Nikishin factorization theorems and their applications

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1. УМН, 1970, т. 25, вып. 6 (156), 129–191

2. Изв. АН СССР, Сер. матем., 1972, т. 36, вып. 4, 795–813

Plan:

(a) Some statements

(б) Two illustrations

X Banach space

Type p, $1 \le p \le 2$:

$$\int_0^1 \left\| \sum r_i(t) x_i \right\|_X dt \le K_p \left(\sum \|x_i\|^p \right)^{1/p},$$

$x_i \in X$.

Cotype q, $2 \le q < \infty$:

$$\left(\sum \|x_i\|^q \right)^{1/q} \le C_p \int_0^1 \left\| \sum r_i(t) x_i \right\|_X dt$$

$\{r_i\}$ Rademacher functions

L^s, $1 \le s < \infty$

type $\min(s, 2)$

cotype $\max(s, 2)$

Every X is of type 1.

$\boxed{\mu(\Omega) = 1}$

Theorem. (Nikishin)

X of type p $\quad (1 \le p \le 2)$

$T : X \longrightarrow L^0(\Omega, \mu)$

linear and continuous

Then $\forall \varepsilon > 0 \ \exists \Omega_\varepsilon \subset \Omega$:

$\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$

$\forall t > 0 \quad \mu\{x \in \Omega_\varepsilon : |(Tu)(x)| > t\}$

$$\le \frac{C_{\varepsilon, T}}{t^p} \|u\|_X^p, \quad u \in X$$

Nikishin: $\quad 1^\circ \ X = L^s$

2° Certain nonlinear T are admissible

3° Also, the case of $p = 1$ and X arbitrary

If $T_\alpha : X \to L^0(\Omega, \mu)$ are equicontinuous, then $C_{\varepsilon, T_\alpha}$ does not depend on α.

Remark.

G compact group

μ Haar measure

$X = L^p(G)$, $1 \leq p \leq 2$

T commutes with translations

$\Downarrow$

We can take $\Omega_\varepsilon = \Omega$,

i.e. T is of weak type (p, p)

Nikishin used this theorem for the study of a.e. convergence of Fourier series and other similar expansions.

An application concerning convergence in measure

$$f \in L^1(\mathbb{T}) \Rightarrow \sum_{|j| \leq N} \hat{f}(j) e^{ijt} \longrightarrow f$$

in measure

Konyagin, 1988:

Nothing similar occurs in higher dimensions

$D \subset \mathbb{R}^n$ closed, convex,
$\operatorname{int} D \neq \emptyset$

$f \in L^1(\mathbb{T}^n)$

$$S_R(f) = \sum_{\nu \in \mathbb{Z}^n \cap RD} \hat{f}(\nu) e^{i\nu x},$$

$R \to 0$

$\exists f \in L^1(\mathbb{T}^n)$:

$S_R(f)$ do not converge to f in measure as $R \to \infty$.

1994, Pichugov

Suppose the contrary.

Then the S_R are uniformly of weak type $(1, 1)$ by the Nikishin theorem (because they commute with translations).

$$\|S_R\|_{L^2 \to L^2} \leq 1$$

$$\Downarrow \qquad \text{Interpolation}$$

$$\|S_R\|_{L^1 \to L^1} \leq C \log R$$

Known estimate:

$$\|S_R\|_{L^1 \to L^1} \geq C' (\log R)^n$$

(Yudin, 1979)

Contribution of the French school of functional analysis.

B. Maurey
J.-L. Krivine

$\sim 1973 - 1975$

A consequence of Nikishin's theorem:

$$T : X \longrightarrow L^0(\Omega, \mu)$$
type p

$$r < p \quad \Rightarrow$$

$$\left(\int_{\Omega_\varepsilon} |(Tf)(\omega)|^r \, d\mu(\omega) \right)^{1/r} \leq C_{\varepsilon, T} \|f\|_X$$

Maurey: such statements can be proved via separation theorem.

Systematic use of separation theorems led to several new statements. Here is just one of them.

Thm. $T : X \longrightarrow L^1(\Omega, \mu)$, X is of type 2. Then $\exists\ T_1 : X \longrightarrow L^2(\Omega, \mu)$ and $a \in L^2(\Omega, \mu)$ such that

$$T x = a \cdot T_1 x \qquad \forall x \in X$$

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "T_1"'] & & L^1(\mu) \\
& L^2(\mu) \arrow[ur, "f \mapsto f \cdot a"'] &
\end{tikzcd}$$

If $X = L^2$, the statement is very close to the Grothendieck theorem.

