Lecture 1: Some precise results in the problem of optimal investment

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Outline

Goal: "precise" (ideally, necessary and sufficient) answers to some mathematical questions arising in the theory of optimal investment.

References

- K. and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950, 1999.
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- K. and M. Sîrbu. On the two-times differentiability of the value functions in the problem of optimal investment in incomplete markets. *Annals of Applied Probability*, Vol. 16, No. 3, 1352-1384, 2006.
- K. and K. Weston. Muckenhoupt's (A_p) condition and the existence of the optimal martingale measure. *arXiv*, July, 2015.

Introduction to optimal investment

Consider an economic agent (an investor) in an *arbitrage-free* financial model.

x: initial capital

Goal: invest x *"optimally"* up to maturity T.

Question

How to compare two investment strategies:

1. $x \longrightarrow X_T = X_T(\omega)$
2. $x \longrightarrow Y_T = Y_T(\omega)$

Clearly, we would prefer 1st to 2nd if $X_T(\omega) \geq Y_T(\omega)$, $\omega \in \Omega$.
However, as the model is arbitrage-free, in this case, $X_T(\omega) = Y_T(\omega)$, $\omega \in \Omega$.

Introduction to optimal investment

Classical approach (Von Neumann - Morgenstern, Savage): an investor is *"quantified"* by

$\mathbb{P}$: "scenario" probability measure

$U = U(x)$: *utility function*

"Quality" of a strategy

$$x \longrightarrow X_T = X_T(\omega)$$

is then measured by *expected utility*: $\mathbb{E}[U(X_T)]$.

Given two strategies: $x \longrightarrow X_T$ and $x \longrightarrow Y_T$ the investor will prefer the 1st one if

$$\mathbb{E}[U(X_T)] \geq \mathbb{E}[U(Y_T)]$$

Introduction to optimal investment

Inputs:

1. Arbitrage-free financial model (all traded securities)
2. Risk-averse investor:

   x: initial wealth

   $\mathbb{P}$: "real world" probability measure

   $U = U(x)$: strictly increasing and strictly concave utility function

Output: the optimal investment strategy $x \longrightarrow \widehat{X}_T$ such that

$$\mathbb{E}[U(\widehat{X}_T)] = u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Here $\mathcal{X}(x)$ is the set of strategies with initial wealth x.

Merton's problem

First papers in continuous time finance: Merton (1969), (1971).

Black and Scholes model: *a savings account* and *a stock*.

1. We assume that the interest rate is 0.
2. The price of the stock:

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right).$$

Here $W = (W_t)_{t \geq 0}$ is a Wiener process and

$\mu \in \mathbf{R}$: drift

$\sigma > 0$: volatility

Merton's problem

In this case, the problem of optimal investment

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)]$$

becomes a *stochastic control problem*:

$$u(x,t) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_{T-t})] = \sup_{\pi} \mathbb{E}[U(X^{\pi}_{T-t})],$$

where the *controlled* process X^{π} is the wealth process:

$$dX^{\pi} = X^{\pi}\pi(\mu dt + \sigma dW) \quad X^{\pi}_0 = x$$

and the *control* process π is the proportion of the capital invested in stock.

Merton's problem

Bellman equation:

$$u_t + \sup_{\pi} \left[\pi x \mu u_x + \frac{1}{2}\pi^2\sigma^2 x^2 u_{xx} \right] = 0.$$

It follows that

$$\begin{cases} u_t(x,t) &= \frac{\mu^2 u_x^2}{2\sigma^2 u_{xx}}(x,t) \\ u_{xx}(x,t) &< 0 \\ u(x,T) &= U(x) \end{cases}$$

and the optimal proportion:

$$\widehat{\pi}(x,t) = -\frac{\mu u_x}{\sigma^2 x u_{xx}}(x,t).$$

Merton's problem

Merton (1969) solved the system for the case, when

$$U(x, \alpha) = \frac{x^\alpha - 1}{\alpha} \quad (\alpha < 1).$$

Here

$$-\frac{U'(x)}{xU''(x)} = \frac{1}{1-\alpha} \quad (= \text{const!})$$

This key property is "inherited" be the solution:

$$\frac{u_x}{xu_{xx}}(x, t) = \text{const.}$$

Merton's problem

After this substitution the first equation in the system becomes

$$u_t = \text{const } x^2 u_{xx}$$

and could be solved analytically.
The optimal strategy (Merton's point):

$$\widehat{\pi} = \frac{\mu}{(1-\alpha)\sigma^2}.$$

Surprisingly, the problem has not been solved for general utility function U for quite a long time (Pliska (1986), Karatzas, Lehoczky, Shreve (1987)).

