PT-symmetric Hamiltonians with parameter. The problem on similarity to self-adjoint ones.

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Based on the joint work with S.N.Tumanov

The work is in progress.

C.Bender was one of the first who suggested to study spectral problems for Hamiltonians of the form

$$\left[-\frac{d}{dx^2} + P(x)\right] y(x) = \lambda y(x), \qquad y(\pm\infty) = 0,$$

provided that PT (parity and time) condition $p(x) = -\overline{p(-x)}$ holds. It is clear that the spectrum of such operators is symmetric with respect to the real axis. Among problems which appear in the study of these operators important role play the following

1. When the spectrum of a PT-symmetric operator is real?

2. When a corresponding operator is similar to a self-adjoint?

The aim of this talk is to cast some light to these problems.

Some papers on this subject.

Bessis, Jinn-Justin' 1992 found numerically that the spectrum of the operator

$$A = -\frac{d}{dx^2} + ix^3 \qquad \text{in} \quad L_2(\mathbb{R})$$

is real and positive.

Dorey, Dunning, Tateo' 2001-2004 approved this hypothesis.

Shin' 2002 generalized this result for

$$A = -\frac{d}{dx^2} + ix^3 + aix, \quad a \geqslant 0.$$

Gunderson' 2001, Shin' 2002, Eremenko and Morenkov'2005, Eremenko and Gabrialov'2008-2013 investigated the operators

$$A = -\frac{d}{dx^2} + ix^3 + aix, \quad a \geqslant 0.$$

and found the pairs of real numbers (a, λ) for which the equation $Ay = \lambda y$ has solutions $y(x)$ whose all zeros, but finitely many, are real.

The same was done for

$$A = -\frac{d}{dx^2} + x^4 + ax^2 + icx, \quad a, c \in \mathbb{R}, \quad a \geqslant 0.$$

In my opinion it is very useful to study these problems introducing a parameter.

For particular polynomials (or analytic functions) $P(x)$ the problem

$$[-i\tau\frac{d}{dx^2} + P(x)]y - \lambda y = 0$$

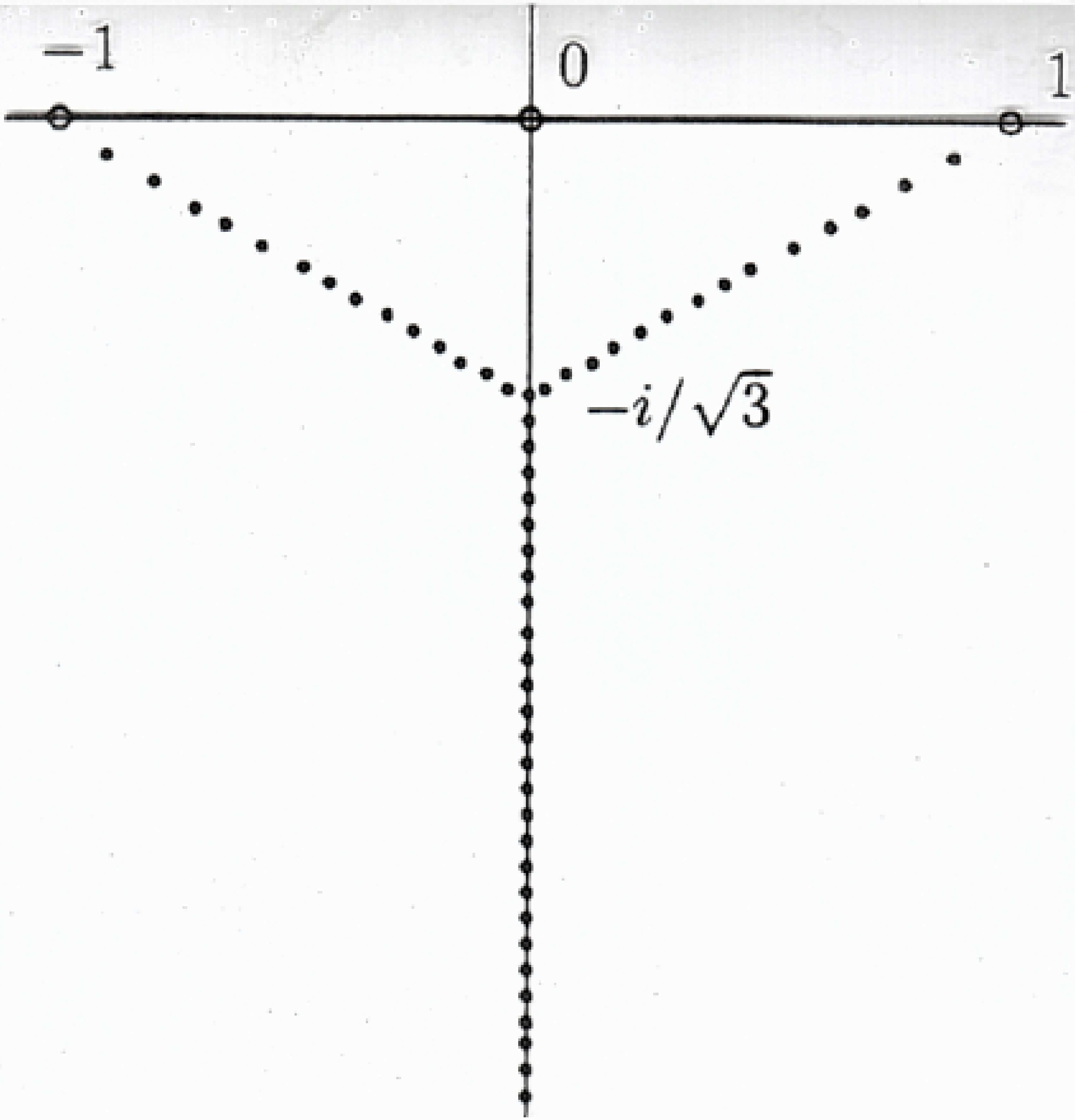
with parameter $\tau \to 0$ was studied by many author. Mainly the problem was of great interest because of the connection with the celebrated Orr-Sommerfeld problem. Certainly we can rewrite it in the form

