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Arbitrage Theory and Stochastic Calculus

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Outline

Classical model and the Harrison–Pliska theorem

- A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t=0,1,\ldots,T}, P)$.
- The price process $S = (S_t^1, \ldots, S_t^d)$, S_t is $\mathcal{F}_t$-measurable ; $S_t^1 = 1 \;\forall\, t$, i.e. the first traded asset is the *numéraire*, say, "bank account" or "cash". Thus, $\Delta S_t^1 = S_t^1 - S_{t-1}^1 = 0$.
- The value process of a self-financing portfolio with zero initial capital : $V_t = H \cdot S_t := \sum_{u \le t} H_u \Delta S_u$, H_t is $\mathcal{F}_{t-1}$-measurable, H_t^i, $i \geq 2$, are holdings in stocks ("visible").

Definition

NA-property : $R_T \cap L_+^0 = \{0\}$ *where* $R_T = \{V_T : \; V_T = H \cdot S_T\}$.

Theorem (Harrison–Pliska (1981))

Suppose that Ω is finite. Then the NA property holds if and only if there is a probability measure $\tilde{P} \sim P$ such that S is a $\tilde{P}$-martingale.

Let $A_T := R_T - L_+^0$. Then NA-property holds iff $A_T \cap L_+^0 = \{0\}$.

Harrison–Pliska theorem : comments

People working in mathematical economics relate the "classical arbitrage theory" with the names of Harrison, Pliska, and Kreps. In fact, the results of these authors are more or less reformulations of already known (but their formulations are very important formulations for the further development). One can say, that they invited stochastic calculus to finance.

Lemma (Stiemke, modern version)

Let K and R be closed cones in $\mathbb{R}^n$ where K is proper. Then

$$R \cap K = \{0\} \quad \Leftrightarrow \quad (-R^*) \cap \operatorname{int} K^* \neq \emptyset.$$

Applying this lemma in $L^0(\mathbb{R})$ with $K = L^0_+ = L^0(\mathbb{R}_+)$ and $R = R_T$ we obtain that NA is equivalent to existence of r.v. $\rho_T > 0$ such that $E\rho H \cdot S_T \leq 0$ for all H. Normalizing and putting $\tilde{P} = \rho_T P$ we get a probability measure under which S is a martingale.

Infinite Ω

Theorem (Dalang–Morton–Willinger (1990), short version)

The NA property holds if and only if there is a probability measure $\tilde{P} \sim P$ such that S is a $\tilde{P}$-martingale.

Looks like the same theorem with the removed assumption... But we enter here into different world !

On the next slide we present an extended version, sometimes called FTAP - Fundamental Theorem of Asset (or Arbitrage) Pricing. The most essential part of it are due to Dalang, Morton, and Willinger, but there are many other contributors (new equivalences, new proofs or improvements of already existing) : Schachermayer, Rogers, Jacod, Shiryaev, Kramkov, Kabanov, Stricker, Engelbert,...

NA criteria for arbitrary Ω

Theorem (Dalang–Morton–Willinger (1990), extended version)

The following conditions are equivalent :

(a) $A_T \cap L^0_+ = \{0\}$ *(NA condition)* ;

(b) $A_T \cap L^0_+ = \{0\}$ *and* $A_T = \bar{A}_T$ *(closure in* L^0*)* ;

(c) $\bar{A}_T \cap L^0_+ = \{0\}$;

(d) *there is* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}(\tilde{P})$;

(e) *there is* $\tilde{P} \sim P$ *with* $d\tilde{P}/dP \in L^\infty$ *such that* $S \in \mathcal{M}(\tilde{P})$;

(f) *there is* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}_{loc}(\tilde{P})$.

(g) $\{\eta \Delta S_t : \ \eta \in L^0(\mathcal{F}_{t-1})\} \cap L^0_+ = \{0\}$ *for all* $t \le T$ *(NA for all 1-step models).*

$S \in \mathcal{M}(\tilde{P})$ *iff* $\rho S \in \mathcal{M}(P)$ *where* $\rho_t = E(\rho_T | \mathcal{F}_t)$.

