Optimal stopping problems arising from Mathematical Finance

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Outline

- Basic formulation
- Mathematical finance
- Markovian approach
- Free-boundary problem
- American call option
    - Black and Scholes model
    - Diffusion-type models

Basic formulation

Optimal stopping theory is part of Stochastic control theory with a wide set of applications in the areas of statistics, economics, mathematical finance, etc and well-developed methods of solution.

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where

- the supremum is taken over all stopping times τ
  $\longrightarrow$ Random variables $\tau(\omega) < \infty$, $\mathbb{P} - a.s.$, such that $\{\omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

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$\exists$ Markov process $X = (X_t)_{t \geq 0}$ such that $G_t = G(t, X_t)$ for

- some measurable function $G(t, x)$,
- $x \in E$,
- E is the state space of X.

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The no-arbitrage price of an American-type contingent claim is given by the *Optimal stopping problem*:

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- $G(x)$ is the *payoff function* of the contingent claim
- written on the underlying risky asset price process X
- r is the constant *discount rate / interest rate*
- the supremum is taken over all stopping times τ of X
- $\mathbb{E}_x$ is taken under some *Equivalent Martingale Measure* $\mathbb{P}_x$ for X given that $X_0 = x \in E$.

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$\longrightarrow$ Formulation by [Dynkin '63] [1]

[1] DYNKIN, E. B. (1963). The optimum choice of the instant for stopping a Markov process. *Soviet Mathematical Doklady* 4 (627-629)

Solution to the Optimal stopping problem

General theory of Optimal stopping for Markov processes [2]

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- We have a set of domains $\mathcal{C}$, such that

$$\mathcal{C} = \{x \in E \mid V(x) > G(x)\} \qquad \textit{(continuation region)}$$
$$\mathcal{D} = \{x \in E \mid V(x) = G(x)\} = E \setminus \mathcal{C} \qquad \textit{(stopping region)}$$

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- Optimal stopping problem

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is equivalent to the problem of

Finding the smallest superharmonic function $\widetilde{V}$ which dominates G on E.

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Free-boundary problem

A traditional way is:

Free-boundary problem [3]

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Basic idea: $\widetilde{V}$ and $\mathcal{C}$ (or $\mathcal{D}$) solve the free-boundary problem:

$$\begin{cases} \mathbb{L}_X \widetilde{V} - r\,\widetilde{V} = 0 \quad \text{in} \quad \mathcal{C} \\ \widetilde{V}|_{\partial\mathcal{C}} = G|_{\partial\mathcal{C}} \quad \textit{(continuous fit)} \\ \frac{\partial \widetilde{V}}{\partial x}\Big|_{\partial\mathcal{C}} = \frac{\partial G}{\partial x}\Big|_{\partial\mathcal{C}} \quad \textit{(smooth fit)}^4 \end{cases}$$

where $\mathbb{L}_X$ is the infinitesimal generator of the process X.

[3] PESKIR, G. and SHIRYAEV, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Birkhauser, Basel.

[4] MIKHALEVICH, V. S. (1958). A Bayes test of two hypotheses concerning the mean of a normal process. *Visnik Kiiv. Univ, 1(101-104).*

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Verification Theorem:

The solution of the free-boundary problem coincides with the solution of the initial optimal stopping problem

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The proof is based on the standard verification **key arguments** of the

1. Change-of-variable formula (i.e. Itô's formula and its extensions) from the stochastic calculus
2. Doob's optional sampling theorem from the martingale theory
3. etc...

American Call option pricing problem

Definition (American Call option)

Contracts which entitle the holder to buy an *underlying asset* X "on or before" a future *expiry* or *maturity* T time for a pre-specified amount called *strike price* K.

- The payoff of a perpetual American Call option is therefore

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- Assume that the underlying asset price X follows the **Black & Scholes model**:

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- The price is given by the solution of the following optimal stopping problem:

$$V_*(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-r\,\tau}\, (X_\tau - K)^+ \right]$$

for all $x > 0$.

Free-boundary problem for American Call option

- It follows from the general theory of optimal stopping problems for Markov processes, that the optimal stopping time is given by

$$\tau_* = \inf\{t \geq 0 \,|\, V_*(X_t) = X_t - K\} = \inf\{t \geq 0 \,|\, X_t \geq b_*\}$$

for some $b_* > 0$ to be specified.

