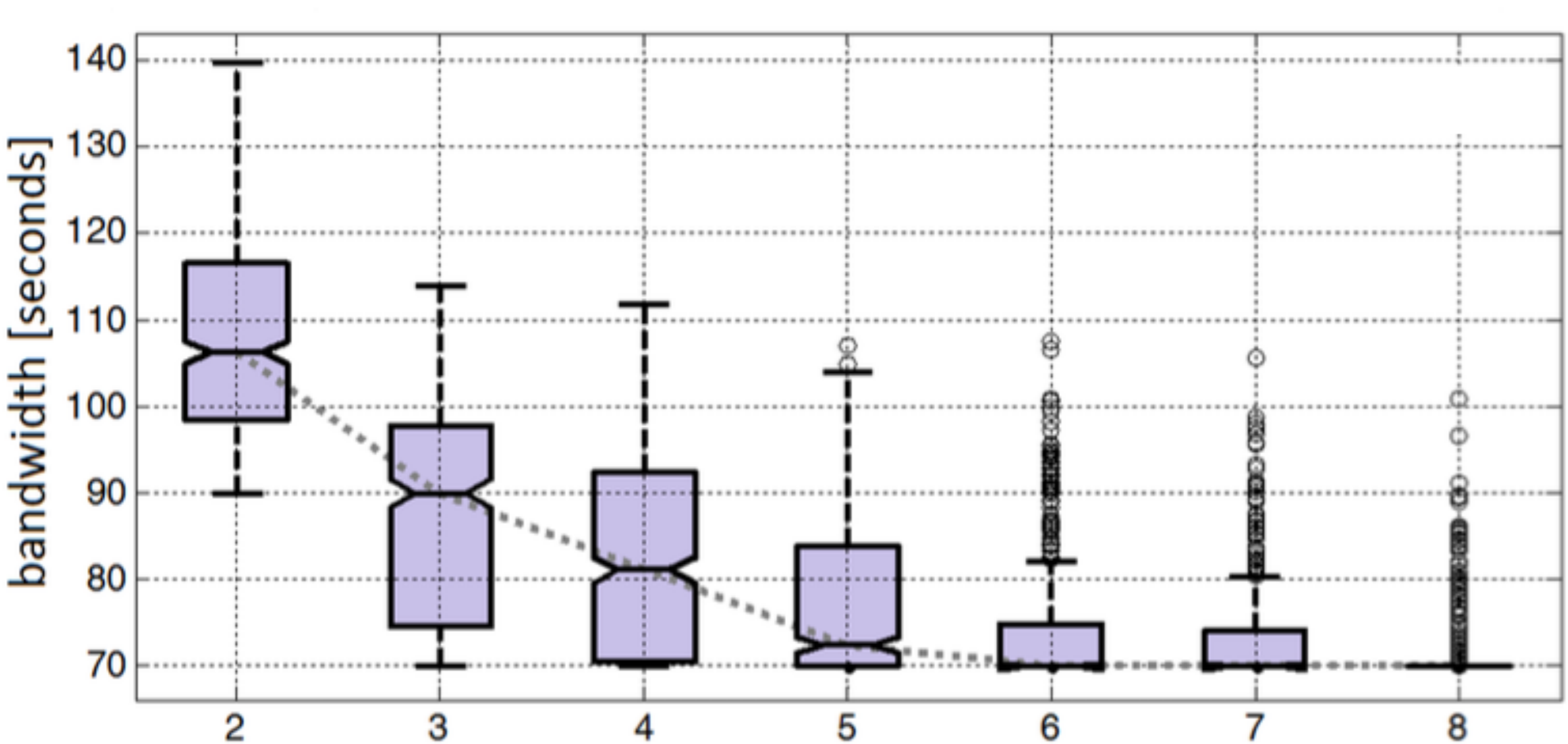
$$\omega^*_{\mathcal{N}} = \begin{cases} \omega^*_{\mathcal{L}} & f_{\mathcal{L}}(\omega^*_{\mathcal{L}}) \geq g^* \\ 0 \text{ or } \Delta & \text{otherwise} \end{cases}$$