(d') *there is a process* $\rho \in \mathcal{M}$, $\rho > 0$, *such that* $\rho S \in \mathcal{M}$;

(e') *there is a bounded process* $\rho \in \mathcal{M}$, $\rho > 0$, *such that* $\rho S \in \mathcal{M}$;

(f') *there is a process* $\rho \in \mathcal{M}$, $\rho > 0$, *such that* $\rho S \in \mathcal{M}_{loc}$;

Outline

Setting

Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a stochastic basis and let $\mathcal{S}$ be the space of semimartingales $X = (X_t)_{t \le T}$, $X_0 = 0$. We fix in $\mathcal{S}$ a convex subset $\mathcal{X}^1$ of processes $X \ge -1$ containing the zero process.
Define the set $\mathcal{X} := \mathrm{cone}\, \mathcal{X}^1 = \mathbf{R}_+ \mathcal{X}^1$. In mathematical finance its elements are interpreted as admissible value processes starting from zero initial capital ; the elements of $\mathcal{X}^\lambda := \lambda X^1$ are called λ-admissible.
Define the set of "strictly" 1-admissible processes $\mathcal{X}^1_> \subseteq \mathcal{X}^1$: with $X, X_- > -1$.
The sets $x + \mathcal{X}^1$, $x + \mathcal{X}^1$ etc., $x \in \mathbf{R}$, have obvious sense. We are particularly interested in the set $1 + \mathcal{X}^1_>$. Its elements are strictly positive value processes ; sometimes they are called numéraires.

NA, NFL, NFLVR, 1

Define the convex sets $\mathcal{X}_T = \{X_T : X \in \mathcal{X}\}$ and $C = (\mathcal{X}_T - L^0_+) \cap L^\infty$. We denote $\bar{C}$, $\tilde{C}^*$, $\bar{C}^*$ the norm closure, the sequential weak* closure, and weak* closure of C in L^∞.

The properties *NA, NFLVR, NFLBR, NFL* mean that $C \cap L^\infty_+ = \{0\}$, $\bar{C} \cap L^\infty_+ = \{0\}$, $\tilde{C}^* \cap L^\infty_+ = \{0\}$, $\bar{C}^* \cap L^\infty_+ = \{0\}$, respectively. Obviously,

$$\begin{array}{ccccccc} C & \subseteq & \bar{C} & \subseteq & \tilde{C}^* & \subseteq & \bar{C}^* \\ NA & \Leftarrow & NFLVR & \Leftarrow & NFLBR & \Leftarrow & NFL. \end{array}$$

Define the *ESM-property* as the existence of $\tilde{P} \sim P$ such that $\tilde{E} X_T \le 0$ for all $X \in \mathcal{X}$. According to the Kreps–Yan separation theorem the properties *NFL* and *ESM* are equivalent.

Lemma

NFLVR $\quad\Leftrightarrow\quad$ *NA holds and the set* $\mathcal{X}^1_T$ *is P-bounded.*

NA, NFL, NFLVR, 2

Recall that $\mathcal{S}$ is a Frechet space with the quasinorm

$$\mathbf{D}(X) := \sup\{E1 \wedge |H \cdot X_T| : \ H \text{ is predictable}, \ \ |H| \le 1\}.$$

The model has *concatenation property* if for any $X, X' \in \mathcal{X}^1$ and any bounded predictable processes $H, G \ge 0$ such that $HG = 0$ and $\tilde{X} := H \cdot X + G \cdot X' \ge -1$, the process $\tilde{X} \in \mathcal{X}^1$.

Theorem (Kabanov, 1997)

Suppose that the concatenation property holds and $\mathcal{X}^1$ is closed in $\mathcal{S}$. Then under the NFLVR condition $C = \bar{C}^*$ and, as a corollary,

$$NFLVR \Leftrightarrow NFLBR \Leftrightarrow NFL \Leftrightarrow ESM.$$

To get a simple proof of this theorem (even for the model based on a price process) is still a challenge (recall the recent progress due to Josef Teichmann).

Model based on a price process

For the "standard model" considered by Delbaen and Schachermayer in their famous papers, $X = H \cdot S$ with S be a fixed d-dimensional semimartingale where H runs through $L(S)$, the closedness follows from the Mémin theorem. So, the hypotheses of the theorem are fulfilled. It is easy to see that under separating measure, the process S is a martingale, if it is bounded, and a local martingale, if it is locally bounded. Without local boundedness, we have only the following :

Theorem (Delbaen–Schachermayer, 1998 ; Kabanov, 1997)

In any neighborhood of a separating measure there is a probability measure under which S is a σ-martingale.

The proof in [Kabanov, 1997] is based on use of local characteristics of semimartingales.