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- The equivalent free-boundary problem for $V_*(x)$ and b_* is given by

$$(r-\delta)\, x\, V'(x) + \frac{\sigma^2}{2} x^2\, V''(x) = r V(x) \quad \text{for} \quad x < b$$

$$V(x)\big|_{x=0+} = 0 \qquad \textit{(natural boundary)}$$

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- *SOLUTION:* The value function $V_*(x)$ and the optimal exercise boundary b_* are given by

$$V_*(x) = \begin{cases} \frac{b_*}{\gamma_1}\left(\frac{x}{b_*}\right)^{\gamma_1} & , \quad \text{for } 0 < x < b_* \\ x - K & , \quad \text{for } x \geq b_* \end{cases} \quad , \text{ with } \quad b_* = \frac{\gamma_1 K}{\gamma_1 - 1}$$

for a known $\gamma_1 > 1$.

Sketch of Verification theorem

- Applying the local time-space formula to the solution $e^{-rt}\,V(X_t)$ combined with the smooth-fit condition and $P(X_t = b_*) = 0$, we get

$$e^{-rt}\,V(X_t) = V(x) + \int_0^t e^{-ru}\Big((r-\delta)\,X_u\,V' + \frac{\sigma^2}{2}\,X_u^2\,V'' - rV\Big)(X_u)\,du + M_t$$

with $M = (M_t)_{t\geq 0}$ defined by

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- It follows from the free-boundary conditions for V with b_* that

$$\Big((r-\delta)\,x\,V' + \frac{\sigma^2}{2}\,x^2\,V'' - rV\Big)(x) \leq 0 \quad \text{and} \quad V(x) \geq (x-K)^+$$

  hold for all $x > 0$ and thus, for any stopping time τ of X started at $x > 0$ we have

$$e^{-r(t\wedge\tau)}\,(X_{t\wedge\tau} - K)^+ \leq e^{-r(t\wedge\tau)}\,V(X_{t\wedge\tau}) \leq V(x) + M_{t\wedge\tau}$$

Verification theorem

- Taking the expectation with respect to $\mathbb{P}_x$ (using Doob's optional sampling theorem) and letting t go to infinity (using Fatou's lemma), we obtain

$$\mathbb{E}_x\left[e^{-r\tau}\,(X_\tau - K)^+\right] \le V(x) \quad \text{for any stopping time } \tau, \ \ \forall\, x > 0.$$

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- By virtue of the structure of τ_*, we observe that
  - If $x \geq b_*$, then we have equality in the above for τ_* instead of τ.

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  - If $x < b_*$, then using the conditions of the free-boundary problem satisfied by V in the continuation region:

$$e^{-r(t\wedge\tau_*)}\,V(X_{t\wedge\tau_*}) = V(x) + M_{t\wedge\tau_*}$$

    Taking the expectation with respect to $\mathbb{P}_x$ (using Doob's optional sampling theorem) and letting t go to infinity (using Lebesgue dominated convergence), we obtain

$$\mathbb{E}_x\left[e^{-r\tau_*}(X_{\tau_*} - K)^+\right] = \mathbb{E}_x\left[e^{-r\tau_*}\,V(X_{\tau_*})\right] = V(x) \quad \forall\ x < b_*$$

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  - Russian option [5] payoff $G(X_t, S_t) = S_t$
  - American lookback option [6] [7]
    - with fixed strike payoff $G(X_t, S_t) = (S_t - K)^+$
    - with floating strike payoff $G(X_t, S_t) = (S_t - K\,X_t)^+$

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[5] SHEPP, L., & SHIRYAEV, A. N. (1993). The Russian option: reduced regret. *The Annals of Applied Probability* **3**(3) (631–640).

[6] PEDERSEN, J. L. (2000). Discounted optimal stopping problems for the maximum process. *Journal of Applied Probability* **37**(4) (972–983).

[7] GUO, X., & SHEPP, L. (2001). Some optimal stopping problems with nontrivial boundaries for pricing exotic options. *Journal of Applied Probability* **38**(3) (647–658).

Other optimal stopping problems

- *Other optimal stop. prob. for running maxima of diffusion processes:* Jacka (1991), Dubins, Shepp, and Shiryaev (1993), and Graversen and Peskir (1998), Peskir (1998), Shepp and Shiryaev (1994), Dai (2001), Guo and Zervos (2010), Glover, Hulley, and Peskir (2013), among others.