Numerical experiment



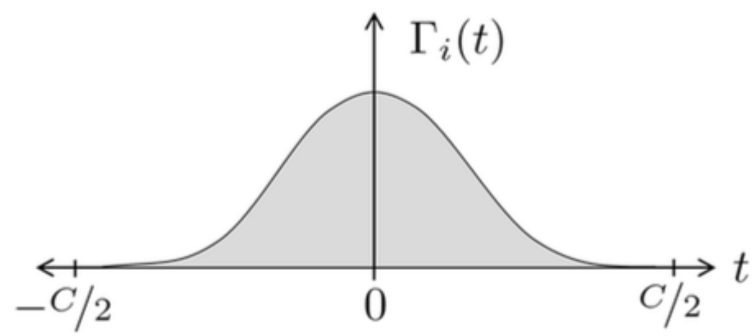
Numerical experiment



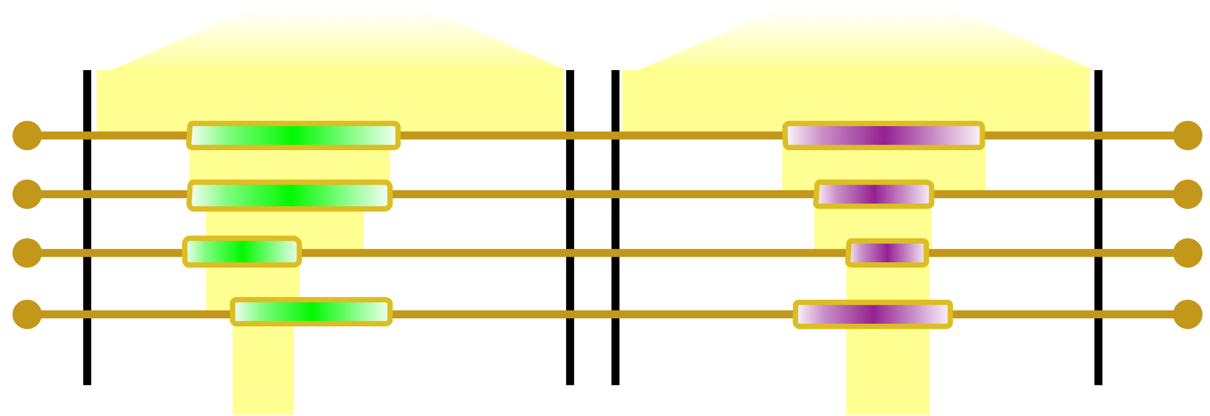
Extensions

Gaussian arrival functions

Platoon dispersion:

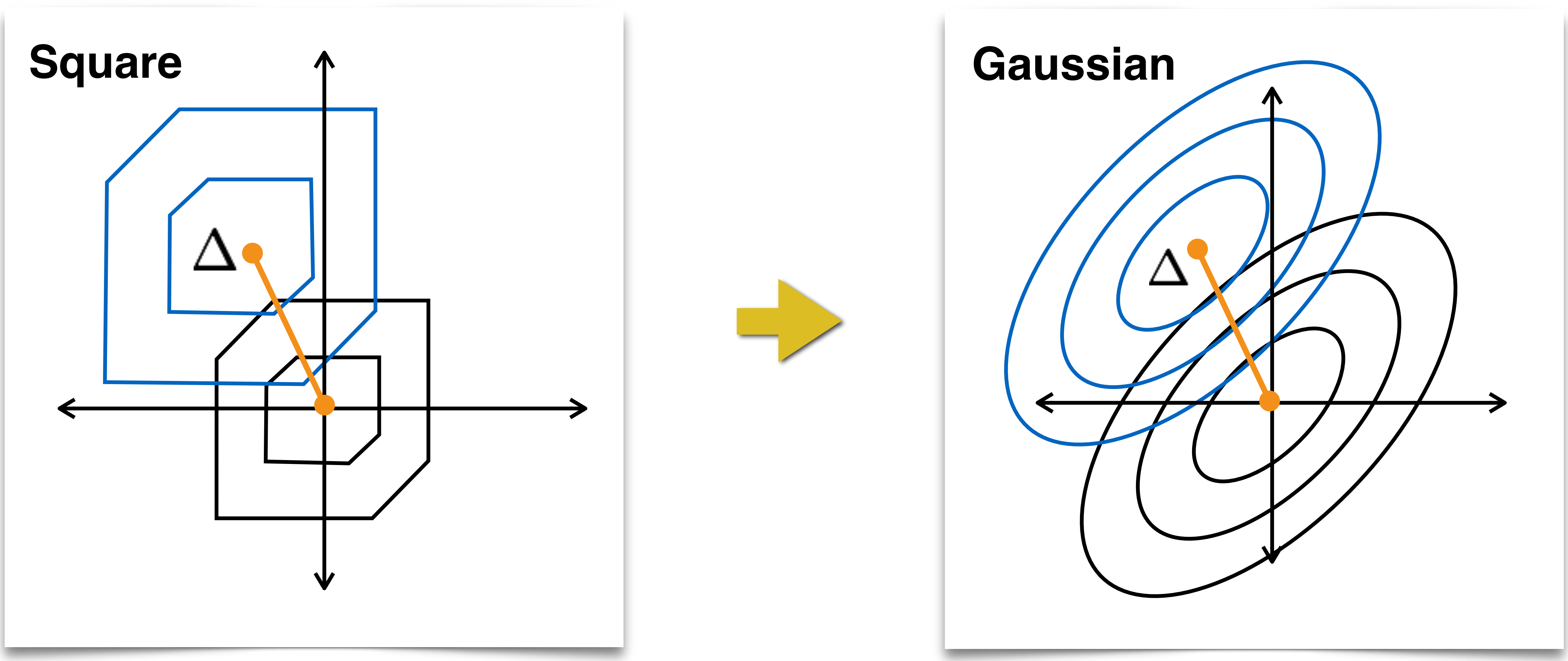


Lenses with varying transparency:



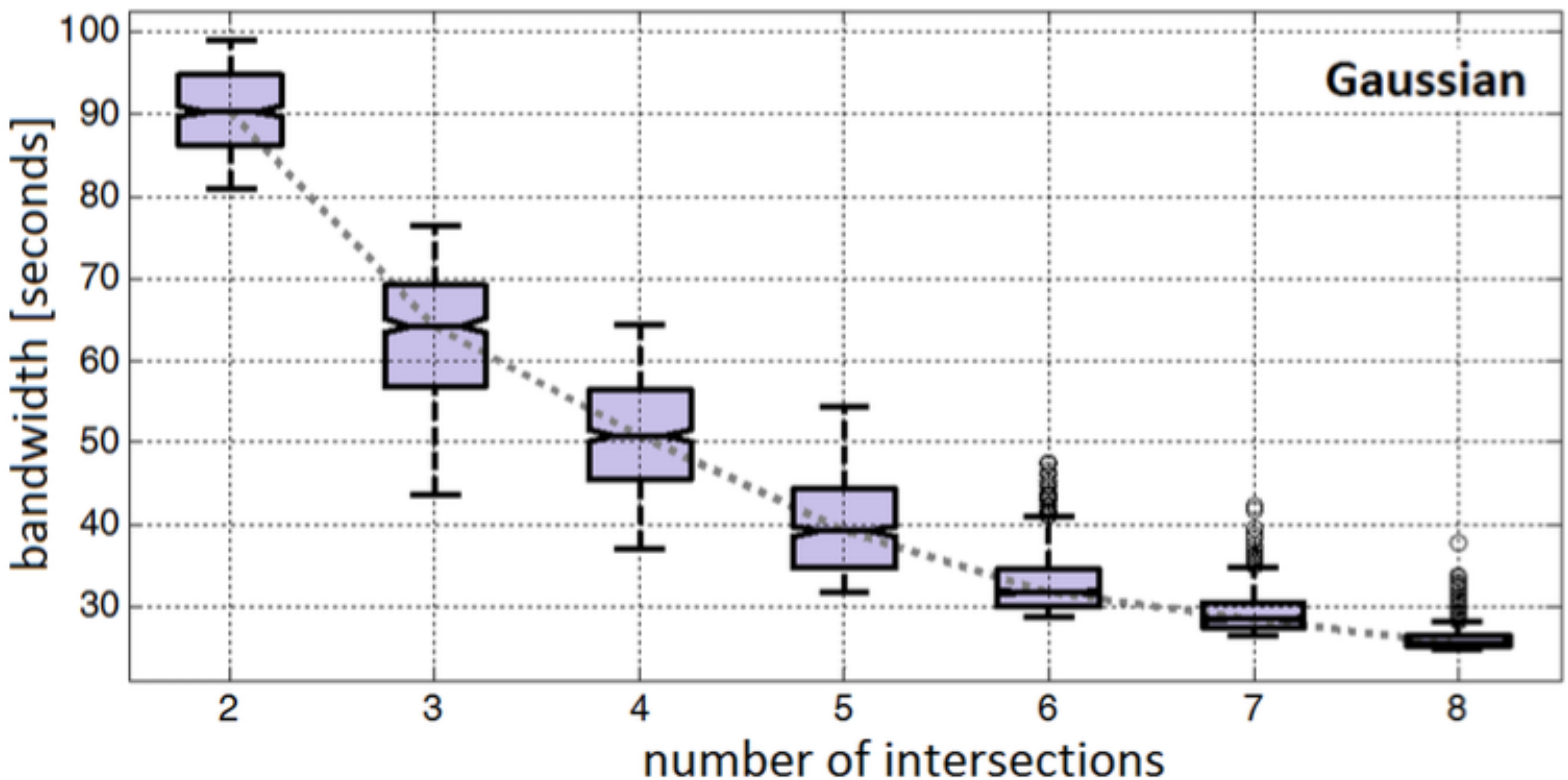
$$\begin{aligned} b(\omega) &= \int \prod_{i=1}^{n} \Gamma_i(t-\omega_i) && = \gamma\, \mathcal{G}(\omega|0,\Sigma) \\ \bar{b}(\omega) &= \int \prod_{i=1}^{n} \bar{\Gamma}_i(t-\omega_i+\delta_i^o) && = \bar{\gamma}\, \mathcal{G}(\omega|\Delta,\bar{\Sigma}) \end{aligned}$$

Gaussian arrival functions



- [Carreira & Perpiñán, 2000]: The maximum is on the line segment $[0, \Delta]$
- The problem is reduced to a line search (Newton's method).

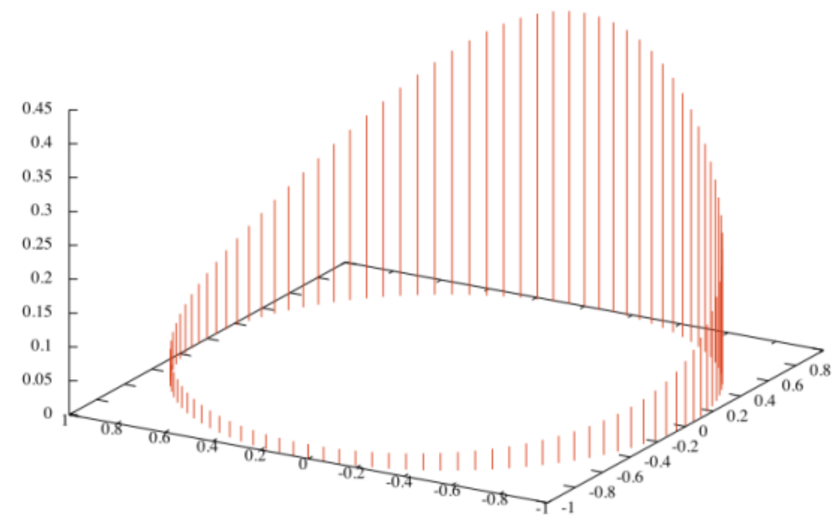
Gaussian arrival functions



von Mises arrival functions

$$\Gamma(\theta|\kappa_i) = \frac{\exp(\kappa_i \cos(\theta))}{2\pi I_0(\kappa_i)}$$