$$-y'' + \varepsilon i P(x)y = \mu y \quad \text{with} \quad \varepsilon \to +\infty,$$

where $i\tau\lambda = \mu$, $\varepsilon = -1/\tau$.

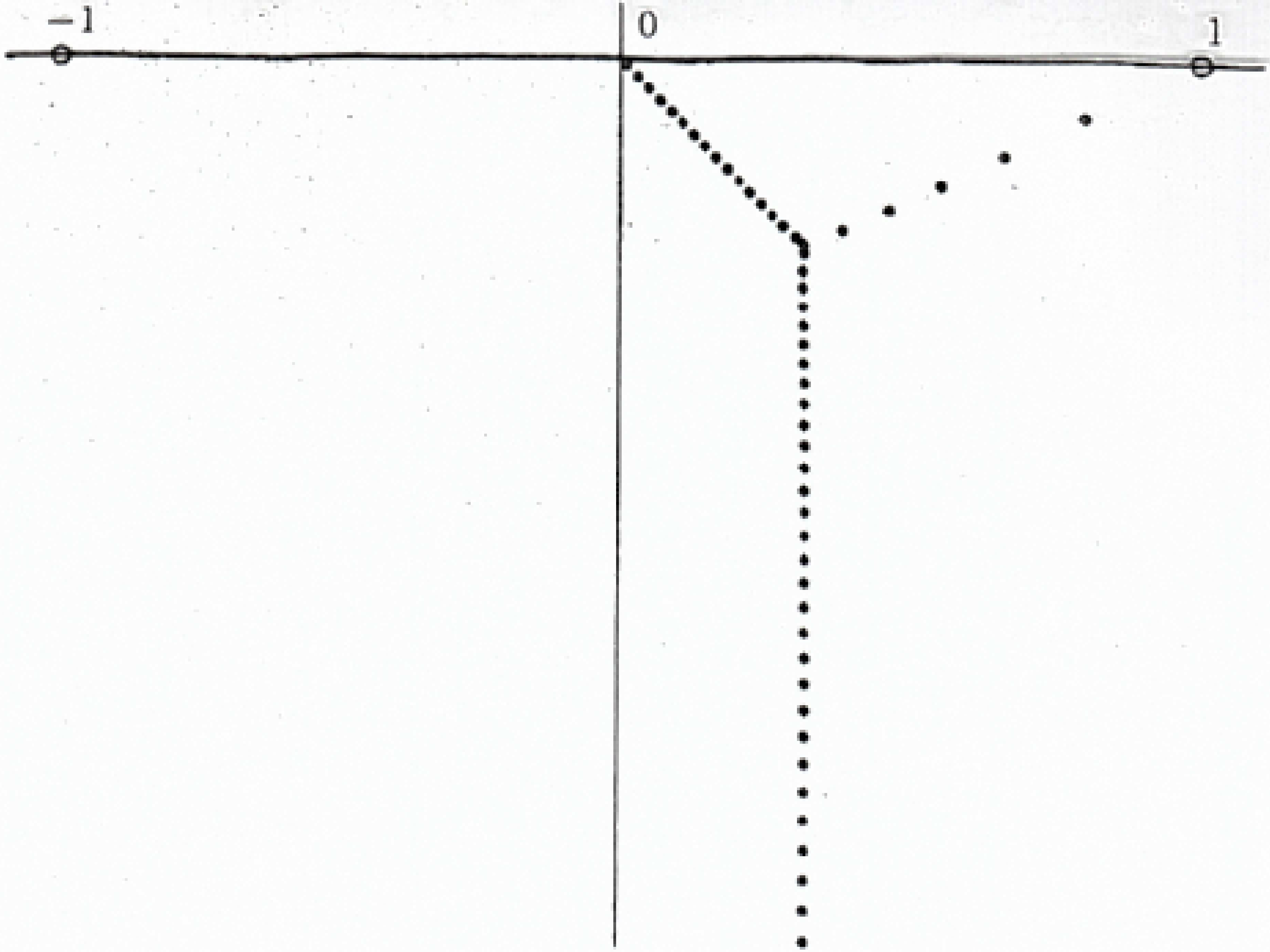
Since the velocity profiles in the Orr-Sommerfeld are not necessarily even, such problems were investigated not assuming the PT-symmetry.