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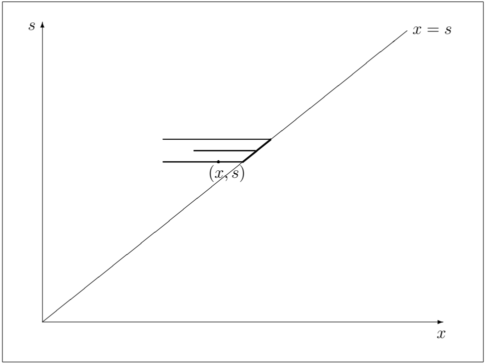

Figure 1. A computer drawing of the state space of the process (X, S)

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where

- $\delta(s)$, $\sigma(s) > 0$ are continuously differentiable bounded functions on $[0, \infty]^2$.
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Main AIM:

Optimal stopping problem for the valuation of an American Call option:

$$V_*(x, s) = \sup_{\tau} E_{x,s}\left[e^{-r\tau}\,(X_\tau - K)^+\right]$$

- supremum over all stopping times τ with respect to the natural filtration of X,
- $E_{x,s}$ denotes the expectation under the assumption that (X, S) starts at (x, s).

The optimal stopping problem in (X, S)-setting

Main purpose: Solve the optimal stopping problem

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$\Rightarrow$ Optimal stopping time is

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The free-boundary problem for Call option

From the general theory of optimal stopping problems for Markov processes we formulate the equivalent free-boundary problem

$$(r-\delta(s))\,x\,\partial_x V(x,s) + \frac{\sigma^2(s)}{2}\,x^2\,\partial_{xx} V(x,s) = rV(x,s) \quad \text{for} \quad 0 < x < b(s) \wedge s$$

$$V(x,s)\big|_{x=0+} = 0 \qquad \textit{(natural boundary)}$$

$$V(x,s)\big|_{x=b(s)-} = b(s) - K \qquad \textit{(continuous fit)}$$

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when $b_*(s) \leq s$ holds.

The free-boundary problem for Call option

From the general theory of optimal stopping problems for Markov processes we formulate the equivalent free-boundary problem

$$(r-\delta(s))\,x\,\partial_x V(x,s) + \frac{\sigma^2(s)}{2}\,x^2\,\partial_{xx} V(x,s) = rV(x,s) \quad \text{for} \quad 0 < x < b(s) \wedge s$$

$$V(x,s)\big|_{x=0+} = 0 \qquad \textit{(natural boundary)}$$

$$V(x,s)\big|_{x=b(s)-} = b(s) - K \qquad \textit{(continuous fit)}$$

$$\partial_x V(x,s)\big|_{x=b(s)-} = 1 \qquad \textit{(smooth fit)}$$

when $b_*(s) \leq s$ holds.

Otherwise, when $s < b_*(s)$ holds, we have

$$V(x,s)\big|_{x=0+} = 0 \qquad \textit{(natural boundary)}$$

$$\partial_s V(x,s)\big|_{x=s-} = 0 \qquad \textit{(normal reflection)}^{8}$$

[8] DUBINS, L., SHEPP, L. A. and SHIRYAEV, A. N. (1993). Optimal stopping rules and maximal inequalities for Bessel processes. *Theory of Probability and Applications* **38**(2) (226–261).

State space of the process (X, S)

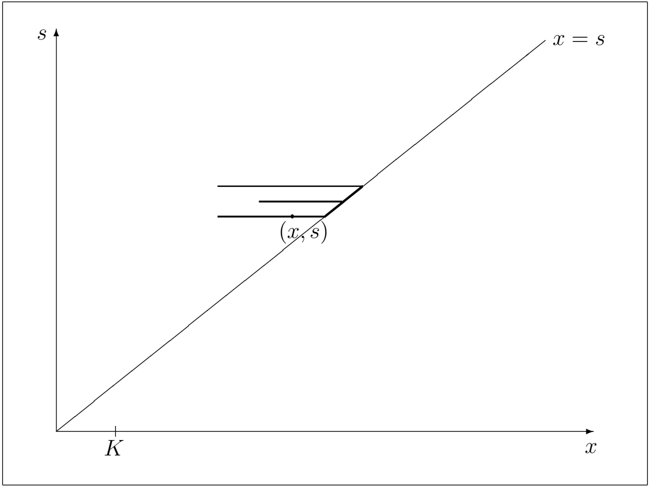

Figure 2. A computer drawing of the state space of the process (X, S)

