Dynamics in systems with constraints

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M $(\dim M = n)$ is the configurational space,

$x = (x_1, \dots, x_n)$ are the local coordinates,

$\dot{x} = (\dot{x}_1, \dots, \dot{x}_n)$ are the velocities,

$\Gamma = TM$ is the phase space,

$T(\dot{x}, x, t)$ is the kinetic energy, $\det\left\| \dfrac{\partial^2 T}{\partial \dot{x}^2} \right\| \neq 0$,

$F = (F_1, \dots, F_n)$ are forces (elements of T^*M).

Equations of motion (Lagrangian equations): $[T] = F$,

$[f] = \dfrac{d}{dt}\dfrac{\partial f}{\partial \dot{x}} - \dfrac{\partial f}{\partial x}$ are Lagrangian derivatives.

Equation of the constraint: $\Phi(\dot{x}, x, t) = 0$, $\dfrac{\partial \Phi}{\partial \dot{x}} \neq 0$.

Usually (in physical applications) $T = \frac{1}{2}(A(x)\dot{x}, \dot{x})$ is positive definite,

$\Phi = (a(x), \dot{x})$ is linear in $\dot{x}$, $a \neq 0$.

How can we determine the motion of the system with constraint?

There are several possibilities.

Nonholonomic model

$$[\boldsymbol{T}] = \boldsymbol{F} + \boldsymbol{\lambda}\,\frac{\partial \boldsymbol{\Phi}}{\partial \dot{\boldsymbol{x}}}, \quad \boldsymbol{\Phi} = 0 \qquad (\boldsymbol{NH})$$

Basic principles used to obtain NH:

1°. Release from the constraint: $[T] = F + R$, R is a new force, reaction of the constraint (or the constraint force).

2°. Principle of ideality: $(R, \delta x) = 0$, δx are virtual displacements (variations) of the system with constraint.
$(R, \delta x)$ is the work of the force R on the virtual displacement δx.

3°. Definition of variations (in the state x, $\dot{x}$ at the time moment t):

$$\left(\frac{\partial \Phi}{\partial \dot{x}}, \delta x\right) = 0. \qquad (*)$$

Equivalent principles

A. The d'Alembert–Lagrange principle:

$$([T] - F, \delta x) = 0 \quad \text{for all } \delta x \text{ which satisfy } (*).$$

B. The generalized Hamilton's principle:

$$\delta \int_{t_1}^{t_2} T\,dt + \int_{t_1}^{t_2} (F, \delta x)\,dt = 0$$

for all variations $\delta x(t)$ $(\delta x(t_1) = \delta x(t_2) = 0)$ which satisfy the equation

$$\left(\frac{\partial \Phi}{\partial \dot{x}} \Big|_{x(t)}, \delta x(t) \right) = 0. \qquad (*)$$

If $F = -\dfrac{\partial V}{\partial x}$, then $L = T - V$ is the Lagrangian.

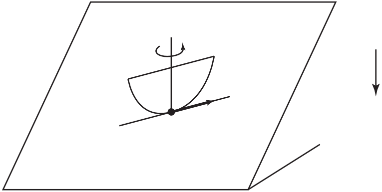
C. The Hölder's principle: an admissible path $t \mapsto x(t)$, $t_1 \leqslant t \leqslant t_2$, is a motion of the given constrained Lagrangian system if and only if it is a critical point (in the sense of Hölder) of the action functional $\displaystyle\int_{t_1}^{t_2} L\,dt$.

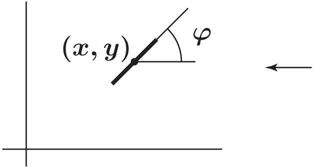
Evidently equation $(*)$ is not equivalent to the equation

$$\delta \Phi = \left(\frac{\partial \Phi}{\partial \dot{x}}, \delta \dot{x} \right) + \left(\frac{\partial \Phi}{\partial x}, \delta x \right) = 0$$

(in general case).

Example: the Chaplygin skate (sleigh)





$$\Phi = \dot{x}\sin\varphi - \dot{y}\cos\varphi = 0$$

$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2) + x$ (for an appropriate choice of mass, length and time units)

$$t = 0: \; x = y = 0, \; \dot{x} = \dot{y} = 0, \; \varphi = 0, \; \dot{\varphi} = \omega$$

$$x = \frac{\sin^2 \omega t}{2\omega^2}, \quad y = \frac{1}{2\omega^2}\left(\omega t - \frac{1}{2}\sin 2\omega t\right), \quad \varphi = \omega t.$$

On the average the skate does not slide down the inclined plane!

