# On exponential utility maximization

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Let a utility function $U$ be an increasing concave function on $\mathbb{R}$ with finite values which is not identically a constant. We consider the problem of maximizing expected utility from (discounted) terminal wealth, i.e. the problem of maximizing the functional

$$\mathsf{E}\, U(x + H \cdot S_T), \quad H \in \mathscr{H},$$

over a convex cone $\mathscr{H} \subseteq L(S)$ of predictable integrands with respect to a semimartingale $S = (S_t)_{0 \le t \le T}$, $x$ is the initial wealth of the agent.

The value function $u$ is given by

$$u(x) = \sup_{H \in \mathscr{H}} \mathsf{E}\, U(x + H \cdot S_T), \quad x \in \mathbb{R}.$$

It is well known that an appropriate choice of the class of strategies of an agent is a difficult task in this setting. The class $\mathscr{H} = L(S)$ is too rich as it typically leads to arbitrage opportunities and hence to a degenerate utility maximization problem in the sense $u(x) \equiv U(+\infty)$. On the other hand, the class $\mathscr{H}^b \subset L(S)$ of strategies $H$ such that the integral process $H \cdot S$ is bounded from below by a constant, may be not sufficiently rich. First, it typically fails to contain the optimal solution. Second, if $S$ is not locally bounded, it may even happen that $\mathscr{H}^b = \{0\}$ and $u(x) \equiv U(x)$, which can be considered as another degenerate case. A proper choice of $\mathscr{H}$ is studied e.g. in Schachermayer (2001, 2003), S. Biagini & Frittelli (2005, 2007, 2008), S. Biagini & Černý (2011), S. Biagini & Sîrbu (2012).

Let $(\Omega, \mathscr{F}, \mathrm{P})$ be a probability space and $\Phi\colon \mathbb{R} \to \mathbb{R}_+$ a nonzero even lower semicontinuous convex function with $\Phi(0) = 0$ (a Young function). Then the Orlicz space $L^\Phi = L^\Phi(\mathrm{P})$ on $(\Omega, \mathscr{F}, \mathrm{P})$, associated with $\Phi$, is defined by

$$L^\Phi = \{\xi \in L^0(\mathrm{P})\colon \mathrm{E}\,\Phi(\varepsilon\xi) < +\infty \text{ for some } \varepsilon > 0\}.$$

$L^\Phi$ is a Banach lattice wrt the Luxemburg norm $N_\Phi$ given by

$$N_\Phi(\xi) = \inf\left\{K > 0\colon \mathrm{E}\,\Phi\left(\xi/K\right) \le 1\right\}, \quad \xi \in L^\Phi.$$

Let $\Psi$ be the Fenchel conjugate of $\Phi$:

$$\Psi(y) = \sup_{x \in \mathbb{R}}\,[xy - \Phi(x)],$$

then $\Psi$ is also a Young function. Note that the Fenchel conjugate of $\Psi$ is $\Phi$. Another norm on $L^\Phi$, equivalent to $N_\Phi$, is the Orlicz norm $\|\cdot\|_\Phi$:

$$\|\xi\|_\Phi = \sup_{\mathrm{E}\,\Psi(\eta) \le 1} |\mathrm{E}\,\xi\eta|.$$

It follows from the general theory of Banach lattices that the norm dual space $(L^\Phi)'$ admits a decomposition into the direct sum

$$(L^\Phi)' = (L^\Phi)'_r \oplus (L^\Phi)'_s,$$

where $(L^\Phi)'_r$ is the band of regular, i.e. order continuous functionals, and $(L^\Phi)'_s$ is the band of singular functionals that are disjoint to every regular functional. $(L^\Phi)'_r$ can be identified with $L^\Psi$: to every $\eta \in L^\Psi$ there corresponds a functional $\xi \rightsquigarrow \mathsf{E}\,\xi\eta$ on $L^\Phi$, which belongs to $(L^\Phi)'_r$; this correspondence is a linear one-to-one isometry between $L^\Psi$ with the Orlicz norm and $(L^\Phi)'_r$ with the dual norm. We consider the above functional associated with $\eta$ as the measure with the density $\eta$ with respect to P. Elements of $(L^\Phi)'_s$ are characterized by the following property: $\mu \in (L^\Phi)'_s$ if and only if $\mu(\xi) = 0$ for all $\xi \in L^\infty$. A sufficient condition for $(L^\Phi)'_s = \{0\}$ is the $\Delta_2$-condition: there are $x_0 > 0$ and $K > 0$ such that