Merton's problem

In general case, we define the conjugate function

$$v(y,t) = \sup_{x>0} [u(x,t) - xy]$$

The function v satisfies

$$v_t = \text{const } y^2 v_{yy}$$
$$v(y,T) = V(y) := \sup_{x>0} [U(x) - xy]$$

Methodology: compute v first and then find u from the inverse duality relationship:

$$u(x,t) = \inf_{y>0} [v(y,t) + xy]$$

Model of a financial market

There are $d + 1$ *traded* or *liquid* assets:

1. a *savings account* with zero interest rate.
2. d *stocks*. The price process S of the stocks is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

Assumption (No Arbitrage)

$$\mathcal{Q} \neq \emptyset$$

where $\mathcal{Q}$ is the family of martingale measures for S.

Economic agent or investor

x: initial capital

U: utility function for consumption at the maturity T such that

1. $U : (0, \infty) \to \mathbf{R}$
2. U is strictly increasing
3. U is strictly concave
4. The Inada conditions hold true:

$$U'(0) = \infty \quad U'(\infty) = 0$$

Problem of optimal investment

The goal of the investor is to maximize **the expected utility of terminal wealth**:

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0$$

Here $\mathcal{X}(x)$ is the set of strategies with initial wealth x.

Assumption

The value function is finite:

$$u(x) < \infty, \quad x > 0.$$

Main approaches

1. HJB equation for stochastic control.
2. Convex duality and martingales.
3. FBSDE.

Convex duality and martingales

Basic idea: as

$$\mathbb{E}[U(\widehat{X}_T(x))] = \max_{X \in \mathcal{X}(0)} \mathbb{E}[U(\widehat{X}_T(x) + X_T)],$$

we have that for any $X \in \mathcal{X}(0)$

$$\mathbb{E}[U'(\widehat{X}_T(x))X_T] = 0.$$

Hence, there is $\mathbb{Q} \in \mathcal{Q}$ (a martingale measure for S) such that

$$U'(\widehat{X}_T(x)) = \text{const}\frac{d\mathbb{Q}}{d\mathbb{P}}$$

Remark

The last identity may not hold for general incomplete markets.

Investment in complete models

Complete model: $|\mathcal{Q}| = 1$

Define the functions

$$V(y) = \max_{x>0} \left[U(x) - xy \right], \quad y > 0.$$
$$v(y) = \mathbb{E}\left[V\left(y(\frac{d\mathbb{Q}}{d\mathbb{P}}) \right) \right], \quad y > 0$$

Theorem

$$u(x) = \inf_{y>0} [v(y) + xy]$$

Investment in complete models

Theorem

The following conditions are equivalent:

1. *The dual value function $v = v(y)$ is finite:*

$$v(y) < \infty, \quad y > 0$$

2. *The primal value function $u = u(x)$ is strictly concave and satisfies the Inada conditions.*

Moreover, in this case, $\widehat{X}(x)$ exists for any $x > 0$ and

$$\widehat{X}_T(x) = -V'\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right), \quad y = u'(x).$$

Investment in complete markets

The optimal terminal wealth $\widehat{X}_T(x)$ is uniquely determined by the equations:

$$\widehat{X}_T(x) = -V'(y\frac{d\mathbb{Q}}{d\mathbb{P}})$$
$$\mathbb{E}_{\mathbb{Q}}[\widehat{X}_T(x)] = x$$

The optimal number of stocks $\widehat{H}$ solves the BSDE:

$$\widehat{X}_t = -V'(y\frac{d\mathbb{Q}}{d\mathbb{P}}) - \int_t^T \widehat{H}_u dS_u,$$

and is given by the martingale representation formula:

$$\widehat{X}_t = \mathbb{E}^{\mathbb{Q}}[-V'(y\frac{d\mathbb{Q}}{d\mathbb{P}})|\mathcal{F}_t] = x(y) + \int_0^t \widehat{H}_u dS_u.$$

Back to Merton's problem

For Black and Scholes model we have

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T) = \exp(-\frac{\mu}{\sigma}W_T^{\mathbb{Q}} + \frac{1}{2}\frac{\mu^2}{\sigma^2}T),$$

where

$$W_t^{\mathbb{Q}} = W_t + \frac{\mu}{\sigma}t,$$

is the $\mathbb{Q}$-Brownian motion.
To find the optimal number of shares $\widehat{H}$ we use the *Clark-Ocone formula*:

$$\sigma \widehat{H}_t S_t = \mathbb{E}^{\mathbb{Q}}[\mathbf{D}_t^{\mathbb{Q}}[-V'(y\frac{d\mathbb{Q}}{d\mathbb{P}})]|\mathcal{F}_t],$$

where $\mathbf{D}^{\mathbb{Q}}$ is the *Malliavin derivative* under $\mathbb{Q}$.