Free-boundary function for Call option

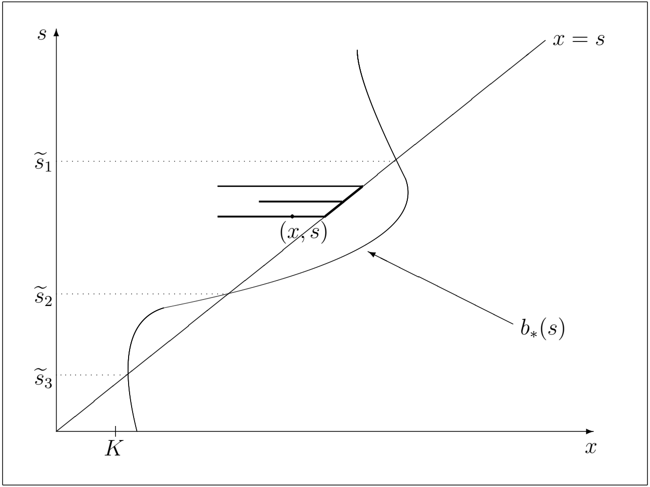

Figure 3. A computer drawing of the state space of the process (X, S) and the boundary function $b_*(s)$.

Solution of the free-boundary problem for Call

The value function takes the form

$$V(x, s; b_*(s)) = \frac{b_*(s)}{\beta_1(s)}\left(\frac{x}{b_*(s)}\right)^{\beta_1(s)} \quad \text{with} \quad b_*(s) = \frac{\beta_1(s)K}{\beta_1(s)-1}$$

for $0 < x < b_*(s) \leq s$ and $s > K$, where

$$\beta_1(s) = \frac{1}{2} - \frac{r-\delta(s)}{\sigma^2(s)} + \sqrt{\left(\frac{1}{2} - \frac{r-\delta(s)}{\sigma^2(s)}\right)^2 + \frac{2r}{\sigma^2(s)}} \; > \; 1\,,$$

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and the form

$$V(x, s; \widetilde{s}_{2k-1}) = \exp\left(-\int_s^{\widetilde{s}_{2k-1}} \beta_1'(q) \ln q \, dq \right) \frac{\widetilde{s}_{2k-1}^{\,1-\beta_1(\widetilde{s}_{2k-1})}}{\beta_1\big(\widetilde{s}_{2k-1}\big)} \, x^{\beta_1(s)}$$

for $0 < x < s < b_*(s)$.

Free-boundary function for Call option

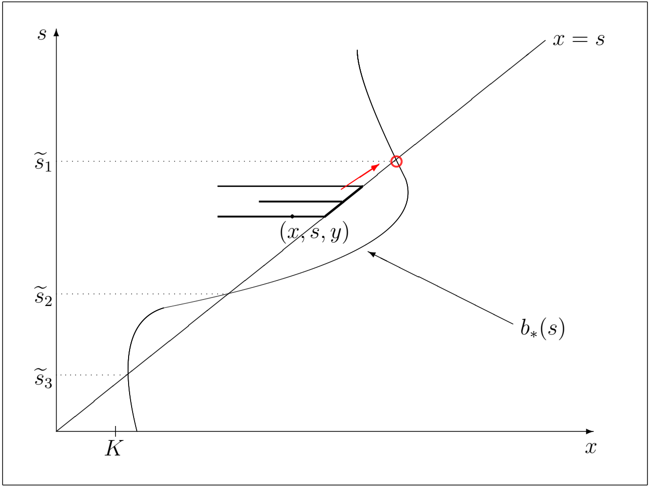

Figure 3. A computer drawing of the state space of the process (X, S) and the boundary function $b_*(s)$.

Main Result of the Call option in the (X,S)-setting

Theorem

The value function of the opt. stop. prob. for Call option is given by

$$V_*(x,s) = \begin{cases} V(x,s;b_*(s)), & \text{if} \quad 0 < x < b_*(s) \le s \\ V(x,s;\widetilde{s}), & \text{if} \quad 0 < x \le s < b_*(s) \\ x - K, & \text{if} \quad b_*(s) \le x \le s \end{cases}$$

and the optimal stopping time is

$$\tau_* = \inf\{t \ge 0 \,|\, X_t \ge b_*(S_t)\}$$

where

$$b_*(s) = \frac{\beta_1(s)K}{\beta_1(s) - 1}$$

Diffusion-type model

$(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$.