Basic definitions, 1

Definition

The family $\mathcal{X}$ has $NAA1$-property (No Asymptotic Arbitrage of the 1st Kind) if for any sequences of $x^n \downarrow 0$ and $X^n \in \mathcal{X}$ with $x^n + X^n \geq 0$ we have $\limsup_n P(x^n + X^n_T \geq 1) = 0$.

The $NAA1$-property admits reformulations :

BK-property, $NUPBR$ (No Unbounded Profit with Bounded Risk) :

$K_0^1 := \{X_T : \; X \in \mathcal{X}^1\}$ is bounded in L^0.

NA_1 $\bar{x}(\xi) > 0$ for every $\xi \in L^0_+ \setminus \{0\}$ where

$$\bar{x}(\xi) := \inf\{x \in \mathbb{R}_+ : \; \text{there is } X \in \mathcal{X}^x \text{ such that } x + X_T \geq \xi\}.$$

Lemma

$NA_1 \Leftrightarrow NAA1 \Leftrightarrow BK(NUPBR)$.

Lemma

(Delbaen–Schachermayer) $NFLVR \quad \Leftrightarrow \quad NA \;\&\; BK$.

Outline

Basic definitions, 2

A supermartingale $Z > 0$ with $Z_0 = 1$ is a *supermartingale deflator* if $Z(1 + X)$ is a supermartingale for each $X \in \mathcal{X}^1$.

Lemma

Let $Z > 0$ be a supermartingale deflator. Then the set $\mathcal{X}^1_T$ is P-bounded.

Proof. Let $X \in \mathcal{X}^1$. As $EZ_T(1 + X_T) \leq 1$, the set $Z_T\mathcal{X}^1_T := \{Z_T X_T : \ X \in \mathcal{X}^1\}$ is bounded in L^1, hence, it is P-bounded. The multiplication by a finite random variable preserve the boundedness in probability. Thus, the set $\mathcal{X}^1_T = Z_T^{-1}(Z_T\mathcal{X}^1_T)$ is P-bounded.

So, the existence of a supermartingale deflator implies NA_1.

Basic definitions, 3

Definition

A process $Z > 0$ with $Z_0 = 1$ is *local martingale deflator* if $Z(1 + X)$ is a local martingale for each $Z \in \mathcal{X}^1$.

Definition

An element V of $1 + \mathcal{X}^1_>$ is called *tradable local martingale (supermartingale) deflator* if $1/V$ is a local martingale *(supermartingale)* deflator.

From the Jensen inequality follows :

Lemma

The tradable supermartingale deflator, if exists, is unique.

Main theorem : “mathematical finance formulation”

”Standard model.

Theorem

The following conditions are equivalent :
(i) NA_1 ;
(ii) *there exists a supermartingale deflator ;*
(iii) *there exists a local martingale deflator ;*
(iv) *in any neighborhood of* P *there exists a probability measure* $\tilde{P} \sim P$ *admitting a tradable local martingale deflator.*
Moreover, if the Lévy measures of S *are concentrated on finite number of points, then the* NA_1*-property ensures the existence of the local martingale deflator under* P.

(iv) for any $\varepsilon > 0$ there are a probability measure $\tilde{P} \sim P$ with $||\tilde{P} - P|| < \varepsilon$ and a process $X' = H' \cdot S \in \mathcal{X}^1_>$ such that the ratio $(1+X)/(1+X')$ is a local martingale with respect to $\tilde{P}$ $\forall X \in \mathcal{X}^1_>$.

Main theorem : comments

The chain of implications $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ is easy and well known.

The implication $(i) \Rightarrow (ii)$ is due to Karatzas and Kardaras.

Moreover they proved another important result : (i) implies the existence of tradable supermartingale deflator.

The implication $(i) \Rightarrow (iii)$ was proven recently by Takaoka.

The implication $(i) \Rightarrow (iv)$ was proven by Kardaras for one-step model, i.e. $d = 1$.

The proof is based on the techniques of local characteristics and does not rely on functional analysis.