First rigorous analytical explanations were obtained by Sh. in 1997. Many authors worked with this problem, recently some general result were obtained by Tumanov and Sh. The illustration of interesting phenomena can be observed on the pictures.



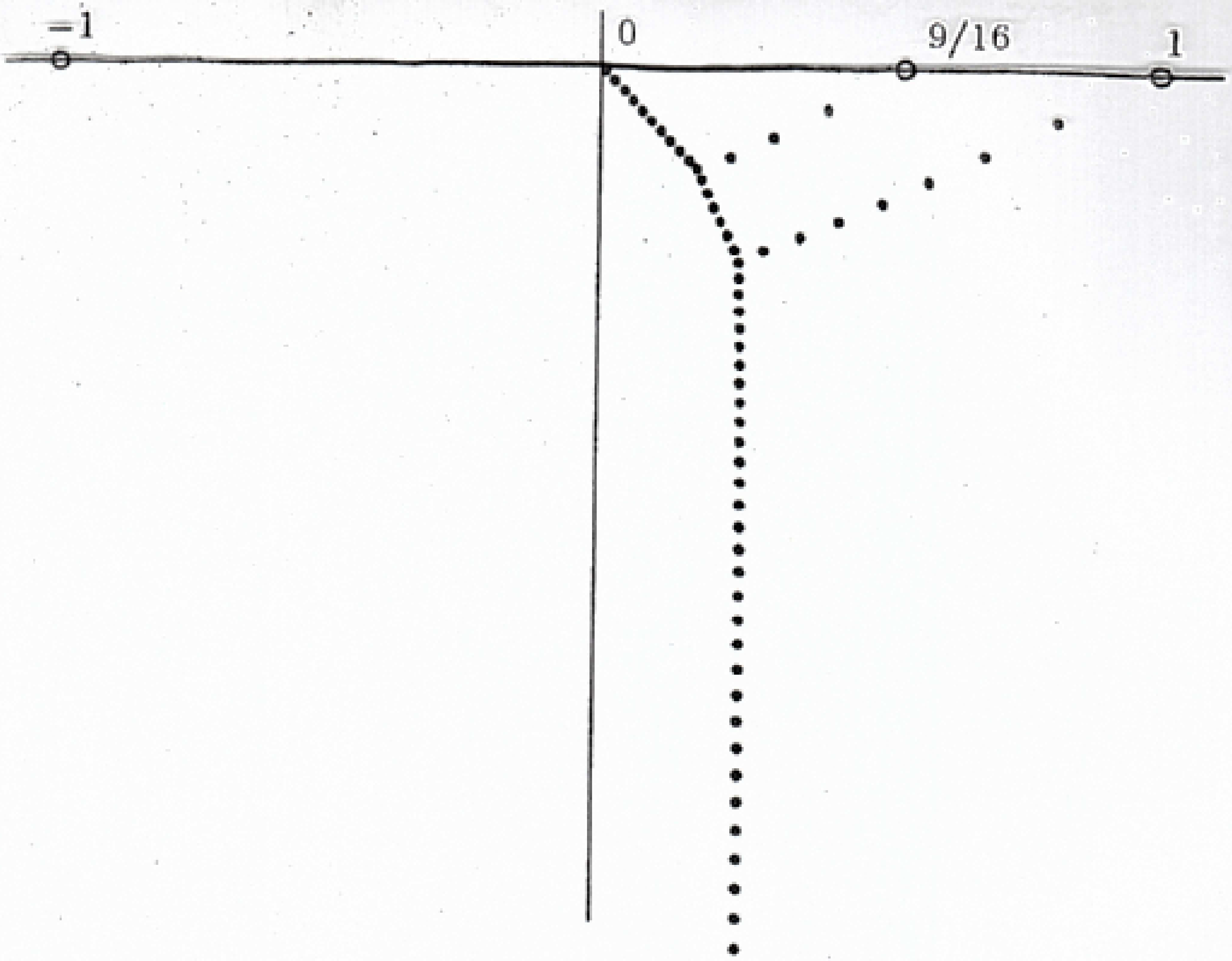
$q(x) = x. \qquad \varepsilon^2 = 10000$