Anisotropic friction

$$[L] = -\frac{\partial R}{\partial \dot{x}}, \tag{1}$$

$L = \frac{1}{2}(A(x)\dot{x}, \dot{x}) - V(x)$, R is a Rayleigh's function—a nonnegative definite quadratic form in velocities,

$$(T + V)^{\cdot} = -2R \leqslant 0.$$

Let $R_N = \dfrac{N}{2}(a(x), \dot{x})^2$, where $N > 0$, $a \neq 0$ is a covector field.

THEOREM 1. *Let* $x_N(t)$, $t \geqslant 0$, *be a solution of equation* (1) *with an initial condition which does not depend on* N *and satisfies the equation* $(a(x), \dot{x}) = 0$. *Then the limit*

$$\lim_{N\to\infty} x_N(t) = \widehat{x}(t)$$

exists on every bounded time interval $0 \leqslant t \leqslant t_0$. *The limiting function satisfies the system of nonholonomic equations*

$$[L] = \lambda a, \qquad (a, \dot{x}) = 0.$$

C. Carathéodory. ***Z. Angew. Math. Mech.*** **1933. V. 13. P. 71–76.**

V. N. Brendelev. ***J. Appl. Math. Mech.*** **1982. V. 45. no. 3. P. 351–355.**

A. V. Karapetyan. ***J. Appl. Math. Mech.*** **1982. V. 45. no. 1. P. 30–36.**

Servo constraints (by H. Beghin)

Let us consider the free system and suppose that we want to realize the motion with constraint $\Phi = 0$ by using control forces λM. Here M is a fixed covector field (for example, a force field), and $\lambda = \lambda(t)$ is a function to be determined below. We have

$$[T] = F + \lambda M, \qquad \Phi = 0.$$

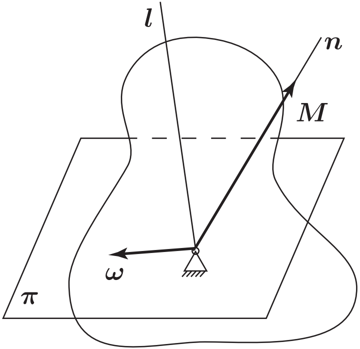
The condition of realizability of the constraint $\Phi = 0$ is

$$\left(A^{-1}M, \frac{\partial \Phi}{\partial \dot{x}}\right) \neq 0, \qquad A = \frac{\partial^2 T}{\partial \dot{x}^2}. \qquad (*)$$

Of course, the condition $(*)$ holds if $M = \dfrac{\partial \Phi}{\partial \dot{x}}$ $(\neq 0)$.

Example. **The axes l and n are fixed in the rigid body.**

Constraint: $\omega \perp l$.



M **is the control torque,**
ω **is an angular velocity,**
$l \perp \pi$

Condition $(*)$ is violated if $l \perp n$. If $l = n$ then we obtain the nonholonomic motion of the rigid body with constraint $(\omega, l) = 0$ (Suslov's problem).

Vakonomic (variational axiomatic kind) model

$$\delta \int_{t_1}^{t_2} T\, dt + \int_{t_1}^{t_2} (F, \delta x)\, dt = 0$$

for all $\delta x(t)$, $t_1 \leqslant t \leqslant t_2$ $(\delta x(t_1) = \delta x(t_2) = 0)$, and

$$\delta \Phi = \left(\frac{\partial \Phi}{\partial \dot{x}}, \delta \dot{x} \right) + \left(\frac{\partial \Phi}{\partial x}, \delta x \right) = 0.$$

Equations of motion:

$$[T] = F + \lambda [\Phi] + \dot{\lambda}\, \frac{\partial \Phi}{\partial \dot{x}}, \qquad \Phi = 0.$$

If $F = -\dfrac{\partial V}{\partial x}$ then any motion of vakonomic system is a solution of the Lagrange variational problem.

Example. **The vakonomic skate:**

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2) + x,$$

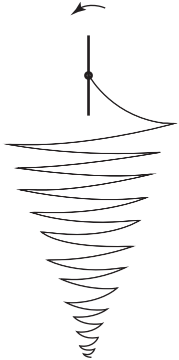
$$\Phi = \dot{x}\sin\varphi - \dot{y}\cos\varphi.$$

Let $\varphi = 0$ and $\dot{y} + \lambda\cos\varphi = 0$ for $t = 0$. Then

$$\ddot{\varphi} = t^2 \sin\varphi\cos\varphi,$$

$$\dot{x} = t\cos^2\varphi \ (>0 \text{ for } t > 0),$$

$$\dot{y} = t\sin\varphi\cos\varphi$$



Added masses

$$L_N = \frac{1}{2}(A(x)\dot{x}, \dot{x}) + \frac{N}{2}(a(x), \dot{x})^2 - V(q), \quad N > 0.$$

Let $x_N(t)$, $t > 0$, denote the motion of such a system with initial condition x_0, $\dot{x}_0$ satisfying $(a(x_0), \dot{x}_0) = 0$.