Canonical form and local characteristics

The semimartingale S can be written as :

$$S = S_0 + S^c + xI_{\{|x|\le 1\}} * (\mu - \nu) + xI_{\{|x|>1\}} * \mu + B^h,$$

where (B^h, C, ν) is the triplet predictable characteristics, $h(x) := I_{\{|x|\le 1\}}$ is the truncation function, $(|x|^2 \wedge 1) * \nu < \infty$.
There is a predictable increasing càdlàg process A with $A_0 = 0$ and $A_T \le 1$ such that

$$B^h = b^h \cdot A, \qquad C = c \cdot A, \qquad \nu(dt, dx) = dA_t K_t(dx).$$

Let $\bar{\mathcal{P}}$ be the completion of the σ-algebra $\mathcal{P}$ with respect to the measure $m(d\omega, dt) := P(d\omega)dA_t(\omega)$.
We write $K_{\omega,t}(Y) = \int Y(x)K_{\omega,t}(dx)$ and omit ω or ω, t ;
$\mathcal{X}^1(S) := \{H \cdot S : \; H \in L(S), H \cdot S \ge -1\}$, $\mathcal{X}^1_>(S)$.

Simplifying assumption

Lemma

Let $\varepsilon > 0$. Then there exists a probability measure $\tilde{P} \sim P$ with the bounded density $d\tilde{P}/dP$ such that $||\tilde{P} - P|| \le \varepsilon$, and $|x|^2 * \mu_T + S_T^{*2} + ...$ belongs to $L^1(\tilde{P})$.

Proof. Let $Z_T^n := c_n(1 + n^{-1}U_T)^{-1}$ where $U_T := |x|^2 * \mu_T + S_T^{*2} + ...$ and c_n is the normalizing constant. The probability $P^n = Z_T^n P$ for sufficiently large n meets the requirements.

NA_1 is invariant under equivalent change of probability measure. On the other hand, if $P' \sim P$ with the density process ρ and Z' is a P'-local martingale, then $\rho Z'$ is a P-local martingale deflator (i.e. $(iv) \Rightarrow (iii)$). We may assume wlg that S is a special semimartingale of the form

$$S = S_0 + S^c + x * (\mu - \nu) + B,$$

where $S^c \in \mathcal{M}^{2,c}$, $\langle X^c \rangle = c \cdot A$, $B = b \cdot A$ with $b = b^h + K(x\bar{h})$, $\nu(dt, dx) = K_t(dx)dA_t$, $|x|^2 * \nu_T < \infty$, and $S^d := x * (\mu - \nu) \in \mathcal{M}^2$. But we shall work assuming only that $K(|x|^2 \wedge |x|) < \infty$.

Ratio of stochastic exponentials

Let X be a semimartingale with $\Delta X > -1$ and let $\mathcal{E}(X)$ denote the solution of the linear equation $Z = 1 + Z_- \cdot X$.
A semimartingale $X = 1 + H \cdot S$ such that $X, X_- > 0$ solves the linear equation $X = 1 + X_- X_-^{-1} \cdot X$ and, hence, admits the representation $X = \mathcal{E}(X_-^{-1} \cdot X) = \mathcal{E}(X_-^{-1} H \cdot S)$.
For the standard model the set $1 + \mathcal{X}^1_>$ can be identified as the set of stochastic exponentials :

$$1 + \mathcal{X}^1_> = \{\mathcal{E}(f \cdot S) : \ f \in L(S), \ f \Delta S > -1\}.$$

An important step is to obtain a convenient representation of the ratio of stochastic exponentials and give answers to questions when this ration is a supermartingale or a local martingale.

When the ratio of two stochastic exponentials is a supermartingale ?

Using Yor's formula we get that

$$\mathcal{E}(f \cdot S)/\mathcal{E}(g \cdot S) = \mathcal{E}(R) \tag{1}$$

where

$$R = (f - g) \cdot S + \langle g \cdot S^c \rangle - \langle f \cdot S^c, g \cdot S^c \rangle - (f - g)\frac{gx}{1 + gx} x * \mu.$$

Lemma

Suppose that $K(|x|^2 \wedge |x|) < \infty$ m-a.e. Let $f, g \in L(S)$ with $f\Delta S > -1,\ g\Delta S > -1$. Let

$$F(f, g) := (f - g)\left[(b - cg) - K\left(x\frac{gx}{1 + gx}\right)\right].$$

If $F(f, g) \le 0$ m-a.e., then $\mathcal{E}(R)$ is a supermartingale. If $F(f, g) = 0$ m-a.e., then $\mathcal{E}(R)$ is a local martingale (and a supermartingale).