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Asset price process $X = (X_t)_{t \geq 0}$ with dynamics

$$dX_t = (r - \delta(S_t, Y_t))\, X_t\, dt + \sigma(S_t, Y_t)\, X_t\, dB_t \quad (X_0 = x > 0)$$

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- *running maximum* process $S = (S_t)_{t \geq 0}$ defined by

$$S_t = \max_{0 \leq u \leq t} X_u \vee s$$

- *running maximum drawdown* process $Y = (Y_t)_{t \geq 0}$ defined by

$$Y_t = \max_{0 \leq u \leq t} (S_u - X_u) \vee y$$

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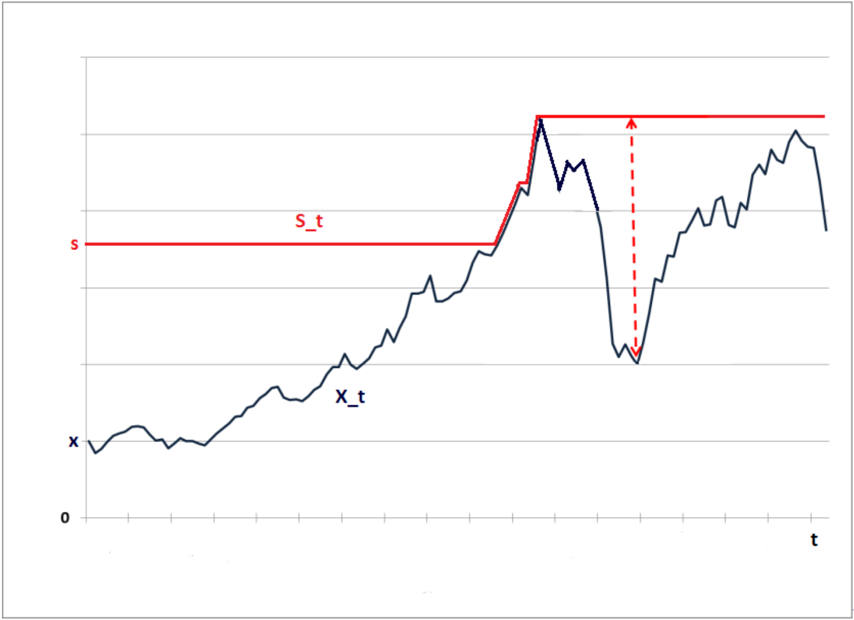
- 3-D state space of the process (X, S, Y): $0 < s - y \leq x \leq s$.

Processes X, S and Y



State space of the process (X, S, Y)

Recall that: $0 < s - y \leq x \leq s$.

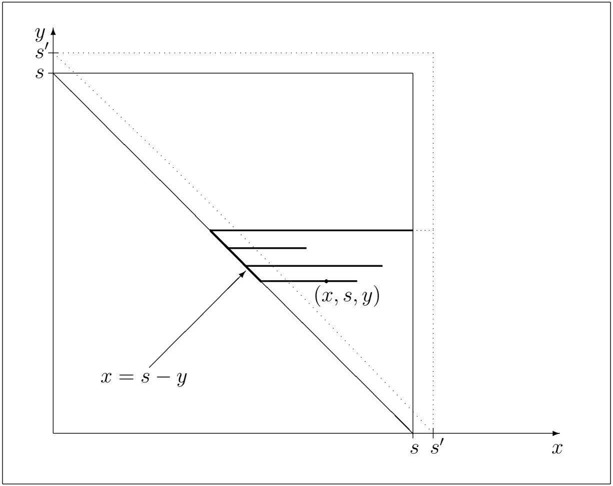


Figure 1. A computer drawing of the state space of the process (X, S, Y), for some s fixed, which increases to s'.

Optimal stopping problem for the process (X, S, Y)

- Main AIM:
  Optimal stopping problem for the valuation of an American Call option for the process (X, S, Y): [9]

$$V_*(x,s,y) = \sup_{\tau} E_{x,s,y}\left[e^{-r\tau}\left(X_\tau - K\right)^+\right]$$

  - supremum over all stopping times τ wrt to the natural filtration of X
  - $E_{x,s,y}$ denotes the expectation under the assumption that (X, S, Y) starts at (x, s, y).