Model problem



$$q(x) = x^2, \qquad \varepsilon^2 = 15000$$

Model problem



$$q(x) = \frac{1}{50}(7x-1)^2, \qquad \varepsilon^2 = 15000$$

Model problem

Now let us first consider the simplest case $P(x) = x$ on the finite segment $[-1, 1]$. Let us study the spectrum of the operator

$$A = T + i\varepsilon B$$

$$T = -\frac{d}{dx^2}$$

on the domain

$$\mathcal{D}(T) = \{y \in W^{2,2}[-1, 1] \mid y(-1) = y(1) = 0\}.$$

and

$$By(x) = xy(x).$$

Notice (it was done by Langer, Tretter'2004) that PT-symmetric operators can be viewed as self-adjoint operators in Krein space with indefinite product

$$[f(x), g(x)] = \int_{-1}^{1} f(x)\overline{g(-x)}\, dx.$$

The spectrum of T in $L_2(-1,1)$ can be easily calculated, it consists of the points

$\mu_k = \mu_k(0) = (\pi k/2)^2$. Let

$$d = min_{k \geqslant 1}(\mu_{k+1} - \mu_k) = \frac{3}{4}\pi^2.$$

According to general stability Theorem the circles Γ_k with radius $r < 3\pi^2/8 = d/2$ centered at the points μ_k contain only one simple eigenvalue $\mu_k(\varepsilon)$, provided that $\varepsilon < d\|B\|/2 = d/2$ and because of symmetry all $\mu_k = \mu_k(\varepsilon)$ lie on the real axis. So, The spectrum of $A = T + i\varepsilon B$ remains to be real provided that parameter ε is sufficiently small, in particular, if $\varepsilon < d/2 = 3\pi^2/8 \sim 3,7$. Is

$d = 3\pi^2/8$ the largest possible (critical) number which guarantee the reality of the spectrum? How do the eigenvalues move, as ε increases? Let us observe this interesting phenomena.

We see that the critical number $\varepsilon = \varepsilon_1$ is defined by the collision of the first and the second eigenvalues. Numerical calculations show that $\varepsilon_1 \sim 12,3124$ and for $\varepsilon > \varepsilon_1$ the non-real eigenvalues necessarily appear.

THEOREM 1. *The number ε_1 which defines the first collision of the eigenvalues, is critical, the eigenvalues do not come through each other but come to the complex plane in the perpendicular directions with respect to the real axis. The eigenvalues $\mu_k(\varepsilon)$ are analytic curves on ε with* *the exception of the collision points* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ *. These curves do not intersect each other. The collision points* $\varepsilon_k \to \infty$*. Complex eigenvalues do not come back, in particular, there are exactly* $2k$ *non-real eigenvalues for* $\varepsilon_k < \varepsilon < \varepsilon_{k+1}$.

THEOREM 2. *The operator* $A = T + i\varepsilon B$ *is similar to a self-adjoint (*$A = USU^{-1}, S = S^*,\ U$ *and* U^{-1} *are bounded) iff* $\varepsilon < \varepsilon_1$

THEOREM 3. *Given a number* $\varepsilon_k < \varepsilon \leqslant \varepsilon_{k+1}$ *there exist* A*-invariant subspaces* H^1_ε *and* H^2_ε*, such* *that rank*$A|_{H^1} = 2k$*,* $A|_{H^1}$ *has exactly* $2k$ *non-real eigenvalues and* $A|_{H^2}$ *has only real eigenvalues and it is similar to a self-adjoint operator.*

Proof follows from Theorem 1 applying the following result: *the system of root functions of* A *forms a Riesz basis*.