THEOREM 2. *The limit*

$$\lim_{N\to\infty} x_N(t) = \widehat{x}(t)$$

exists on every bounded time interval $0 \leqslant t \leqslant t_0$.

The limit $t \mapsto \widehat{x}(t)$ *is an extremal of Lagrange's variational problem*

$$\delta \int_{t_1}^{t_2} L_0 \, dt = 0, \quad L_0 = \frac{1}{2}(A\dot{x}, \dot{x}) - V,$$

under the linear constraint $(a, \dot{x}) = 0$.

V. V. Kozlov. I–V. ***Mosc. Univ. Mech. Bull.***
37, no. 3–4, p. 27–34 (1982); 37, no. 3–4, p. 74–80 (1982);
38, no. 3, p. 40–51 (1983); 42, no. 5, p. 40–49 (1987);
43, no. 6, p. 23–29 (1988).

Theorem 2 is connected with the method of penalty functions.

Servo constraints

Consider the motion of the system with kinetic energy

$$\boldsymbol{T_* = T + \lambda N},$$

where $\boldsymbol{N}$ is some function of $\boldsymbol{x}$, $\boldsymbol{\dot{x}}$ and $\boldsymbol{t}$, and $\boldsymbol{\lambda = \lambda(t)}$ a function to be determined.

Equations of motion:

$$\boldsymbol{[T_*] = F, \quad \Phi = 0}$$

or

$$\boldsymbol{[T] = F - \dot{\lambda}\,\frac{\partial N}{\partial \dot{x}} - \lambda[N], \quad \Phi = 0.} \qquad (*)$$

The condition of solvability of this system w.r.t. $\boldsymbol{\ddot{x}}$ and $\boldsymbol{\dot{\lambda}}$ is

$$\boldsymbol{\left(A^{-1}\frac{\partial N}{\partial \dot{x}}, \frac{\partial \Phi}{\partial \dot{x}}\right) \neq 0.} \qquad (**)$$

Let $N = \Phi$. If $\dfrac{\partial \Phi}{\partial \dot{x}} \neq 0$ then condition $(**)$ holds. In this case we obtain the vakonomic motion:

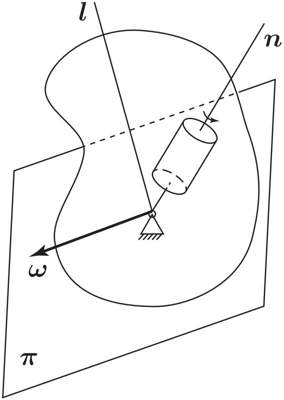
$$\delta \int_{t_1}^{t_2} T\,dt + \int_{t_1}^{t_2} (F, \delta x)\,dt = 0, \quad \Phi = 0$$

for all variations δx with fixed extremities and

$$\delta\Phi = \left(\frac{\partial \Phi}{\partial \dot{x}}, (\delta x)^{\cdot}\right) + \left(\frac{\partial \Phi}{\partial x}, \delta x\right) = 0.$$

See: V. V. Kozlov. *Mosc. Univ. Mech. Bull.* 1989. № 5. P. 59–66.

Example.



Control angular momentum

$$l \perp \pi$$

Condition $(**)$ is violated if $l \perp n$. If $l = n$ then we obtain the vakonomic motion of the rigid body with constraint $(\omega, l) = 0$. For unique determination of the body motion we must know the initial velocity of the gyroscope as well.

$(*) \implies [T] = F + R, \quad R = -\dot{\lambda}\,\frac{\partial N}{\partial \dot{x}} - \lambda[N]$ **is "the reaction of constraint".**

This is a version of a release from the constraint.

$$\int_{t_1}^{t_2} (R, \delta x)\, dt = \int_{t_1}^{t_2} (\lambda\, \delta N)\, dt = 0 \quad \text{if} \tag{1}$$

$$\delta N = \left(\frac{\partial N}{\partial \dot{x}}, \delta \dot{x}\right) + \left(\frac{\partial N}{\partial x}, \delta x\right) = 0. \tag{2}$$

(1) is the principle of ideality, and (2) is the definition of variations.

Системы на группах Ли

G — n-мерная группа Ли, g — ее алгебра Ли.