A reformulation of the Grothendieck theorem useful for what follows:

X, Y Banach lattices

$S : X \to Y$ bounded linear

$\Downarrow$

$$\Big\| \Big(\sum |S x_i|^2 \Big)^{1/2} \Big\|_Y \leq K_G \|T\| \cdot \Big\| \Big(\sum |x_i|^2 \Big)^{1/2} \Big\|_X$$

In other words: $X(\ell^2) \xrightarrow{S} Y(\ell^2)$

Krivine used the same duality methods to prove some quite abstract statements. Here is one of them:

$$\begin{tikzcd} V \arrow[rr, "T"] \arrow[dr, dashed] & & X \arrow[rr, "S"] & & W \\ & L^p(\mu) \arrow[urrr, dashed] & & & \end{tikzcd}$$

V, W Banach spaces

X Banach lattice

T p-convex:

$$\Big\| \Big(\sum |Tv_i|^p \Big)^{1/p} \Big\|_X \le C \Big(\sum \|v_i\|_V^p \Big)^{1/p}$$

S p-concave:

$$\Big(\sum \|Sx_i\|_W^p \Big)^{1/p} \le C \Big\| \Big(\sum |x_i|^p \Big)^{1/p} \Big\|_X$$

The last definition makes sense if S is defined on a subspace of X.

A version (also can be attributed to Krivine – Maurey – Nikishin)

X Banach lattice, id_X p-convex

$Y \subset X$ closed subspace

W Banach space

$$Y \xrightarrow{S} W \quad p\text{-concave}$$

$$\Downarrow$$

$$\exists\, f \in (X^p)^* : \quad \|f\| \leq C \quad \text{and}$$

$$\|Sy\| \leq C\, f(|y|^p)^{1/p}$$

$$X^p = \{ y : |y|^{1/p} \in X \}$$

$$\|y\|_{X^p} = \| |y|^{1/p} \|_X^p .$$

(In fact, K. — Anisimov, 2006)

Corollary. If $p=2$, then S extends to an operator from X to W.

Proof. $x \mapsto f(|x|^2)^{1/2}$ is a Hilbert norm on X. $\square$

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id_X p-convex $\overset{\text{def}}{\iff}$ X p-convex:

$$\left\| \left(\sum |v_i|^p\right)^{1/p} \right\|_X \leq C \left(\sum \|v_i\|_X^p\right)^{1/p}$$

id_X p-concave $\overset{\text{def}}{\iff}$ X p-concave

$$\left(\sum \|x_i\|^p\right)^{1/p} \leq C \left\| \left(\sum |x_i|^p\right)^{1/p} \right\|$$

X p-convex $\iff$ X' q-concave

$p^{-1}+q^{-1}=1$, $1 \leq p \leq \infty$.

L^p is 2-convex for $2 \leq p \leq \infty$

We pass to another application of factorization theorems

(K.; Anisimov, 2006)

X — Banach lattice of measurable functions on the unit circle $\mathbb{T}$

X_A — the set of functions in X that are the boundary values of a function in the Smirnov class N_+:

$$X_A = X \cap N_+$$

$$L^p(m)_A = H^p, \quad 1 \le p \le \infty.$$

There is a version of the Grothendieck theorem for spaces of analytic functions on $\mathbb{T}$:

$$S : X_A \longrightarrow W,$$

W a Banach lattice

$$\Downarrow$$

$$\left\| \left(\sum |S x_i|^2 \right)^{1/2} \right\|_W \leq A \|S\| \left\| \left(\sum |x_i|^2 \right)^{1/2} \right\|_X$$

(Basically, Bourgain, ~ 1982, though he did not state the result in this form)

Before it was resolved, the question had circulated during more than one decade and had been viewed quite difficult.

So, some partial questions had also been put forward, as "preliminary steps" so-to-say.

The main one:

is $L^1(\mathbb{T})/H^1$ of cotype 2?

Surely, after Bourgain's result this also was answered in the positive.

Recall that L^1 is of cotype 2.

A more general question:

X a lattice of measurable functions on $\mathbb{T}$, of cotype 2.

IS X/X_A of cotype 2?

Yes

Fact: X is of cotype 2

$\Updownarrow$

X is 2-concave

L^p

$1 \le p \le 2$

A "peculiar" example:

$e \subset \mathbb{T}$, $me > 0$, $m(\mathbb{T} \setminus e) > 0$

$$X = \left\{ f : \int_e |f| + \left(\int_{\mathbb{T} \setminus e} |f|^2 \right)^{1/2} < \infty \right\}.$$

For simplicity, I show only that X/X_A has the Orlicz property.

Def. Z has the Orlicz property if $\sum \|x_i\|^2 < \infty$ for every unconditionally convergent series $\sum x_i$ in Z.

Equivalently: for every bounded linear operator $S : c_0 \to Z$ we have $\sum \|Se_i\|^2 < \infty$.

Fact.

$$\begin{tikzcd} c_0 \arrow[r, "S"] \arrow[dr, dashed, "S_1"'] & X/X_A \\ & X \arrow[u, "\pi"'] \end{tikzcd}$$

Always there is a lifting.

Proof. We dualize:

X^* is a lattice of functions,

$$(X/X_A)^* = Z(X^*)_A$$

$$\begin{tikzcd}
Z(X^*)_A \arrow[r, "S^*"] \arrow[d, hook] & \ell^1 \\
X^* \arrow[ur, dashed, "\sigma"'] &
\end{tikzcd}$$

?

S^* is 2-concave automatically (Grothendieck theorem for spaces of analytic functions + the 2-concavity of ℓ^1).

By the version of Nikishin-(Krivine-Maurey-_ _ .) theorem, S^* extends to an operator σ...