$$\Phi(2x) \leq K\Phi(x), \quad x \geq x_0.$$

Let us return to the utility maximization problem. Define a Young function $\Phi$ by

$$\Phi(x) = -U(-|x|) + U(0), \quad x \in \mathbb{R}.$$

Put $\mathscr{A} := \{x + H \cdot S_T \colon H \in \mathscr{H}\}$, $\mathscr{C} := (\mathscr{A} - L^0_+) \cap L^\Phi$ and introduce the set

$$\mathscr{R} = \{\mu \in (L^\Phi)' \colon \mu(\mathbb{1}) = 1 \text{ and } \mu(\xi) \le 0 \text{ for every } \xi \in \mathscr{C}\}$$

of "separating" functionals. Note that functionals in $\mathscr{R}$ are positive since any negative random variable from $L^\Phi$ belongs to $\mathscr{C}$. In particular, if $\mu \in \mathscr{R}$ then $\mu^r$ and $\mu^s$ are positive, where $\mu = \mu^r + \mu^s$ is the decomposition of $\mu$ into the sum of a regular and singular components.

As usual, min stands for the infimum that is attained and $\min \varnothing = +\infty$. Define

$$V(y) = \sup_{x \in \mathbb{R}} [U(x) - xy], \quad y \in \mathbb{R}.$$

In the next theorem, (1) is a special case of Morozov (2010). A more restrictive statement is due to S. Biagini & Frittelli (2008). Equality between (1) and (2) is proved in G. (2009). The full proof of the theorem can be found in G. & Khasanov & Morozov (2014).

## Theorem

*We have*

$$u(x) = \min_{y \geq 0} [v(y) + xy], \quad x \in \mathbb{R},$$

*where* $v(0) = V(0)$ *and*

$$v(y) = \min_{\mu \in \mathscr{R}} \left[ y\|\mu^s\| + \mathrm{E}\, V\left( y \frac{d\mu^r}{d\mathrm{P}} \right) \right] \tag{1}$$

$$= \min_{\mathrm{Q} \in \mathscr{Q}} \left[ ya(\mathrm{Q}) + \mathrm{E}\, V\left( y \frac{d\mathrm{Q}}{d\mathrm{P}} \right) \right], \quad y > 0, \tag{2}$$

*where*

$$\mathscr{Q} = \{\mathrm{Q} \text{ is a probability measure} \colon \mathrm{Q} \ll \mathrm{P},\ d\mathrm{Q}/d\mathrm{P} \in L^{\Psi},\ a(\mathrm{Q}) < \infty\},$$

$$a(\mathrm{Q}) = \sup_{\xi \in \mathscr{C} \colon \mathrm{E}\, U(-\xi^-) > -\infty} \mathrm{E}_{\mathrm{Q}} \xi.$$

In what follows,

$$U(x) = -e^{-x}, \quad x \in \mathbb{R}.$$

Then

$$V(y) = y \log y - y, \quad y > 0,$$

$$\Phi(x) = e^{|x|} - 1, \quad \Psi(x) = \big(|x| \log |x| - |x| + 1\big) \mathbb{1}_{|x|>1}, \quad x \in \mathbb{R}.$$

Note that

$$u(x) = e^{-x} u(0), \quad x \in \mathbb{R}, \qquad v(y) = yv(1) + y \log y, \quad y > 0,$$

and

$$u(0) = e^{-v(1)-1}.$$

Exponential utility maximization and the duality in this special case is considered in Frittelli (2000), Bellini & Frittelli (2002), Delbaen & Grandits & Rheinländer & Samperi & Schweizer & Stricker (2002), Kabanov & Stricker (2002), Esche & Schweizer (2005), Acciaio (2005), . . .