$$\theta \in [0, 2\pi]$$



von Mises arrival functions

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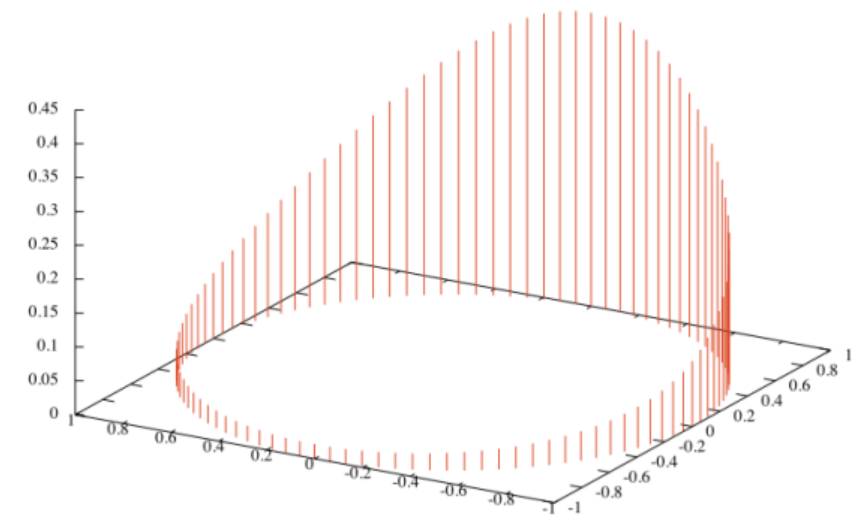
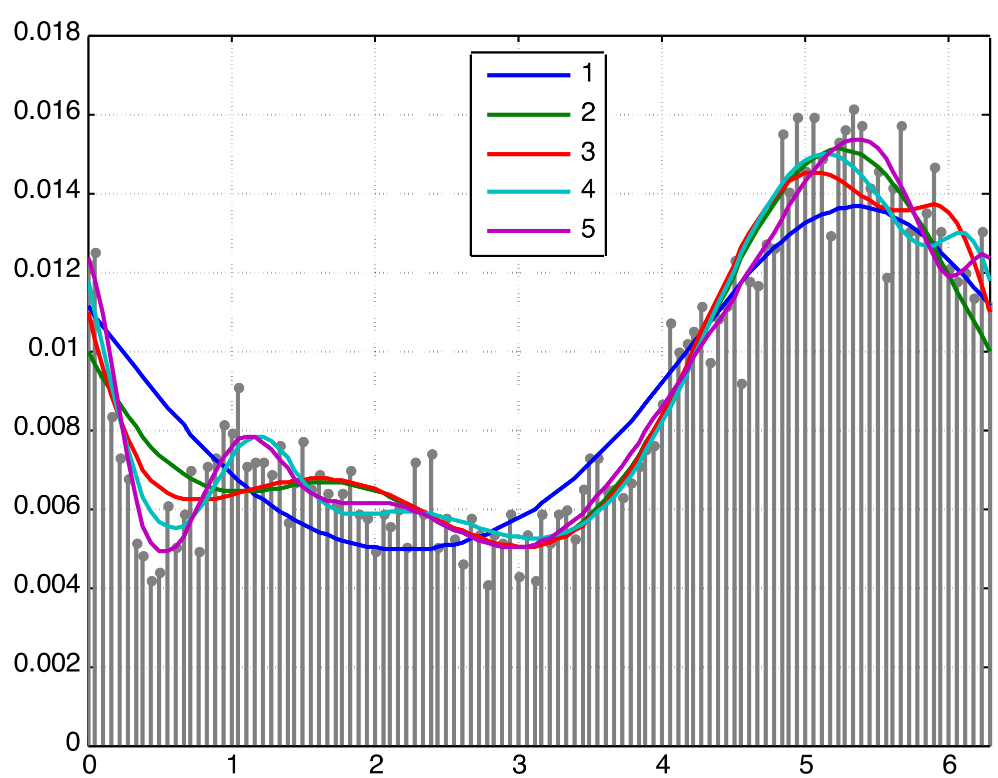
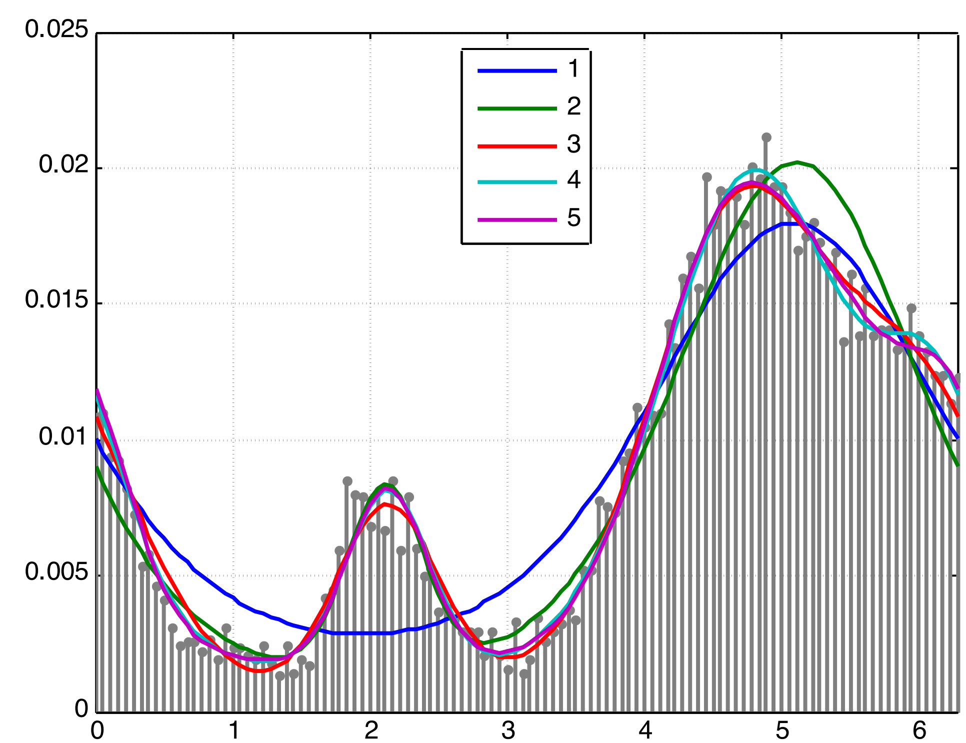