The Maximization Problem in $\mathbb{R}^d$

$$\Psi(v) := bv - \frac{1}{2}|c^{1/2}v|^2 - K(vx - \ln(1+vx)) \to \max, \quad vx > -1\ K\text{-a.e.}$$

Let

$$N := \{v \in \mathbb{R}^d :\ vx = 0\ K\text{-a.e.},\ cv = 0,\ vb = 0\},$$
$$J := \{v \in \mathbb{R}^d :\ vx \geq 0\ K\text{-a.e.},\ cv = 0,\ vb \geq K(vx)\} \setminus N.$$

Proposition

Let $J = \emptyset$. Then there is $v^0 \in N^\perp$, $v^0x > -1$ K-a.e., such that

$$\Psi(v^0) = \sup \Psi(v).$$

For any point $v^0 \in D$ at which the supremum is attained

$$(v - v^0)\left(b - cv^0 - K\left(x\frac{v^0x}{1+v^0x}\right)\right) \leq D\Psi(v^0, v - v^0) \leq 0 \quad \forall\, v \in D.$$

Tradable supermartingale deflator, 1

Proposition

NA_1 implies that $\mathcal{E}(g)$ is a tradable supermartingale deflator.

Proof. **Step 1** : NA_1 implies that $m\{(\omega,t): \ J(\omega,t)\neq\emptyset\}=0$, where $J(\omega,t)$ is the set of $v\in\mathbb{R}^d$ such that $cv=0$, $vx\geq 0$ K-a.e., $vb\geq K(vx)$ and at least one inequality is strict. Thus, we get a candidate for g by solving the maximization problem for every (ω,t) except a m-null set (only measurable selection is needed here).

Step 2 Checking that $g\in L(S)$ using the criterion of integrability due to Cherny and Shiryaev : $g\in L(S)$ iff

$$\left(|c^{1/2}g|^2+K(|gx|^2\wedge 1)+\left|gb-K\left(gx\left(1-I_{\{|x|\leq 1,\ |gx|\leq 1\}}\right)\right)\right|\right)\cdot A_T<\infty.$$

and "laws of large numbers".

Laws of Large Numbers

Let J be a locally square integrable martingale, that is $J \in \mathcal{M}^2_{loc}$, and let H be a predictable process. Put $\Gamma_\infty := \{H^2 \cdot \langle J \rangle_T = \infty\}$ and $M^n := H^n \cdot J$ where $H^n := HI_{\{|H| \le n\}}$.

Lemma

The sequence of random variables $M^n_T/(1 + \langle M^n \rangle) \to 0$ in probability on Γ_∞ and is bounded in probability on Γ^c_∞.

Let N^n be a sequence of counting processes with the compensators $\tilde{N}^n = I_{\{G \le n\}} \cdot \tilde{N}$ where $\tilde{N}$ is a predictable increasing process and $G \ge 0$ is a predictable process. Let $B_\infty := \{\tilde{N}_T = \infty\}$.

Lemma

The sequence of random variables $N^n_T/(\tilde{N}^n_T) \to 1$ in probability on B_∞ and is bounded in probability on B^c_∞.

Tradable supermartingale deflator, 2

Proposition

Let g be a d-dimensional process such that $K(gx \le -1) = 0$ m-a.e. and let $g^n := gI_{\{|g| \le n\}}$. Suppose that for every $n \ge 1$

$$-g^n(b - cg^n) - K\left(\frac{(g^n x)^2}{1 + g^n x}\right) \le 0 \quad m\text{-a.e.}$$

and the sequence of random variables $g^n \cdot S_T$ is bounded in probability. Then $g \in L(S)$.

The inequality above is due to the general inequality for g with $f = 0$. The boundedness in probability follows from NA_1 :

Lemma

Let $\mathcal{R}$ be a set of scalar semimartingales R with $R_0 = 0$, $\Delta R > -1$, and such that $\mathcal{E}^{-1}(R)$ is a supermartingale. Then $\mathcal{R}$ is bounded in probability iff the set $\{\mathcal{E}_T(R) : \ R \in \mathcal{R}\}$ is bounded in probability ;

Again on the ratio of stochastic exponentials, 1

Let

$$\bar{S} = S - cg \cdot A - \sum_{s \le .} \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \Delta S_s.$$

Then

$$S = \bar{S} + cg \cdot A + \sum_{s \le .} (g_s \Delta S_s) \Delta \bar{S}_s.$$

Then

Lemma

$L(S) = L(\bar{S})$.