[9] GAPEEV P.V. & RODOSTHENOUS N. (2014) Optimal stopping problems in diffusion-type models with running maxima and drawdowns. *Journal of Applied Probability* **51**(3) (799–817).

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- *Other problems for the drawdown process:*
  Pospisil, Vecer, and Hadjiliadis (2009), Mijatovic and Pistorius (2012), Zhang (2015), among others.

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The optimal stopping problem in the (X,S,Y)-setting

$$V_*(x,s,y) = \sup_{\tau} E_{x,s,y}\left[e^{-r\tau}\,(X_\tau - K)^+\right]$$

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General theory of optimal stopping for Markov processes

$\Rightarrow$ Optimal stopping time is

$$\tau_* = \inf\{t \geq 0 \,|\, V_*(X_t, S_t, Y_t) = X_t - K\}$$

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$$\tau_* = \inf\{t \geq 0 \,|\, X_t \geq b_*(S_t, Y_t)\}$$

for some function $K < b_*(s,y)$ to be determined.

The optimal stopping problem in the (X,S,Y)-setting

$$V_*(x,s,y) = \sup_{\tau} E_{x,s,y}\left[e^{-r\tau}\left(X_\tau - K\right)^+\right]$$

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The free-boundary problem

From the general theory of optimal stopping problems for Markov processes we formulate the equivalent free-boundary problem

$$(r - \delta(s,y))\, x\, \partial_x V(x,s,y) + \frac{\sigma^2(s,y)}{2}\, x^2\, \partial_{xx} V(x,s,y) = rV(x,s,y)$$

such that $s - y < x < b(s,y) \wedge s$

$$\partial_y V(x,s,y)\big|_{x=(s-y)+} = 0 \qquad \textit{(normal reflection)}$$

$$V(x,s,y)\big|_{x=b(s,y)-} = b(s,y) - K \qquad \textit{(continuous fit)}$$

$$\partial_x V(x,s,y)\big|_{x=b(s,y)-} = 1 \qquad \textit{(smooth fit)}$$

when $b_*(s,y) \le s$ holds.

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Call option in the (X,S,Y)-setting

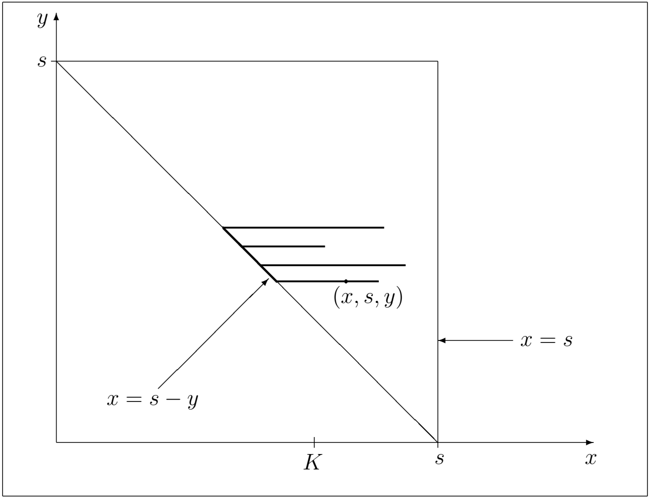

Figure 2. A computer drawing of the state space of the process (X, S, Y), for some s fixed.

Call option in the (X,S,Y)-setting

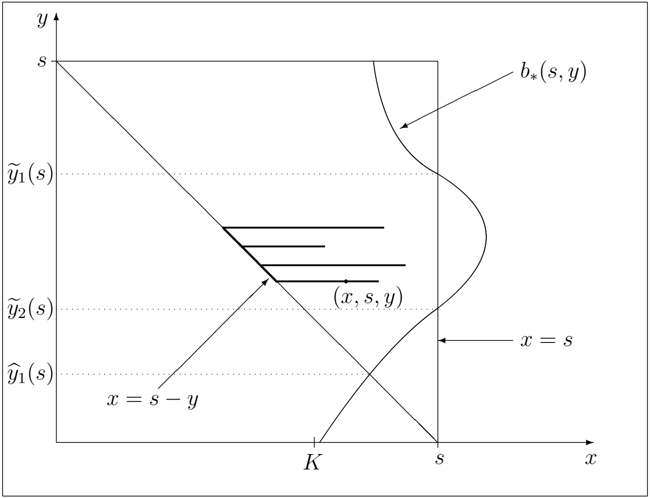

Figure 3. A computer drawing of the state space of the process (X, S, Y), for some s fixed, and the boundary function $b_*(s, y)$.