Fit von Mises mixtures to arrival data

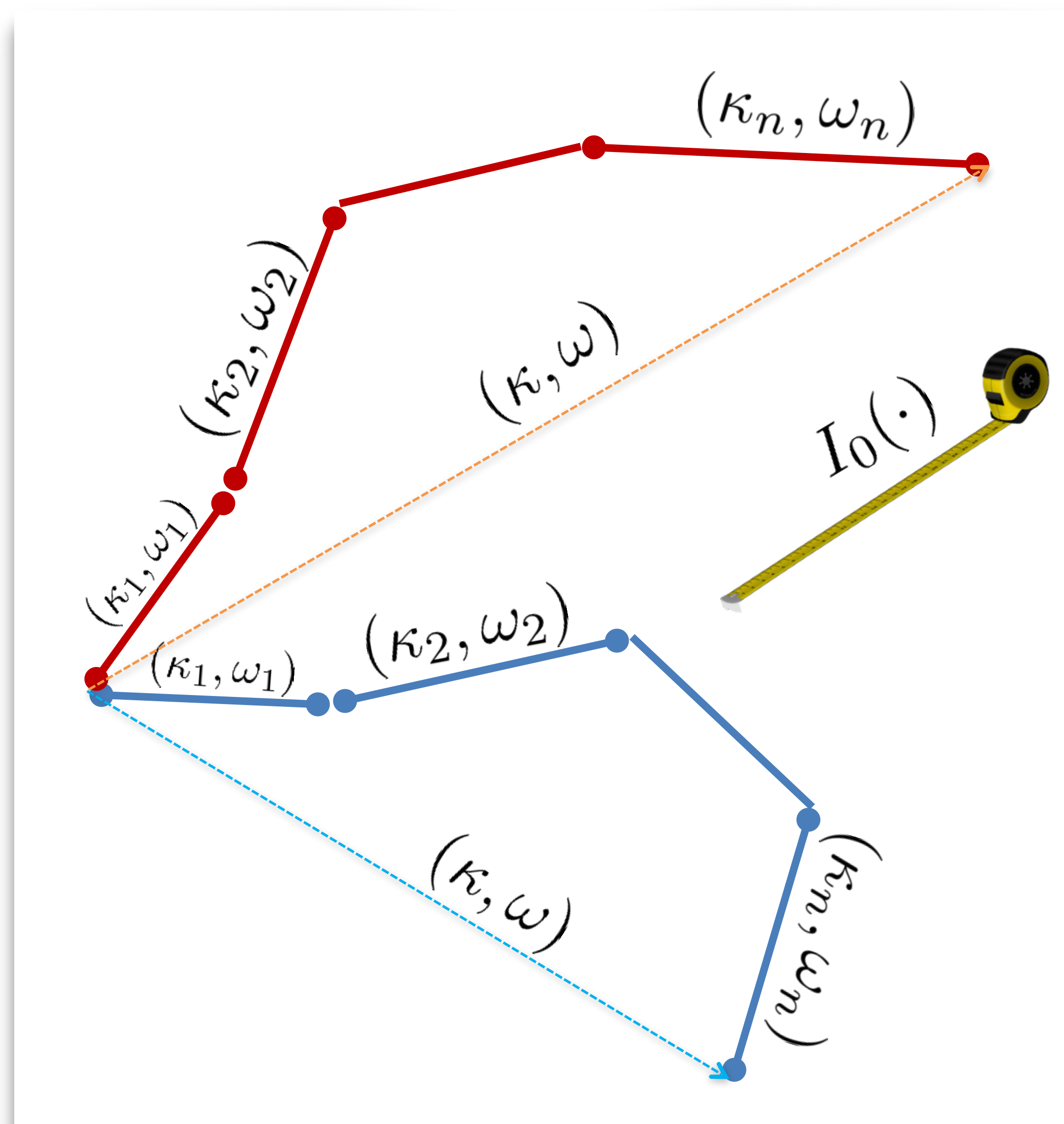




von Mises bandwidth maximization problem

$$\begin{aligned} \text{maxmize} \quad & \alpha\, I_0\left(\sqrt{\sum_{i,j} \kappa_i \kappa_j \cos(\omega_i - \omega_j)}\right) + \bar{\alpha}\, I_0\left(\sqrt{\sum_{i,j} \bar{\kappa}_i \bar{\kappa}_j \cos(\bar{\omega}_i - \bar{\omega}_j)}\right) \\ \text{subject to} \quad & \omega_i - \bar{\omega}_i = \text{given} \end{aligned}$$