Proof. We show only $\subseteq$. Let $f \in L(S)$. Then

$$|(f, cg)| \cdot A_T \le \frac{1}{2} |c^{1/2} f|^2 \cdot A_T + \frac{1}{2} |c^{1/2} g|^2 \cdot A_T < \infty,$$

$$\sum_{s<T} \frac{|g_s \Delta S_s f_s \Delta S_s|}{1 + g_s \Delta S_s} \le \frac{1}{2} \sum_{s<T} \frac{|f_s \Delta S_s|^2}{1 + g_s \Delta S_s} + \frac{1}{2} \sum_{s<T} \frac{|g_s \Delta S_s|^2}{1 + g_s \Delta S_s} < \infty.$$

Again on the ratio of stochastic exponentials, 2

Fix $g \in L(S)$ with $g\Delta S > -1$.

Lemma

$\{h \in L(\bar{S}),\ h\Delta\bar{S} > -1\} = \{f \in L(S),\ f\Delta S > -1\} - g.$
If $f \in L(\bar{S})$, $f\Delta S > -1$, *then*

$$\frac{\mathcal{E}(f \cdot S)}{\mathcal{E}(g \cdot S)} = \mathcal{E}((f-g) \cdot \bar{S}).$$

Thus,

$$1 + \mathcal{X}^1_>(\bar{S}) = \frac{1}{\mathcal{E}(g \cdot S)}(1 + \mathcal{X}^1_>(S)).$$

One can prove that

$$1 + \mathcal{X}^1(\bar{S}) = \frac{1}{\mathcal{E}(g \cdot S)}(1 + \mathcal{X}^1(S)).$$

Supermartingale deflator

Theorem (Karatzas–Kardaras)

NA_1 holds if and only if there exists a tradable supermartingale deflator.

Write this deflator as the stochastic exponential $\mathcal{E}(g \cdot S)$ in the above formula. This means that for the corresponding $\bar{S}$ the measure P is the separating one and, hence, there exists $\tilde{P} \sim P$ arbitrary close to P such that $\bar{S}$ is a σ-martingale with respect to $\tilde{P}$. But this means that $\mathcal{E}(g \cdot S)$ is tradable local martingale deflator with respect to $\tilde{P}$.

Separating measure, 1

Theorem

In any neighborhood of a separating measure P there is a probability measure $\tilde{P}$ under which S is a σ-martingale.

Proof. We start with the construction of the density process for $\tilde{P}$.

Lemma

Let $\varepsilon > 0$ and let $Y : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}_+ \setminus \{0\}$ be a $\tilde{\mathcal{P}}$-measurable function such that $P \otimes A$-a.e. :
(a) $K_t(|Y - 1|) \le \varepsilon/2$,
(b) $I_{\{\Delta A > 0\}} K_t(Y - 1) = 0$.
Then the process $Z := \mathcal{E}((Y - 1) * (\mu - \nu)) > 0$ is uniformly integrable martingale and $\tilde{P} := Z_T P$ is a probability such that $||\tilde{P} - P|| \le \varepsilon$. The $\tilde{P}$-triplet of S :
$\tilde{B} = B + (Y - 1)xI_{\{|x| \le 1} * \nu$, $\tilde{C} = C$, $\tilde{\nu} = Y\nu$.

Separating measure, 2

Proof. Note that

$$|Y-1|*\nu_T=\int_{[0,T]}K_t(|Y-1|)dA_t\le(\varepsilon/2)A_T\le\varepsilon/2$$

in view of (b). The process $M:=(Y-1)*(\mu-\nu)$ is a martingale,

$$\Delta M_t=\int(Y(t,x)-1)\mu(\{t\},dx)-K_t(Y-1)\Delta A_t>-1.$$

Thus, $Z=\mathcal{E}(M)>0$ is a martingale of bounded variation, $Z=1+Z_-\cdot M$. Since

$$\begin{aligned}E\sup_{t\le T}|Z_t-1| &= E\sup_{t\le T}|Z_-(Y-1)*(\mu-\nu)_t|\le EZ_-|Y-1|*(\mu+\nu)_T\\ &= 2EZ_-|Y-1|*\nu_T=2EZ_T|Y-1|*\nu_T\le\varepsilon/2,\end{aligned}$$

Z is uniformly integrable martingale and $||\tilde{P}-P||=E|Z_T-1|\le\varepsilon/2$. The form of the triplet of predictable characteristics of S follows from the Girsanov theorem.

