Квантовое управление в томографическом представлении

Маркович Любовь

Московский физико технического институт
Государственный университет, Москва, Россия

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Outlook

Problems and Application

- **Problem**
  the steering property (Einstein A., Podolsky Yu. and Rosen N., (1935)) known for two-qubit state in terms of specific inequalities for the correlation function is translated for the state of the single qudit with the spin $j = 3/2$.

- **Method**
  the tomographic probability representation for the qudit states is applied.

- **Application**
  the study of the information and the entropic properties of the superconducting multilevel circuits (Kiktenko et all (2015), Glushkova et all (2015)) where the notion of the artificial two-level atoms playing the role of the qubits is used.

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What is the quantum steering?

The density matrix of the system state in a four-dimensional Hilbert space $\mathcal{H}_{AB}$ is ρ_{AB}, $\rho_{AB} = \rho_{AB}^{\dagger}$, $Tr\,\rho_{AB} = 1$.

Definition

The correlations in such system can be described by the joint probability function

$$P(a,b|A,B) = \int p_{\lambda} P(a|A,\lambda) P(b|B,\lambda) d\lambda.$$

$P(a|A,\lambda)$ - the probability distribution of the measurement outcomes a under setting A for a hidden variable λ.

p_{λ} - the probability distribution of the hidden variable λ, ρ_{λ} - its hidden state (a local hidden state (LHS)).

What is the quantum steering?

Definition

If the following model of the correlation

$$P(a,b|A,B) \quad = \quad \int p_\lambda P(a|A,\lambda) Tr(\widehat{\pi}(b|B)\rho_\lambda^{(b)}) d\lambda$$

does not exist, then the state is steerable (Zukowski et all, (2014)).
$\widehat{\pi}(b|B)$ - projection operator for an observable parameterized by the setting B,
$\rho_\lambda^{(b)}$ -some pure state of the system B.

The correlation function

The quantum correlation function for the two-qubit state is determined by

$$E(\overrightarrow{k_1}, \overrightarrow{k_2}) \quad = \quad Tr(\overrightarrow{k_1} \cdot \overrightarrow{\sigma} \otimes \overrightarrow{k_2} \cdot \overrightarrow{\sigma} \rho),$$

$\overrightarrow{\sigma}$ - the vector built out of the Pauli matrix,
$\overrightarrow{k_1}$, $\overrightarrow{k_2}$ - the unit Bloch vectors of the measurement directions equal to ± 1.

Definition

The EPR steering can be detected through the violation of the steering inequalities.

Definition

The spin tomogram

$$\omega(\mathbf{x}) = \omega(m_1, m_2, u) \quad = \quad < m_1 m_2 | u \rho u^\dagger | m_1 m_2 >$$

is the probability to obtain $m_1 = -j_1, -j_1+1, \ldots, j_1$, $m_2 = -j_2, -j_2+1, \ldots, j_2$, $j_{1,2} = 0, 1/2, 1 \ldots$ as the spin projections on directions given by the unitary matrix u.

u - the unitary rotation matrix of the size $N \times N$, $N = (2j_1+1)(2j_2+1)$. We choose $u = u_1 \otimes u_2$, $u_i = u_i(\theta_i, \varphi_i, \psi_i) = u_i(\vec{n}_i), i = 1, 2$ of irreducible representations of the $SU(2)$ - group.

The conditional probability of projections of spins m_1, m_2 on vectors $\vec{n}_1$, $\vec{n}_2$ on the Bloch sphere

$$\omega(m_1, m_2 | \vec{n}_1, \vec{n}_2) =< m_1 m_2 | u_1(\vec{n}_1) \otimes u_2(\vec{n}_2) \rho u_1^\dagger(\vec{n}_1) \otimes u_2^\dagger(\vec{n}_2) | m_1 m_2 > .$$