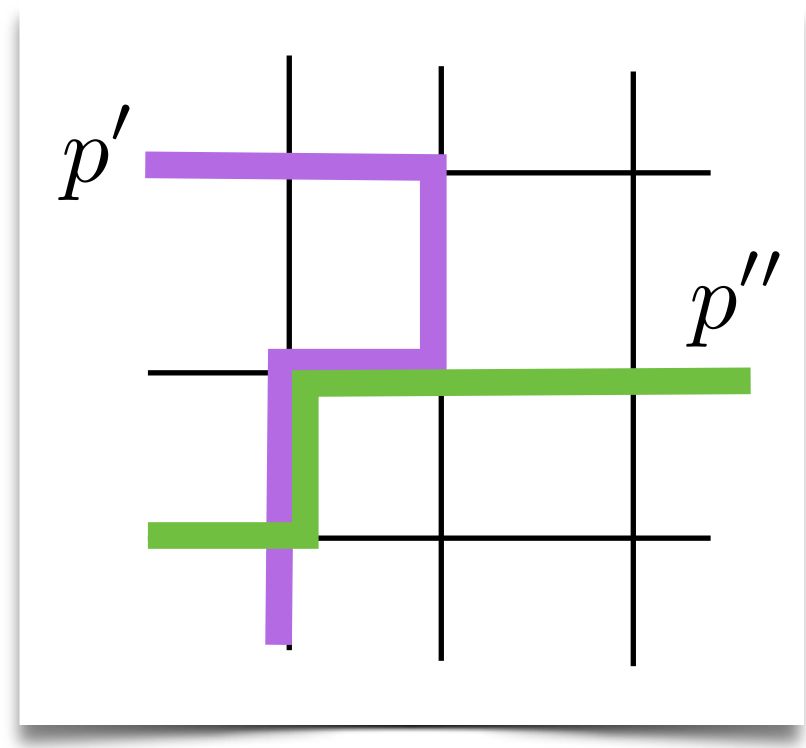
geometric interpretation:



Extension to networks

	arterial	network
topology	linear	general
# paths	2	n
# movements per intersection	2	m
each path uses all intersections	yes	no
solvable with LP	yes	no

Extension to networks



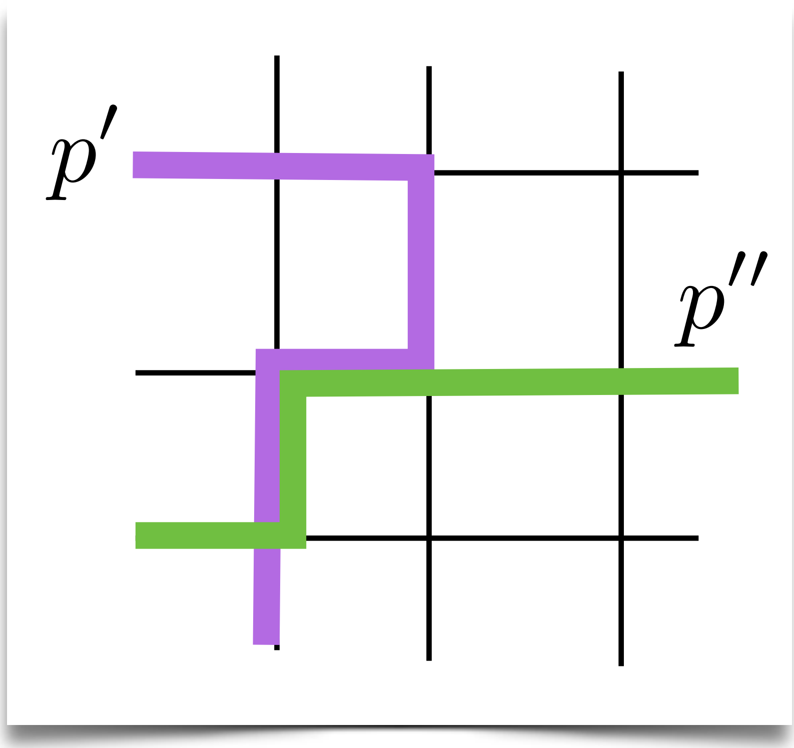
A path p is a sequence of *movements* m: $m \in \mathcal{M}_p$

Offsets are defined for each movement, relative to each path: $\omega(p, m)$

Path bandwidth, as a function of offsets

$$b_p(\omega) = \max\left(0, \min_{m', m'' \in \mathcal{M}_p} \left(\omega(p, m') - \omega(p, m'') + g(m', m'')\right)\right)$$