Уравнения Эйлера–Пуанкаре: $\dot m_k = \sum c^i_{jk} m_i \omega_j$, $1 \leqslant k \leqslant n$,

$\omega_1, \ldots, \omega_n$ — квазискорости (декартовы координаты на g)

$m_1, \ldots, m_n$ — импульсы (координаты на g^*),

$m_k = \sum I_{kj}\omega_j$, $I = \|I_{kj}\|$ — тензор инерции,

$c^i_{jk} = -c^i_{kj}$ — структурные постоянные алгебры Ли.

$T = \dfrac{1}{2}(I\omega, \omega) = \dfrac{1}{2}(I^{-1}m, m)$

— кинетическая энергия — первый интеграл.

Пусть $\Phi = (a, \omega) = \sum a_i \omega_i = 0$ — левоинвариантная связь,

$\lambda b = (\lambda b_1, \ldots, \lambda b_n)$ — управляющая сила, $b \in g^* \setminus \{0\}$.

Уравнения движения:

$$\dot m_k = \sum c^k_{ji} m_i \omega_j + \lambda b_k \quad (1 \leqslant k \leqslant n), \qquad \sum a_i \omega_i = 0.$$

Условие реализуемости связи: $(I^{-1}a, b) \neq 0$.

Общие свойства приведенных систем

$$\dot{x}_j = v_j(x_1, \ldots, x_p), \qquad 1 \leqslant j \leqslant p$$

v_j — однородные квадратичные многочлены.

• Если $t \mapsto x(t)$ — решение, то $t \mapsto -x(-t)$ — решение.

• Если $v(x) \neq 0$ при $x \neq 0$, то имеется инвариантная прямая $x = \lambda\xi$, $\xi \in \mathbb{R}^p \setminus \{0\}$ и $\dot{\lambda} = c\lambda^2$, $c \neq 0$.

• Если a и b коллинеарны, то $x = 0$ устойчиво и имеется прямая, целиком состоящая из положений равновесия.

• Пусть снова $v(x) \neq 0$ при $x = 0$. Если количество инвариантных прямых (считая кратности) конечно, то оно нечетно и не превосходит $2^p - 1$.

• Пусть $p = 2$ и имеется прямая $ax_1 + bx_2 = 0$ $(a^2 + b^2 \neq 0)$, состоящая из положений равновесия. Тогда заменой времени

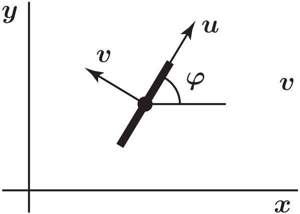
$$d\tau = (ax_1 + bx_2)\, dt$$

она приводится к линейной системе.

• Пусть $p = 2$ и имеется невырожденный квадратичный первый интеграл. Тогда система имеет прямую равновесных состояний.

Сервосани Чаплыгина

$$G = E(2)$$



$$v = 0$$

$$T = \frac{m}{2}(u^2 + v^2) + \frac{I\omega^2}{2}$$

— кинетическая энергия,

$$m\dot{u} = \lambda a,\ 0 = -mu\omega + \lambda b,\ I\dot{\omega} = \lambda c$$

— уравнения движения,

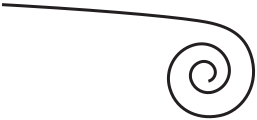
$$\dot{u} = \alpha u\omega,\ \dot{\omega} = \beta u\omega;$$

$$\alpha = \frac{a}{b}, \quad \beta = \frac{mc}{Ib},$$

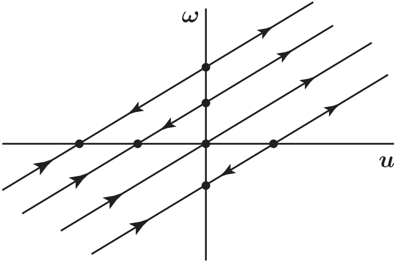
$\beta u - \alpha\omega = \gamma = \mathrm{const}$ — интеграл

$$\dot{u} = u(\beta u - \gamma),$$

$$u(t) = \frac{\gamma}{\beta + e^{\gamma t}}$$



Траектория саней на плоскости



Фазовый портрет

Сервосвязи второго рода

$$(m_k + \lambda b_k)^{\cdot} = \sum c^i_{jk}(m_i + \lambda b_i)\omega_j \quad (1 \leqslant k \leqslant n), \quad \sum a_i \omega_i = 0$$

— уравнения на алгебре Ли $g = \{\omega\}$.