Now we assume that

$$S = \mathscr{E}(L),$$

where $L$ is a Lévy process with $\Delta L > -1$. Let

$$f(\vartheta) := \mathsf{E}e^{\vartheta L_T}, \quad \Theta := \{\vartheta \in \mathbb{R} \colon f(\vartheta) < \infty\}, \quad \vartheta_0 := \inf \Theta.$$

Then

$$\vartheta_0 \in [-\infty, 0], \qquad (\vartheta_0, +\infty) \subseteq \Theta \subseteq [\vartheta_0, +\infty).$$

Moreover, the function $f$ attains its minimum at some $\vartheta^* \geq \vartheta_0$.

Exponential utility maximization in exponential Lévy models is considered in Kallsen (2000). It is shown that the "optimal" portfolio assigns a constant amount of money to the risky asset. In other words, the optimal portfolio is such that $HS_- = \text{const}$, i.e the wealth process can be written as

$$V_t = x + \int_0^t H_s \, dS_s = x + \int_0^t H_s S_{s-} \, dL_s = x + \vartheta L_t,$$

and the maximum is attained at $H = \vartheta^* / S_-$. This is shown under the assumption

$$f'_+(\vartheta^*) = 0.$$

Consider the dual minimization problem (2) for $\mathscr{H}_c = \{H = \vartheta/S_- \colon \vartheta \in \mathbb{R}\}$. If $\Theta = \mathbb{R}$, then

$$a(\mathsf{Q}) = \begin{cases} 0, & \text{if } \mathsf{E}_\mathsf{Q} L_T = 0; \\ +\infty, & \text{otherwise,} \end{cases}$$

so the dual minimization problem in this case is to minimize the relative entropy

$$\mathsf{E}\frac{d\mathsf{Q}}{d\mathsf{P}} \log \frac{d\mathsf{Q}}{d\mathsf{P}}$$

over all $\mathsf{Q} \ll \mathsf{P}$ with $\mathsf{E}_\mathsf{Q} L_T = 0$. The infimum is attained at

$$\frac{d\mathsf{Q}^*}{d\mathsf{P}} = \frac{e^{\vartheta^* L_T}}{\mathsf{E}e^{\vartheta^* L_T}}.$$

Now consider the dual minimization problem (2) for $\mathscr{H}_c = \{H = \vartheta/S_- \colon \vartheta \in \mathbb{R}\}$ in the case $\vartheta_0 \in \mathbb{R}$. Then

$$a(\mathsf{Q}) = \begin{cases} \vartheta_0 \mathsf{E}_{\mathsf{Q}} L_T, & \text{if } \mathsf{E}_{\mathsf{Q}} L_T \leq 0; \\ +\infty, & \text{otherwise,} \end{cases}$$

so the dual minimization problem in this case is to minimize

$$\mathsf{E} \frac{d\mathsf{Q}}{d\mathsf{P}} \log \frac{d\mathsf{Q}}{d\mathsf{P}} + \vartheta_0 \mathsf{E}_{\mathsf{Q}} L_T$$

over all $\mathsf{Q} \ll \mathsf{P}$ with $\mathsf{E}_{\mathsf{Q}} L_T \leq 0$. The infimum is attained at

$$\frac{d\mathsf{Q}^*}{d\mathsf{P}} = \frac{e^{\vartheta^* L_T}}{\mathsf{E} e^{\vartheta^* L_T}}.$$

## Remarks

1. *If* $f'_+(\vartheta^*) > 0$*, the minimal entropy martingale measure does not exist. However, the dual minimization problem has a unique solution.*

2. *If* $\vartheta_0 \in \mathbb{R}$ *and the minimal entropy martingale measure exists, the dual problem is different from that of minimizing the relative entropy over the set of (local,* $\sigma$*)-martingale measures.*

3. *Let* $\mathscr{H}$ *be an arbitrary convex cone in* $L(S)$ *containing* $\mathscr{H}_c$*. The utility maximization problem over* $\mathscr{H}$ *has the same value as that over* $\mathscr{H}_c$ *if and only if the values of the corresponding dual problems coincide. In particular, if* $\vartheta_0 \in \mathbb{R}$*, it is necessary and sufficient that, for every* $H \in \mathscr{H}$ *with* $\mathsf{E}e^{(H\cdot S_T)^-} < \infty$*,*

$$\mathsf{E}_{\mathsf{Q}^*} H \cdot S_T \leq \vartheta_0 \mathsf{E}_{\mathsf{Q}^*} L_T.$$

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Thank you for your attention!