Back to Merton's problem

We deduce that

$$\widehat{H}_t S_t = \frac{\mu}{\sigma^2} \mathbb{E}^{\mathbb{Q}}[y \frac{d\mathbb{Q}}{d\mathbb{P}} V''(y \frac{d\mathbb{Q}}{d\mathbb{P}}) | \mathcal{F}_t] = \frac{\mu}{\sigma^2} R_t,$$

where R is the *risk-tolerance wealth process* defined as the wealth process replicating the payoff:

$$R_T = y \frac{d\mathbb{Q}}{d\mathbb{P}} V''(y \frac{d\mathbb{Q}}{d\mathbb{P}}) = -\frac{U'(\widehat{X}_T)}{U''(\widehat{X}_T)}.$$

Basic questions for incomplete models

1. Does the optimal investment strategy $X(x)$ exist?
2. Does the value function $u = u(x)$ satisfy the *standard* properties of a utility function? In other words,
   2.1 Is u strictly concave?
   2.2 Do Inada conditions

   $$u'(0) = \infty, \quad u'(\infty) = 0$$

   hold true?

Basic questions for incomplete models

3. Does the conjugate function

$$v(y) = \sup_{x>0}\{u(x) - xy\}, \quad y > 0,$$

have the representation:

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})],$$

where

$$V(y) = \sup_{x>0}\{U(x) - xy\}, \quad y > 0?$$

References

These questions were studied in

1. Cox and Huang (89), (91)
2. He and Pearson (91a), (91b)
3. Karatzas, Lehoczky, Shreve and Xu (91)

Goal: get *minimal* (ideally *necessary and sufficient*) conditions.

Asymptotic elasticity

Recall that the *elasticity* for U is defined as

$$E(U)(x) = \frac{xU'(x)}{U(x)}$$

The crucial role is played by the *asymptotic elasticity*:

$$AE(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)}.$$

We always have $AE(U) \leq 1$.

Assumption

$$AE(U) < 1.$$

Minimal market independent condition

Theorem (K.& Schachermayer (99))

The following conditions are equivalent :

1. $AE(U) < 1$.
2. *For any financial model the "qualitative" properties 1–3 hold true.*

In addition, in this case

$$AE(u) \leq AE(U) < 1.$$

Remark

The condition $AE(U) < 1$ is similar to Δ_2-condition in the theory of Orlicz spaces.

Necessary and sufficient conditions

Theorem (K.& Schachermayer (03))

The following conditions are equivalent for given financial model:

1. *For any* $y > 0$ *there is* $\mathbb{Q} \in \mathcal{Q}$ *such that*

$$\mathbb{E}[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)] < \infty.$$

2. *The "qualitative" properties 1–3 hold true.*

Dual space of supermartingales

The lower bound in

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}\left[V\left(y \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]$$

is, in general, not attained. However, if we extend the space of density processes of martingale measures to the space $\mathcal{Y}(y)$ of strictly positive supermartingales Y such that

1. $Y_0 = y$
2. XY is a supermartingale for any $X \in \mathcal{X}(x)$

then (without any extra assumptions!) we have

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)]$$

and the lower bound above is attained by $\widehat{Y}(y) \in \mathcal{Y}(y)$.
Sometimes, this property is even more convenient for computations!

Duality characterization

$$\widehat{X} \text{ is optimal} \quad \Leftrightarrow \quad U'(\widehat{X}_T) = \widehat{Y}_T,$$

where the process $\widehat{Y}$ is such that

1. $X\widehat{Y}$ is a supermartingale, $\forall X \in \mathcal{X}$,
2. $\widehat{X}\widehat{Y}$ is a UI martingale.

In complete case, where there is only one $\mathbb{Q} \in \mathcal{Q}$,

$$\widehat{Y}_T = y_0 \frac{d\mathbb{Q}}{d\mathbb{P}} \text{ with } y_0 = \widehat{Y}_0.$$

Problem: in incomplete case, find conditions $\Longrightarrow$

$$\widehat{Y}_T = y_0 \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \text{ for some } \widehat{\mathbb{Q}} \in \mathcal{Q}.$$

Motivation: uniqueness of utility based prices.

Mukenhoupt's (A_p)-condition

Definition

Let $p > 1$. A process $R > 0$ satisfies (A_p) if there is a constant $c > 0$ such that for every stopping time τ

$$\mathbb{E}\left[\left(\frac{R_\tau}{R_T}\right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau\right] \le c.$$

Known fact: if $R > 0$ satisfies (A_p) and $\mathbb{E}[R_T] < \infty$, then R is of class $(\mathbf{D})$:

$$\{R_\tau : \tau \text{ is a stopping time}\} \text{ is UI.}$$

In particular, if R is a local martingale, then R is a UI martingale.

Local martingale property for $\widehat{Y}$

In general, $\widehat{Y}$ is not a local martingale; see example for logarithmic utility in single-period model in (K. Schachermayer, 1999).