Separating measure, 3

Let $C(\overline{\mathbb{R}}^d)$ denote the compact space of continuous functions on $\overline{\mathbb{R}}^d$ equipped by the uniform norm and the Borel σ-algebra $\mathcal{B}(C(\overline{\mathbb{R}}^d))$ and let $\mathbf{Y} = \mathbf{Y}(\overline{\mathbb{R}}^d)$ be its subset formed by the strictly positive continuous functions. We define, for every (ω, t), the convex subsets of $\mathbf{Y}$

$$\begin{aligned} \Gamma^{\varepsilon}_{\omega,t} &= \big\{ Y : \ K_t((|x| \wedge |x|^2)Y) < \infty, \ K_t(|Y-1|) \le \varepsilon/2, \\ & \quad I_{\{\Delta A > 0\}} K_t(Y-1) = 0 \big\}, \\ \Gamma_{\omega,t} &= \big\{ Y : \ K_t(|xY - xI_{\{|x|\le 1\}}|) < \infty, \ b_t + K_t(xY - xI_{\{|x|\le 1\}}) = 0 \big\}. \end{aligned}$$

The graphs of the set-valued mappings $(\omega, t) \mapsto \Gamma^{\varepsilon}_{\omega,t}$ and $(\omega, t) \mapsto \Gamma_{\omega,t}$ are $\mathcal{P} \otimes \mathcal{B}(C(\overline{\mathbb{R}}^d))$-measurable sets (they are intersections of level sets of functions $\mathcal{P}$-measurable in (ω, t) and continuous in Y, hence, $\mathcal{P} \otimes \mathcal{B}(C(\overline{\mathbb{R}}^d))$-measurable).

Separating measure, 4

Proposition

$$m(\{(\omega,t):\ \Gamma^{\varepsilon}_{\omega,t}\cap\Gamma_{\omega,t}\neq\emptyset\})=0.$$

With this the theorem follows easily. Applying the measurable selection theorem to the mapping $(\omega,t)\mapsto\Gamma^{\varepsilon}_{\omega,t}\cap\Gamma_{\omega,t}$ we find a $\mathcal{P}$-measurable function $(\omega,t)\mapsto Y(\omega,t,.)$ with values in $C(\overline{\mathbb{R}}^d)$ and such that $Y(\omega,t,.)\in\Gamma^{\varepsilon}_{\omega,t}\cap\Gamma_{\omega,t}$ m-a.e. The mapping $(\omega,t,x)\mapsto Y(\omega,t,x)$ from $\Omega\times[0,T]\times\mathbb{R}^d$ into $\mathbb{R}_+\setminus\{0\}$ is measurable with respect to $\tilde{P}=\mathcal{P}\otimes\mathcal{B}^d$ (due to continuity in x). Using Lemma we define a probability $\tilde{P}$ under which the local characteristics of S are : $\tilde{b}_t=b_t+K_t((Y-1)xI_{\{|x|\le 1\}})$, $\tilde{c}=c$, $\tilde{K}_t(dx)=Y(t,x)K_t(dx)$. The criterion : S is a $\tilde{P}-\sigma$-martingale iff

$$\tilde{K}_t(|x|\wedge|x|^2)<\infty,\qquad \tilde{b}_t+\tilde{K}_t(xI_{\{|x|>1\}})=0\qquad m\text{-a.e.}$$

is fulfilled.

Proof of the crucial proposition for $d = 1$

On $\Gamma^{\varepsilon}_{\omega,t}$ we have $K_t(|xY - xI_{\{|x|\le 1\}}|) < \infty$, the image of affine mapping $\Psi_{\omega,t} \mapsto K_t(xY - xI_{\{|x|\le 1\}})$ of the set $\Gamma^{\varepsilon}_{\omega,t}$ is an interval and we need to check that $-b_t$ belongs to it.
Put $r = \sup\{x :\ K(]-\infty, x[) = 0\}$, $R = \sup\{x :]x, \infty[= 0\}$. Then

$$\begin{aligned} EI_{\{r>-n\}}x^- I_{\{x\le -n\}} * \mu_T &= EI_{\{r>-n\}}x^- I_{\{x\le -n\}} * \nu_T \\ &= \int_{[0,T]} I_{\{r>-n\}} K_t(x^- I_{\{x\le -n\}}) dA_t = 0. \end{aligned}$$

Thus, the process $I_{\{r>-n\}}x^- I_{\{x<-1\}} * \mu$ is locally bounded (jumps do not exceed n). Also $I_{\{r>-n\}} \cdot M^c$, $I_{\{r>-n\}} \cdot M^d$, $I_{\{r>-n\}} \cdot |b| \cdot A$ are locally bounded. If they would be bounded, the process $I_{\{r>-n\}} \cdot S$ would be bounded from below and, by the hypothesis, $EI_{\{r>-n\}} \cdot S \le 0$, i.e.