The marginal probability distributions of the first and the second qubit are

$$\omega_1(m_1|\vec{n}_1) = \sum_{m_2=-j_2}^{j_2} \omega_1(m_1, m_2|\vec{n}_1), \quad \omega_1(m_2|\vec{n}_2) = \sum_{m_1=-j_1}^{j_1} \omega_1(m_1, m_2|\vec{n}_2).$$

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The LHS Model in the Form of the Tomogram

Definition

The state is called separable if and only if the density operator of the composite system ρ can be written as

$$\rho = \sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)}, \quad \sum_k p_k = 1.$$

The tomogram can be written as

$$\begin{aligned} \omega(m_1, m_2|\vec{n}_1, \vec{n}_2) &= \sum_k p_k \omega_1^{(k)}(m_1|\vec{n}_1)\omega_2^{(k)}(m_2|\vec{n}_2) \\ &= \sum_\lambda p(\lambda)\omega_1(m_1|\vec{n}_1, \lambda)\omega_2(m_2|\vec{n}_2, \lambda). \end{aligned}$$

The spin tomogram

Definition

The dequantizer and the quantizer operators are defined as as

$$\omega(m_1, m_2, \vec{n}_1, \vec{n}_2) = Tr(\widehat{\rho}\widehat{U}(m_1, m_2, \vec{n}_1, \vec{n}_2)),$$
$$\widehat{\rho} = \sum_{m_1, m_2} \int \omega(m_1, m_2, \vec{n}_1, \vec{n}_2)\widehat{D}(m_1, m_2, \vec{n}_1, \vec{n}_2) d\vec{n}_1 d\vec{n}_2,$$

For the two-qubit state (Filippov, Manko, (2009))

$$\widehat{U} = \left(\frac{1}{2}\widehat{I} + m_1 F(\varphi_1, \theta_1)\right) \otimes \left(\frac{1}{2}\widehat{I} + m_2 F(\varphi_2, \theta_2)\right),$$
$$\widehat{D} = \left(\frac{1}{8\pi^2}\left(\frac{1}{2}\widehat{I} + 3m_1 F(\varphi_1, \theta_1)\right)\right) \otimes \left(\frac{1}{8\pi^2}\left(\frac{1}{2}\widehat{I} + 3m_2 F(\varphi_2, \theta_2)\right)\right),$$

where $\widehat{I}$ is the 2×2 identity matrix and

$$F(\varphi, \theta) = \begin{pmatrix} \cos\theta & -e^{i\varphi}\sin\theta \\ -e^{-i\varphi}\sin\theta & -\cos\theta \end{pmatrix}.$$

The Tomographic Symbol

Any observable A can be identified with a Hermitian operator $\widehat{A}$.

Definition

The tomographic symbol $\omega_A(\mathbf{x})$ of the operator $\widehat{A}$ is

$$\omega_A(\mathbf{x}) = Tr(\widehat{A}\widehat{U}(\mathbf{x})), \quad \widehat{A} = \int \omega_A(\mathbf{x})\widehat{D}(\mathbf{x})d\mathbf{x},$$

where $\widehat{U}(\mathbf{x})$, $\widehat{D}(\mathbf{x})$ are the dequantizer and quantizer operators, respectively.

Definition

The dual tomographic symbol $\omega_A^d(\mathbf{x})$ of the operator $\widehat{A}$ is

$$\omega_A^d(\mathbf{x}) = Tr(\widehat{A}\widehat{U}'(\mathbf{x})) = Tr(\widehat{A}\widehat{D}(\mathbf{x})),$$
$$\widehat{A} = \int \omega_A^d(\mathbf{x})\widehat{D}'(\mathbf{x})d\mathbf{x} = \int \omega_A^d(\mathbf{x})\widehat{U}(\mathbf{x})d\mathbf{x}.$$