Extension to networks



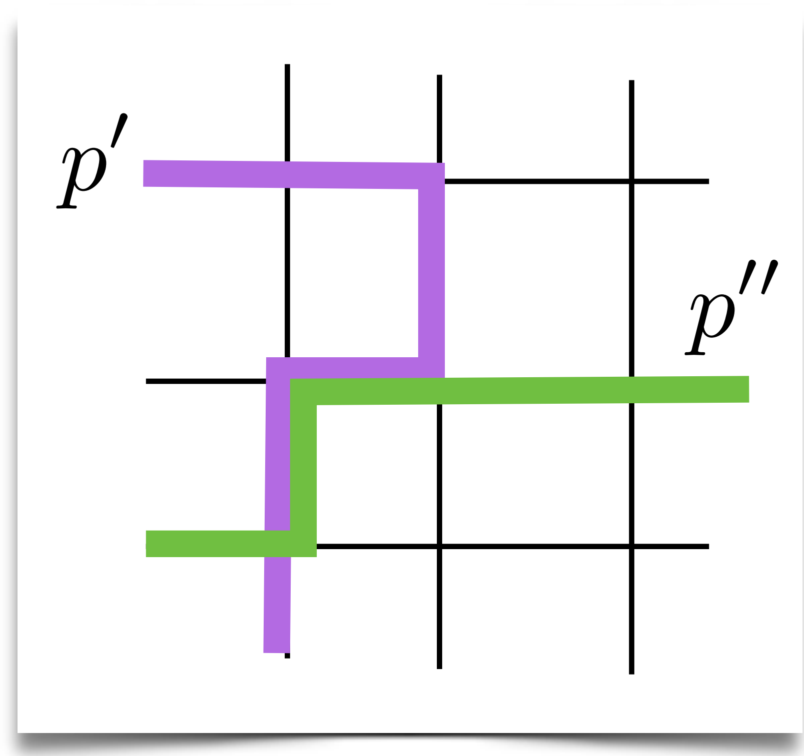
Internal offset constraint

$$\omega(p', m') - \omega(p'', m'') = \delta(m', m'') - T(p', n) + T(p'', n)$$

where

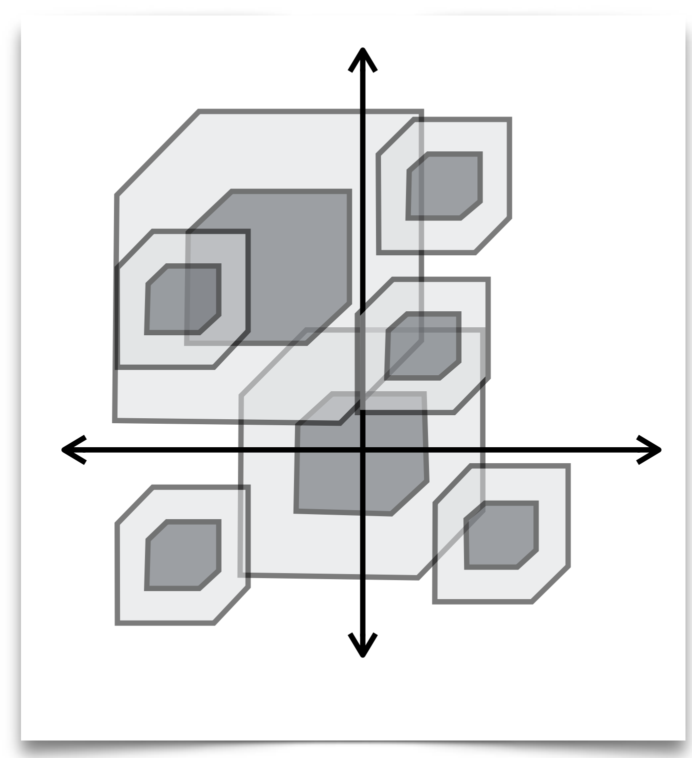
- m' and m'' are movements on the same node n,
- p' and p'' are paths that go through m' and m'',
- $T(p, n)$ is the travel time along p to node n.

Extension to networks

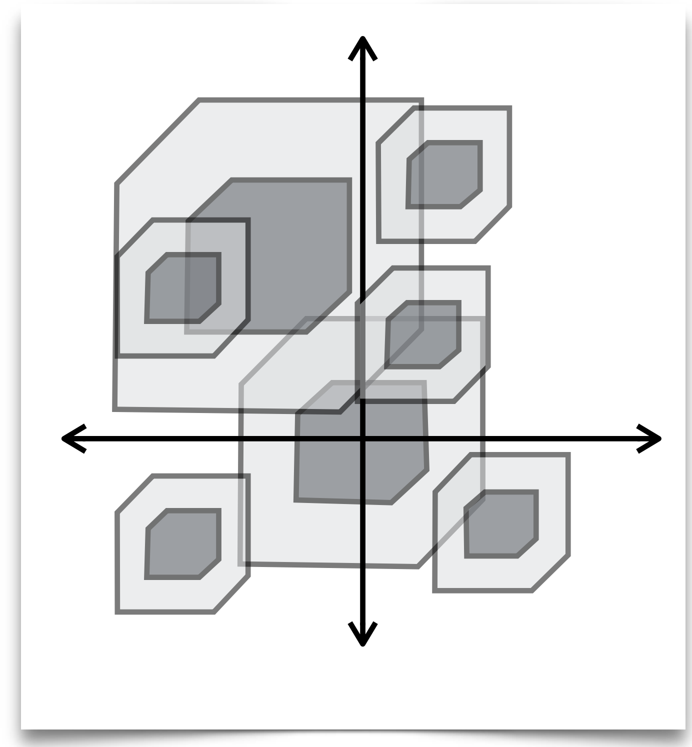


Optimization problem

$$\begin{aligned} &\text{maxmize} \quad \sum_p w_p\, b_p(\omega) \\ &\text{subject to} \quad \text{internal offset constraints} \end{aligned}$$



Generalization of the previous theorem



- Define $\mathcal{P}$ the set of all paths. $(p \in \mathcal{P})$
- Define $\mathcal{P}^*$ the set of all subsets of $\mathcal{P}$.
- Each element of $\mathcal{P}^*$ represents a *selection of paths.*
- For each $\mathcal{U} \in \mathcal{P}^*$ define a *relaxed linear problem.*