Call option in the (X,S,Y)-setting

$$V(x,s,y;b_*(s,y)) = \sum_{i=1}^{2} \frac{(\gamma_{3-i}(s,y)-1)b_*(s,y) - \gamma_{3-i}(s,y)K}{(\gamma_{3-i}(s,y)-\gamma_i(s,y))b_*(s,y)^{\gamma_i(s,y)}} \, x^{\gamma_i(s,y)}$$

for $0 < s - y \leq x < b_*(s,y) \leq s$ and $s > K$, for

$$\gamma_i(s,y) = \frac{1}{2} - \frac{r - \delta(s,y)}{\sigma^2(s,y)} - (-1)^i \sqrt{\left(\frac{1}{2} - \frac{r-\delta(s,y)}{\sigma^2(s,y)}\right)^2 + \frac{2r}{\sigma^2(s,y)}}$$

and $0 < y < s$.

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and $0 < y < s$. The boundary $b_*(s,y)$ solves

$$\partial_y b(s,y) = \sum_{i=1}^{2} \frac{((\gamma_{3-i}(s,y)-1)b(s,y)-\gamma_{3-i}(s,y)K)b(s,y)}{(\gamma_i(s,y)-1)(\gamma_{3-i}(s,y)-1)b(s,y)-\gamma_i(s,y)\gamma_{3-i}(s,y)K} \partial_y \gamma_i(s,y)$$

$$\times \left(\frac{1}{\gamma_{3-i}(s,y)-\gamma_i(s,y)} + \frac{((s-y)/b(s,y))^{\gamma_i(s,y)} \ln((s-y)/b(s,y))}{((s-y)/b(s,y))^{\gamma_i(s,y)} - ((s-y)/b(s,y))^{\gamma_{3-i}(s,y)}} \right)$$

Call option in the (X,S,Y)-setting

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$$\times \left(\frac{1}{\gamma_{3-i}(s,y)-\gamma_i(s,y)} + \frac{((s-y)/b(s,y))^{\gamma_i(s,y)} \ln((s-y)/b(s,y))}{((s-y)/b(s,y))^{\gamma_i(s,y)} - ((s-y)/b(s,y))^{\gamma_{3-i}(s,y)}} \right)$$

The starting condition is given by

$$b_*(s,s-) = b_*(s) = \frac{\beta_1(s)K}{\beta_1(s)-1} \quad \text{as } y \uparrow s.$$

Call option in the (X,S,Y)-setting

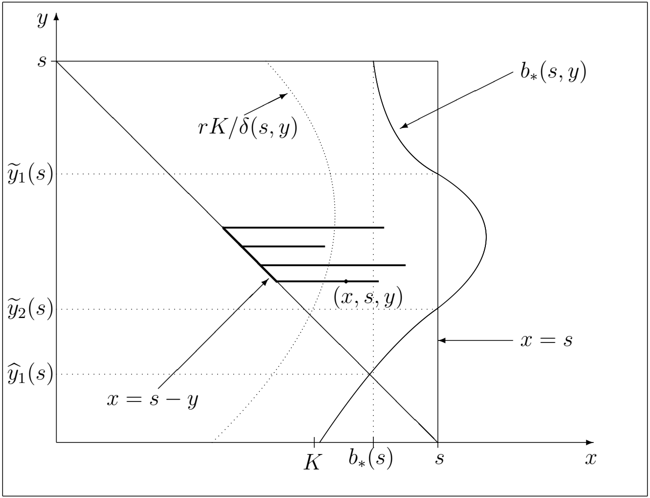

Figure 4. A computer drawing of the state space of the process (X, S, Y), for some s fixed, and the boundary function $b_*(s, y)$.