Tomographic form of the quantum correlation function

The quantum correlation function can be rewritten in the tomographic form

$$E(\overrightarrow{k_1}, \overrightarrow{k_2}) = \int \omega_B(\mathbf{x})\omega_\rho^d(\mathbf{x})d\mathbf{x}$$

or in the equivalent form

$$\begin{aligned} E(\overrightarrow{k_1}, \overrightarrow{k_2}) &= \int \omega_\rho(\mathbf{x})\omega_B^d(\mathbf{x})d\mathbf{x} \\ &= \sum_{m_1, m_2} \int \omega_\rho(m_1, m_2, \vec{n}_1, \vec{n}_2)\omega_B^d(m_1, m_2, \vec{n}_1, \vec{n}_2)d\vec{n}_1 d\vec{n}_2, \end{aligned}$$

where we used the notation $B = k_1\sigma \otimes k_2\sigma$.
Thus, we can write any steering inequality that contains the correlation function in terms of the spin tomograms.

The Single Qudit and the Two-qubit States Tomograms

The tomographic representation for the single qudit with the spin $j = 3/2$ is

$$\widehat{\rho} = \sum_{m=-3/2}^{3/2} \int W(m, \vec{n}) \widehat{\mathcal{D}}(m, \vec{n}) d\vec{n},$$

$$W(m, \vec{n}) = Tr \left(\widehat{\mathcal{U}}(m, \vec{n}) \widehat{\rho} \right).$$

The tomogram $W(m, \vec{n})$ is the conditional probability of projections of the spin m on vector $\vec{n}$ on the Bloch sphere.

Remark

The later tomogram depends on less amount of numbers comparable to the two-qubit state tomogram.

The Single Qudit and the Two-qubit States Tomograms

In (Manko, (1997)) it was obtained that any qudit state with the spin j can be represented as

$$\widehat{\rho}^{(j)} = \sum_{m=-j}^{j} \int \frac{d\omega}{8\pi^2} W(m, \alpha, \beta) \widehat{B}^{j}_{m}(\alpha, \beta).$$

$\widehat{B}^{j}_{m_1}(\alpha, \beta)$ - the quantizer operator.
For the single qudit state the it is defined by the following matrix

$$B^{\frac{3}{2}}_{m}(\alpha, \beta) = B_{1m}(\alpha, \beta) + \frac{i(-1)^m}{2(m+\frac{3}{2})!(\frac{3}{2}-m)!} \left(5mB_{2m}(\alpha, \beta) + \frac{21}{2} \sin\beta B_{3m}(\alpha, \beta) \right).$$

The Single Qudit and the Two-qubit States Tomograms

The quantizer for the single qudit system is

$$\widehat{\mathcal{D}}(m,\alpha,\beta) = \frac{1}{8\pi^2}\widehat{B}_m^{3/2}(\alpha,\beta)$$

The rotation matrix $U^{(3/2)}(\alpha,\beta,\gamma)$ for the single qudit state can be written using Wigner's D-function

$$D^{(j)}_{m',m}(\alpha,\beta,\gamma) \quad = \quad e^{im'\gamma} d^{(j)}_{m',m}(\beta) e^{im'\alpha},$$

where it holds

$$d^{(j)}_{m',m}(\beta) = \left(\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right)^{\frac{1}{2}} \cos\left(\frac{\beta}{2}\right)^{m'+m} \sin\left(\frac{\beta}{2}\right)^{m'-m} P^{(m'-m,m'+m)}_{j-m'}(\cos\beta).$$

Hence, the rotation matrix for the single qudit state has elements

$$U^{(3/2)}_{m',m}(\alpha,\beta,\gamma) = D^{(3/2)}_{m',m}(\alpha,\beta,\gamma).$$

The dequantizer operator is defined by the matrices

$$\mathcal{U}(m,\alpha,\beta) \quad = \quad U^{(3/2)\dagger}(\alpha,\beta,\gamma)|m><m|U^{(3/2)}(\alpha,\beta,\gamma).$$

The relation between the two-qubit system and the single qudit system tomograms is

$$\omega(m_1, m_2, \vec{n}_1, \vec{n}_2) = \int W(m, \vec{n}) K_{12} d\vec{n},$$

where we introduce the kernel function

$$K_{12} \equiv K_{12}(m_1, m_2, m, \vec{n}_1, \vec{n}_2, \vec{n}) = Tr\widehat{\mathcal{D}}(m, \vec{n})\widehat{U}(m_1, m_2, \vec{n}_1, \vec{n}_2).$$