THEOREM 4. *All the non-real eigenvalues concentrated (exponentially close) to the segments* $[\pm i\gamma, \gamma/\sqrt{3}]$*. The number* k *of the non-real eigenvalues is defined by*

$$2k = \frac{2\sqrt{\varepsilon}}{\pi} f(\frac{i}{\sqrt{3}}) + O(\log \varepsilon), \quad f(\lambda) = \int_{-1}^{1} e^{-i\pi/4} \sqrt{(x-\lambda)}\, dx$$

The analogues of Theorems 1-4 can be formulated for much more general PT-symmetric potentials. This can be done for all $q(x) \in L_2$ moreover, for all $q \in W^{-1,2}$), i.e. for $\int q(x)\,dx \in L_2$.

This can be done for operators

$$A = -\frac{d}{dx^2} + x^4 + ax^2 + icx, \quad a, c \in \mathbb{R}$$

acting in $L_2(\mathbb{R})$.

This can be done for the operators

$$A = -\frac{d}{dx^2} + |x|^\beta + ic\,\text{sign}x|x|^\alpha, \qquad c \in \mathbb{R},$$

provided that β is sufficiently large with respect to α (an explicit relation can be written.

How to determine the critical points ε_k? This is a separate and delicate problem. It turns out that for the potential $P(x) = x$ these points can be found explicitly. Consider the Airy equation

$$y'' = zy, \quad y = y(z).$$

Let $Ai(z)$ and $Bi(z)$ be the standard solutions of this equation

$$Ai(z) = Ai(0)\left(1 + \frac{1}{3!}z^3 + \frac{1\cdot 4}{6!}z^6 + \frac{1\cdot 4\cdot 7}{9!}z^9 + \dots\right)$$
$$+Ai'(0)\left(z + \frac{2}{4!}z^4 + \frac{2\cdot 5}{7!}z^7 + \frac{2\cdot 5\cdot 8}{10!}z^{10} + \dots\right),$$

$$Ai(0) = \frac{\Gamma(1/3)}{3^{1/6}2\pi}, \qquad Ai'(0) = \frac{3^{1/6}\Gamma(2/3)}{2\pi},$$

$$Bi(z) = e^{\pi i/6}Ai\left(e^{2\pi i/3}z\right) + e^{-\pi i/6}Ai\left(e^{-2\pi i/3}z\right).$$

Let us consider the function

$$S(z) = -\sqrt{3}Ai(z) + Bi(z).$$

Lemma. *The zeros of* $S(z)$ *lie on the rays* $\arg z = \frac{\pi}{3} + \frac{2\pi k}{3}, k = -1, 0, 1$. *There is the triple symmetry.*

THEOREM *Let* $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots$ *be the modulas of the zeros of the function* $S(z)$. *Then*

$$\varepsilon_k = \left(\alpha_k \frac{\sqrt{3}}{2}\right)^3, \qquad k = 1, 2, \ldots.$$

Now let us look for abstract PT-symmetric operators. The abstract analogue of a PT-symmetric operator is a block matrix operator of the form

$$\mathbb{L} = \begin{pmatrix} A & B \\ -B^* & C \end{pmatrix}, \quad \text{or} \quad \mathbb{L} = \begin{pmatrix} A & B \\ -B^* & C \end{pmatrix} \qquad \text{in } \mathbb{H} = H_1 \oplus H_2$$

THEOREM.(Albeverio-Motovilov-Sh'2009) In the first and in the second case the condition $\|B\| < d/2$ *is sufficient to guarantee the disjointness of the spectral zones and the similarity of* $\mathbb{L}$ *to a self-adjoint operator (the* *estimate is sharp). In the "many island"case the sufficient condition is*

$$\|B\| < d/\pi.$$

The estimate is probably, not sharp. The method of the proof uses essentially the (McEachin) theorem:

The norm of the Sylvester operator

$$S(X) = AX - XC,$$

acting in the Banach space $\mathcal{B}(H)$ of bounded operators on H admits the estimate

$$\|S^{-1}\| \leqslant \frac{\pi}{2d},$$

where d is the distance between the spectra of A and C.

(K.Veselich)

THEOREM. (Motovilov-Shkalikov, 2014) In the general case (i.e. in "many island"case) the condition $\|B\| < d/2$ is sufficient to guarantee that the spectrum of $\mathbb{L}$ remains to be real and the spectral zones remain disjoint. Moreover, $\mathbb{L}$ is similar to a self-adjoint operator, provided that the spectrum of the unperturbed operator is discrete.