$$EI_{\{r>-n\}}x^+ I_{\{x>1\}} * \mu_T - EI_{\{r>-n\}}x^- I_{\{x<-1\}}) * \mu_T + EI_{\{r>-n\}} b \cdot A_T \le 0.$$

Proof of the crucial proposition for $d = 1$

It follows that

$$EI_{\{r>-n\}}xI_{\{x>1\}} * \mu_T = EI_{\{r>-n\}}xI_{\{x>1\}} * \nu_T < \infty.$$

This implies that $I_{\{r>-n\}}K(|x|) < \infty$ m-a.e. In the general case we may conclude that the increasing processes $I_{\{r>-n\}}xI_{\{x>1\}} * \mu$ and $I_{\{r>-n\}}xI_{\{x>1\}} * \nu$ are locally integrable. Applying the similar arguments to the integrand $I_{\{r>-n\}}I_D$ where $D \in \mathcal{P}$, we infer that

$$I_{\{r>-n\}}(K(xI_{\{|x|>1\}}) + b) \le 0 \quad m\text{-a.e.}$$

It follows that

$$K(|x|I_{\{|x|>1\}}) < \infty, \quad K(xI_{\{|x|>1\}}) + b \le 0 \quad m\text{-a.e. on } \{r > -\infty\}.$$

Arguing in the same way with $-I_{\{R<n\}}$ we obtain that

$$K(|x|I_{\{|x|>1\}}) < \infty, \quad K(xI_{\{|x|>1\}}) + b \ge 0 \quad m\text{-a.e. on } \{R < \infty\}.$$

Proof of the crucial proposition for $d = 1$

This means that, modulo m-null subset, we have :
(a) on $\{r > -\infty\}$ the function $1 \in \Gamma^\varepsilon$ and
$-b \geq \Psi(1) = K(xI_{\{|x|>1\}})$,
(b) on $\{R < \infty\}$ the function $1 \in \Gamma^\varepsilon$ and
$-b \geq \Psi(1) = K(xI_{\{|x|>1\}})$
Thus, on the intersections of these sets $-b = \Psi(1)$. The conclusion of the proposition in the case of $d = 1$ from the following (purely deterministic) assertion.

Lemma

If $R = \infty$, then the interval $\Psi(\Gamma^\varepsilon)$ is unbounded from above.

Proof of the crucial proposition for arbitrary d

The sets

$$\Xi_{\omega,t} := \Psi_{\omega,t}(\Gamma_{\omega,t}) + b_t(\omega) \subseteq \mathbb{R}^d$$

are convex and $\{(\omega,t,x):\ x \in \Xi_{\omega,t}\} \in \mathcal{P} \otimes \mathcal{B}^d$. By the measurable version of the separation theorem, there is a predictable process l with values in $(\mathbb{R}^d)^* = \mathbb{R}^d$ such that, outside a m-negligible set, $|l_{\omega,t}| = 1$ and $l_{\omega,t}x < 0$ for every $x \in \Xi_{\omega,t}$ if $0 \notin \Xi_{\omega,t}$, and $l_{\omega,t} = 0$, otherwise. We use the superscript l to denote objects related to the scalar semimartingale $S^l := l \cdot S$. Obviously, $\nu^l(\omega,dt,dx) = K^l_{\omega,t}(dx)dA_t(\omega)$, $K^l_{\omega,t}(dx) = (K_{\omega,t}l^{-1}_{\omega,t})(dx)$,

$$B^l = lb \cdot A + K(lI_{\{|x|\le 1\}} - I_{\{|lx|\le 1\}}|) \cdot A,$$

P is a separating measure for S^l. We have proved that for every fixed (ω,t) outside of a m-negligible set the equation $\Psi^l_{\omega,t}(Y) = -b^l_t(\omega)$ has a solution $Y \in \Gamma^{\varepsilon l}_{\omega,t}$. Due to the above relations, the function $Y(l_{\omega,t}x)$ belongs to $\Gamma^{\varepsilon}_{\omega,t}$ and solves the equation $\Psi_{\omega,t}(Y(l_{\omega,t}x)) = -b_t(\omega)$. Thus, $l = 0$ m-a.e.