The kernel depends on three quantum numbers and six angles.
The inverse transformation is

$$W(m, \vec{n}) = \int \omega(m_1, m_2, \vec{n}_1, \vec{n}_2) K_{21} d\vec{n}_1 d\vec{n}_2,$$

$$K_{21} \equiv K_{21}(m_1, m_2, m, \vec{n}_1, \vec{n}_2, \vec{n}) = Tr\widehat{D}(m_1, m_2, \vec{n}_1, \vec{n}_2)\widehat{\mathcal{U}}(m, \vec{n}).$$

For the dual tomographic symbols

$$\omega^d(m_1, m_2, \vec{n}_1, \vec{n}_2) = \int W^d(m, \vec{n}) K^d_{12} d\vec{n},$$

$$W^d(m, \vec{n}) = \int \omega^d(m_1, m_2, \vec{n}_1, \vec{n}_2) K^d_{21} d\vec{n}_1 d\vec{n}_2,$$

where the kernels are $K^d_{12} = K_{21}$ and $K^d_{21} = K_{12}$.

Steering Inequality for the Two-qubit State

In (Zukowski, 2014) the steering inequality for the two-qubit state is based on the maxima of the correlation function

$$E(\overrightarrow{k}_1, \overrightarrow{k}_2) = \sum_{i,j=1}^{3} T_{ij} k_{1i} k_{2j},$$

T_{ij} - components of the correlation matrix.

Definition

If the bipartite state is non-steerable, then the following inequality is fulfilled

$$\max_{\overrightarrow{k}_1, \overrightarrow{k}_2} (E(\overrightarrow{k}_1, \overrightarrow{k}_2)) \geq \frac{2}{3} \sum_{i,j=1}^{3} T_{ij}.$$

The Single Qudit Steering in the Spin Tomographic Representation

We can write the correlation function of the single qudit state as

$$\mathcal{E}(\overrightarrow{k}_1, \overrightarrow{k}_2) = \int W_{k_1\sigma\otimes k_2\sigma}(\mathbf{y}) W^d_\rho(\mathbf{y}) d\mathbf{y}.$$

Using the intertwining kernels we can deduce that the correlation functions are mathematically completely equivalent.

$$\begin{aligned} E(\overrightarrow{k}_1, \overrightarrow{k}_2) &= \sum_{m_1, m_2=-\frac{1}{2}}^{\frac{1}{2}} \iint d\vec{n}_1 d\vec{n}_2 \sum_{m=-\frac{3}{2}}^{\frac{3}{2}} \int W_\rho(m, \vec{n}) W^d_{k_1\sigma\otimes k_2\sigma}(m, \vec{n}) K_{12} K_{21} d\vec{n} \\ &= \mathcal{E}(\overrightarrow{k}_1, \overrightarrow{k}_2). \end{aligned}$$

Conclusions

- The quantum correlations reflected by the phenomenon of the quantum steering available in the two-qubit system take place also in the single qudit $j = 3/2$.
- We demonstrate the inequalities for the correlation function detecting the presence of the steering not only for the two-qubit states but also for the single qudit $j = 3/2$ state.
- The physical meaning of these hidden correlations is different from the case of two qubits correlations. The observables to be measured for the obtained correlation being mathematically completely equivalent to observables measured in the experiment with two qubits are different for the single qudit.

Conclusions

V.I. Man'ko and L.A. Markovich (2015) Steering and correlations for the single qudit state on the example of $j = 3/2$, J. Russ. Laser Res.,36(4), 2015, 343–349 (DOI 10.1007/s10946-015-9508-x) (arXiv:1503.02296)

V.I. Man'ko and L.A. Markovich (2015) Steering in spin tomographic probability representation, (submitted to Physica A)