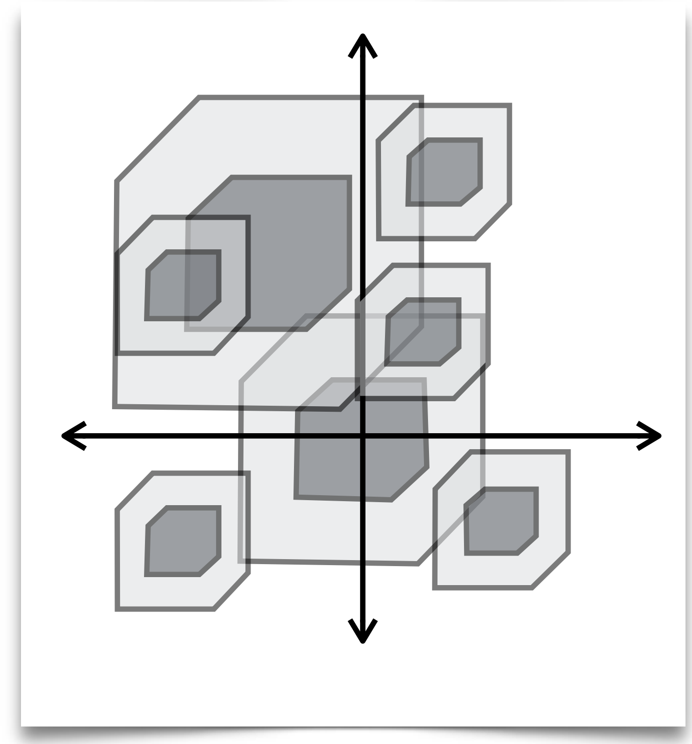
$$\begin{array}{rl} \text{maxmize} & \displaystyle\sum_{p \in \mathcal{U}} w_p \, b_p \\ \text{subject to} & \text{internal offset constraints} \\ & b_p \leq \omega(p, m') - \omega(p, m'') + g(m', m'') \end{array}$$

Theorem

The solution to the original problem *equals* the best of all relaxed problems.

A more tractable solution

NLP → { LP } → Mixed non-linear → Mixed linear

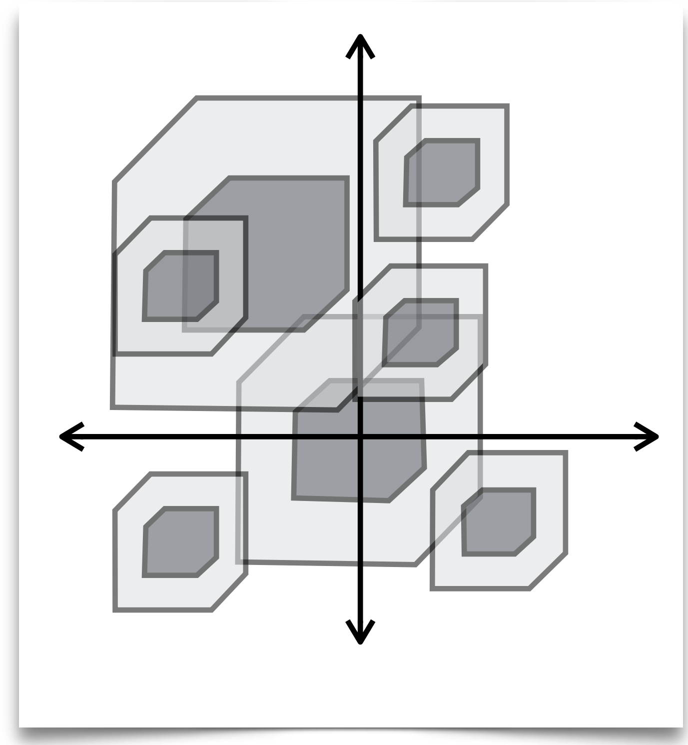


{ LP } → Mixed non-linear

Introduce binary variables α_p to indicate membership of p in $\mathcal{U}$,

$$\begin{aligned} \text{maxmize} \quad & \sum_{p \in \mathcal{P}} \alpha_p \, w_p \, b_p \\ \text{subject to} \quad & \text{internal offset constraints} \\ & b_p \leq \omega(p, m') - \omega(p, m'') + g(m', m'') \end{aligned}$$

Mixed non-linear → Mixed linear



- Due to Glover [1975]
- We can put α_p into the constraints.

$$
\begin{array}{rl}
\text{maxmize} & \displaystyle\sum_{p \in \mathcal{P}} w_p \, b_p \\
\text{subject to} & \text{internal offset constraints} \\
 & b_p \leq \alpha_p(\omega(p, m') - \omega(p, m'') + g(m', m''))
\end{array}
$$

- Define: $K(p, m', m'') = \omega(p, m') - \omega(p, m'') + g(m', m'')$
- Then $K(p, m', m'') \in [0, 2C]$
- Following Glover [1975], the problem is equivalent to,

$$
\begin{array}{rl}
\text{maxmize} & \displaystyle\sum_{p \in \mathcal{P}} w_p \, b_p \\
\text{subject to} & \text{internal offset constraints} \\
 & b_p \leq \alpha_p(2C) \\
 & b_p \leq (1 - \alpha_p)(2C) + K(p, m', m'')
\end{array}
$$

Questions

спасибо!