Call option in the (X,S,Y)-setting

$$V(x,s,y) = C_1(s,y)\, x^{\gamma_1(s,y)} + C_2(s,y)\, x^{\gamma_2(s,y)}$$

for $0 < s - y \le x \le s < b_*(s,y)$, where $C_1(s,y)$ and $C_2(s,y)$ solve

$$\sum_{i=1}^{2} \Big(\partial_s C_i(s,y)\, s^{\gamma_i(s,y)} + C_i(s,y)\, \partial_s \gamma_i(s,y)\, s^{\gamma_i(s,y)} \ln s \Big) = 0$$

$$\sum_{i=1}^{2} \Big(\partial_y C_i(s,y)\, (s-y)^{\gamma_i(s,y)} + C_i(s,y)\, \partial_y \gamma_i(s,y)\, (s-y)^{\gamma_i(s,y)} \ln(s-y) \Big) = 0$$

for each $0 < y < s$, with boundary conditions...

Call option in the (X,S,Y)-setting

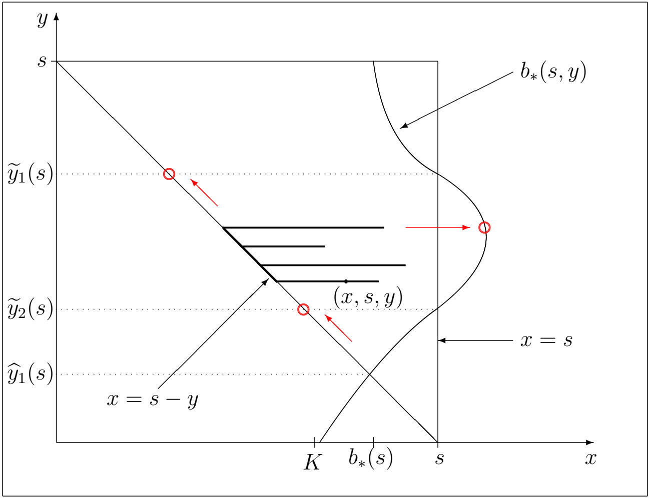

Figure 6. A computer drawing of the state space of the process (X, S, Y), for some s fixed, and the boundary function $b_*(s, y)$.

Call option in the (X,S,Y)-setting

with boundary conditions...

$$C_1(s,\widetilde{y}_{2l}(s)+)\,((s-\widetilde{y}_{2l}(s))-)^{\gamma_1(s,\widetilde{y}_{2l}(s)+)}$$
$$+\,C_2(s,\widetilde{y}_{2l}(s)+)\,((s-\widetilde{y}_{2l}(s))-)^{\gamma_2(s,\widetilde{y}_{2l}(s)+)}$$
$$=V(s-\widetilde{y}_{2l}(s),s,\widetilde{y}_{2l}(s);b(s,\widetilde{y}_{2l}(s)))\ ,$$

$$C_1(s,\widetilde{y}_{2l-1}(s))\,(s-\widetilde{y}_{2l-1}(s))^{\gamma_1(s,\widetilde{y}_{2l-1}(s))}$$
$$+\,C_2(s,\widetilde{y}_{2l-1}(s))\,(s-\widetilde{y}_{2l-1}(s))^{\gamma_2(s,\widetilde{y}_{2l-1}(s))}$$
$$=V((s-\widetilde{y}_{2l-1}(s))-,s,\widetilde{y}_{2l-1}(s)+;b_*(s,\widetilde{y}_{2l-1}(s)+))$$

and

$$C_1(\bar{s}(y)-,y)\,(\bar{s}(y)-)^{\gamma_1(\bar{s}(y)-,y)}+C_2(\bar{s}(y)-,y)\,(\bar{s}(y)-)^{\gamma_2(\bar{s}(y)-,y)}$$
$$=V(\bar{s}(y),\bar{s}(y),y;\ b_*(\bar{s}(y),y))\equiv\bar{s}(y)-K$$

Main Result of the Call option in (X,S,Y)-setting

Proposition 2. *The value function of the opt. stop. prob. for Call option is*

$$V_*(x,s,y) = \begin{cases} V(x,s,y;b_*(s,y)), & \text{if} \quad s-y \le x < b_*(s,y) \le s \\ V(x,s,y), & \text{if} \quad s-y \le x \le s < b_*(s,y) \\ x-K, & \text{if} \quad b_*(s,y) \le x \le s \end{cases}$$

and the optimal stopping time is

$$\tau_* = \inf\{t \ge 0 \,|\, X_t \ge b_*(S_t, Y_t)\}$$

where $b_*(s,y)$ is the minimal solution of an ODE with starting condition $b_*(s,s-) = b_*(s)$, such that it stays strictly above the surface $x = K \vee (rK/\delta(s,y))$.

Thank you!