$T^* = \dfrac{1}{2}(I\omega, \omega) + \lambda(b, \omega)$ — измененная кинетическая энергия

$(I^{-1}a, b) \neq 0$ — условие реализации связи

$\{m_i, m_j\} = \sum c^k_{ij} m_k$ — скобка Ли–Пуассона

$\{m_i, F\} = \sum \dfrac{\partial F}{\partial m_j} c^k_{ij} m_k = 0, \quad F$ — функция Казимира.

- Если $F(m_1, \ldots, m_n)$ — функция Казимира, то $F(m_1 + \lambda b_1, \ldots, m_n + \lambda b_n)$ — первый интеграл.

Если a и b коллинеарны, то имеем ваконовные уравнения движения:

$$\delta \int_{t_1}^{t_2} \frac{1}{2}(I\omega, \omega)\, dt = 0, \qquad (a, \omega) = 0.$$

Твердое тело с левоинвариантной сервосвязью

$a = (0,0,1) \;\Rightarrow\; \omega_3 = 0$ — уравнение связи

$b = (0, b_2, b_3)$

Уравнения движения:

$$
\begin{aligned}
&I_{11}\dot\omega_1 + I_{12}\dot\omega_2 + \omega_2(I_{13}\omega_1 + I_{23}\omega_2 + b_3\lambda) = 0,\\
&I_{12}\dot\omega_1 + I_{22}\dot\omega_2 - \omega_1(I_{13}\omega_1 + I_{23}\omega_2 + b_3\lambda) = 0,\\
&I_{13}\dot\omega_1 + I_{23}\dot\omega_2 + b_3\dot\lambda + \omega_1(I_{12}\omega_1 + I_{22}\omega_2 + b_2\lambda)\\
&\qquad\qquad - \omega_2(I_{11}\omega_1 + I_{12}\omega_2) = 0.
\end{aligned}
$$

Условие реализуемости: $b_3(I_{11}I_{22} - I_{12}^2) + b_2(I_{12}I_{13} - I_{11}I_{23}) \neq 0.$

$$(I_{11}\omega_1 + I_{12}\omega_2)^2 + (I_{12}\omega_1 + I_{22}\omega_2 + b_2\lambda)^2 + (I_{13}\omega_1 + I_{23}\omega_2 + b_3\lambda)^2$$

— первый интеграл.

- **При $b_2 = 0$ существует линейная замена переменных**

$$\omega_1, \omega_2, \lambda \ \mapsto \ m_1, m_2, m_3,$$

приводящая уравнения движения к динамическим уравнениям Эйлера

$$\begin{aligned} \dot m_1 &= (J_3 - J_2) m_2 m_3, \\ \dot m_2 &= (J_1 - J_3) m_3 m_1, \\ \dot m_3 &= (J_2 - J_1) m_1 m_2. \end{aligned}$$

Инвариантные многообразия

$$(I\omega + \lambda b)^{\cdot} + \omega \times (I\omega + \lambda b) = 0, \qquad (a, \omega) = 0 \tag{1}$$

$$\dot{\alpha} + \omega \times \alpha = 0, \qquad \dot{\beta} + \omega \times \beta = 0, \qquad \dot{\gamma} + \omega \times \gamma = 0$$

$$(I\omega + \lambda b, \alpha) = c_1, \qquad (I\omega + \lambda b, \beta) = c_2, \qquad (I\omega + \lambda b, \gamma) = c_3$$

$$I\omega + \lambda b = c_1\alpha + c_2\beta + c_3\gamma$$

Пусть $c_1 = c_2 = 0$, $c_3 = k$. Тогда $I\omega + \lambda b = k\gamma$.

$$\lambda = k\,\frac{(a, I^{-1}\gamma)}{(a, I^{-1}b)}, \qquad \omega = kI^{-1}\left(\gamma - \frac{(a, I^{-1}\gamma)}{(a, I^{-1}b)}\,b\right)$$

$$\dot{\gamma} + k\left[I^{-1}\left(\gamma - \frac{(a, I^{-1}\gamma)}{(a, I^{-1}b)}\,b\right)\right] \times \gamma = 0 \tag{2}$$

- **При $k \neq 0$ системы (1) и (2) изоморфны: они переходят друг в друга линейной заменой переменных.**

Если a и b коллинеарны (вакономная модель), то имеется интеграл

$$H = \frac{1}{2}(I\omega, \omega) = \frac{k^2}{2}\left[(\gamma, I^{-1}\gamma)^2 - \frac{(a, I^{-1}\gamma)^2}{(a, I^{-1}a)}\right]$$

- **Сохраняется мера Хаара $\iiint \sin\theta \, d\theta \, d\varphi \, d\psi$.**