Definition

A semimartingale R is *σ-bounded* if there is a predictable process $h > 0$ such that the stochastic integral $\int hdR$ is bounded.

Assumption

For all wealth processes $X > 0$ and $X' \geq 0$, the process X'/X is σ-bounded.

Proposition

If Assumption holds, then $\widehat{Y}$ is a local martingale.

Assumption on jumps

Assumption holds in the following cases, see (K. Sîrbu, 2006):

1. S is continuous (easy).
2. $\exists$ a *finite-dimensional* local martingale M such that every bounded *purely discontinuous* martingale N admits the integral representation:

$$N = N_0 + \int H dM.$$

3. $\exists$ a *complete* financial market on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$.

Existence of optimal martingale measure

Theorem (K., Weston (15))

Suppose that there are bounds $a \in (0,1)$ *and* $b \geq a$ *for the relative risk-aversion:*

$$a \leq -\frac{xU''(x)}{U(x)} \leq b, \quad x > 0,$$

and there is $\mathbb{Q} \in \mathcal{Q}$ *whose density process satisfies* (A_p) *with*

$$p = \frac{1}{1-a}.$$

Then $\widehat{Y}$ *satisfies* $(A_{p'})$ *with*

$$p' = 1 + \frac{b}{1-a} \qquad (\geq \frac{1}{1-a}).$$

If, in addition, σ*-boundedness Assumption holds, then* $\widehat{Y}_T/y_0$ *is the density process of the optimal martingale measure* $\widehat{\mathbb{Q}}$*.*

Example: bounded market price of risk

Suppose that the maturity T is bounded and

$$dS = S\sigma(\lambda dt + dB),$$

where B is a Brownian motion, $\sigma = (\sigma_t)$ is the volatility, and $\lambda = (\lambda_t)$ is the *market price of risk*.
Recall that the *minimal* martingale measure $\widetilde{\mathbb{Q}}$ has the density

$$\widetilde{Z} = \mathcal{E}(-\int \lambda dB) = \exp(-\int \lambda dB - \frac{1}{2}\int \lambda^2 ds).$$

We have that

$$|\lambda| \leq \text{const} \quad \Rightarrow \quad \widetilde{Z} \text{ satisfies } (A_p), \forall p > 1.$$

In this case, we get the result ($\widehat{Y} \in y_0\mathcal{Z}$) as soon as the relative risk-aversion of U is bounded away from 0 and ∞.

Sharpness of the (A_p) condition

Theorem (K., Weston (15))

Let constants a and p be such that

$$0 < a < 1 \text{ and } p > \frac{1}{1-a}.$$

Then there exists a financial market with a continuous stock price S such that

1. *There is $\mathbb{Q} \in \mathcal{Q}$, whose density process Z satisfies (A_p).*
2. *In the optimal investment problem with the power utility function*

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

the dual minimizer $\widehat{Y}$ with $\widehat{Y}_0 = 1$ is well-defined but does not belong to $\mathcal{Z}$ ($\widehat{Y}$ is a strict local martingale).

Differentiability of optimal investment strategies

1. Does $u''(x)$ exist?
2. Is $u''(x)$ strictly negative?
3. Does
$$\widehat{X}'(x) = \lim_{\Delta x \to 0} \frac{\widehat{X}(x + \Delta x) - \widehat{X}(x)}{\Delta x}$$
exist?

Remark
$\widehat{X}'(x)$ shows what the investor does with "extra penny".

Motivation

"Theoretical". Classical solution of Bellman equation.

"Practical". Asymptotic analysis of financial problems with "imperfections":

1. Sensitivity analysis of utility based prices for small quantities of random endowments.
2. Sensitivity analysis of equilibrium problems for small quantities of random endowments.
3. Sensitivity analysis for small transaction costs.

Smoothness of $u(x)$

Theorem (K.,Sirbu (06))

Suppose that there are bounds $a \in (0,1)$ *and* $b \geq a$ *for the relative risk-aversion:*

$$a \leq -\frac{xU''(x)}{U(x)} \leq b, \quad x > 0,$$

and assume that the σ*-boundedness Assumption holds. Then* $u''(x)$ *and* $\widehat{X}'(x)$ *exist for every* $x > 0$ *and*

$$a \leq -\frac{xu''(x)}{u'(x)} \leq b, \quad x > 0.$$

Remark

Both assumptions are *essential* for the assertion of the theorem to hold true.

Counterexample on sigma-boundedness

There are

One-period model: $S_0 = 1$ and S_1 takes the values
$\{2, 1, \frac{1}{2}, \frac{1}{4}, \dots \frac{1}{2^n}, \dots\}$.

Utility function:

$$1 < -\frac{xU''(x)}{U'(x)} < 2, \quad x > 0.$$

such that $u''(1)$ and $X'(1)$ are not defined.