EXACT DISTRIBUTION OF THE GENERALIZED SHIRYAEV–ROBERTS STOPPING TIME UNDER THE BROWNIAN MOTION SETUP

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Quickest change-point detection: The Brownian motion drift-shift scenario

- Suppose one is able to observe "live" a continuous-time random process, $(X_t)_{t\geqslant 0}$, governed by the stochastic differential equation (SDE):

$$dX_t = \mu\,\mathbb{1}_{\{t>\nu\}}\,dt + dB_t \text{ for } t > 0 \text{ with } X_0 = 0,$$

  where

  - $\mu \neq 0$ is a known constant referred to as the drift magnitude;
  - $\nu \in [0, \infty]$ is an **unknown** parameter referred to as the *change-point*;
  - $(B_t)_{t\geqslant 0}$ is standard Brownian motion, i.e., $\mathbb{E}[dB_t] = 0$, $\mathbb{E}[(dB_t)^2] = dt$, and $B_0 = 0$.

- That is, the process $(X_t)_{t\geqslant 0}$ is such that

$$\mathbb{E}[X_t] = 0 \text{ for } 0 \leqslant t \leqslant \nu \text{ but } \mathbb{E}[X_t] = \mu t \text{ for } t > \nu,$$

  and one's aim is to detect the onset of the drift.

- More concretely, one's aim is to detect the presence of the drift as quickly as is possible within a given level of the "false alarm" risk.

- This problem has been considered by Pollak and Siegmund (1985), Shiryaev (1961, 1963, 1978, 1996, 2002, 2006, 2011), Burnaev et. al (2009), Moustakides (2004), Feinberg and Shiryaev (2006), Beibel (1996)—to name a few.

The Brownian motion quickest change-point scenario (cont'd)

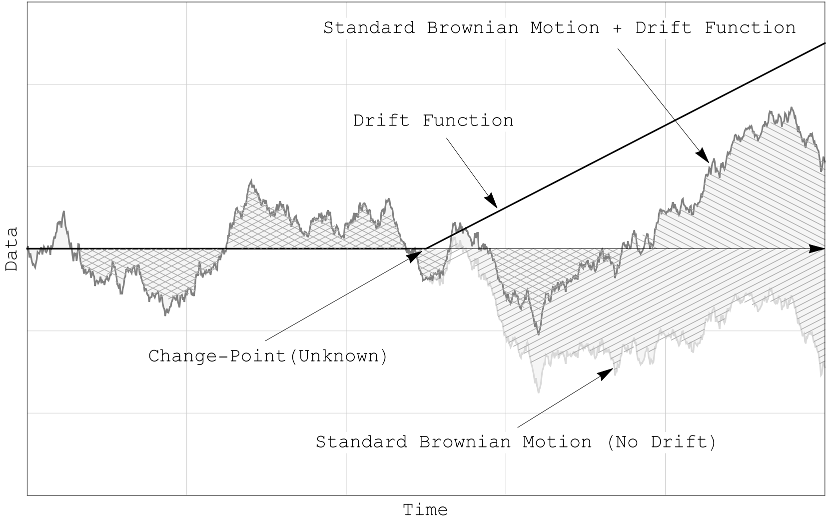


Figure 1. Standard Brownian motion gaining a drift.

The minimax Brownian motion quickest change-point scenario

- The **minimax version** of the problem assumes that the change-point ν is **unknown, but not random**.
- To perform change-point detection in this case the standard approach has been to employ Page's (1954) Cumulative Sum (CUSUM) "inspection scheme".
- This choice may be justified by the fact that the CUSUM scheme is exactly optimal (i.e., the best one can do) in the minimax sense of Lorden (1971). See Moustakides (1986, 2004), Beibel (1996) and Shiryaev (1996).
- However, when one is interested in minimax optimality in the sense of Pollak (1985), a sensible alternative to using the CUSUM scheme would be to devise the Generalized Shiryaev–Roberts (GSR) procedure due to Moustakides et. al (2011).
- For the discrete-time setup, it has been demonstrated that the GSR procedure is a carefully designed headstart is exactly Pollak-minimax optimal in two specific scenarios and is otherwise asymptotically nearly optimal. Needless to say, the GSR procedure outperforms the CUSUM scheme. See Tartakovsky & P. (2010), P. & Tartakovsky (2010), Tartakovsky et. al (2012). An attempt to extend these results to the Brownian motion model has been recently made by Burnaev (2009).

The nomenclature

To continue, introduce

- Notation $\nu = \infty$ is to be understood as the case when $\mathbb{E}[X_t] = 0$ for all $t \geqslant 0$;
- Filtration $(\mathcal{F}_t)_{t \geqslant 0}$ with $\mathcal{F}_t \triangleq \sigma(X_s \colon 0 \leqslant s \leqslant t)$, i.e., the σ-algebra generated by the observed process, $(X_t)_{t \geqslant 0}$, up to time t, where $\mathcal{F}_0 = \{\varnothing, \Omega\}$;
- Probability measure $\mathbb{P}_\nu(\mathcal{A}) \triangleq \mathbb{P}(\mathcal{A}|\nu)$, $\nu = [0, \infty]$; note that $\mathbb{P}_\infty$ is the Wiener measure;
- Expectation operator $\mathbb{E}_\nu[\mathcal{A}]$ with respect to probability measure $\mathbb{P}_\nu(\mathcal{A})$;
- Generic stopping time T.

Pollak's (1985) minimax change-point detection problem

- The false positive risk is measured via Page's (1954) and Lorden's (1971) Average Run Length (ARL) to false alarm:

$$\mathsf{ARL}(T) \triangleq \mathbb{E}_\infty[T],$$

which captures the average sample size under the assumption of zero drift (no change).

- The speed of detection is measured via Pollak's (1985) Supremum (conditional) Average Detection Delay (SADD):

$$\mathsf{SADD}(T) \triangleq \sup_{0\leqslant\nu<\infty} \mathsf{ADD}_\nu(T),$$

where $\mathsf{ADD}_\nu(T) \triangleq \mathbb{E}_\nu[T-\nu|T>\nu]$, $0 \leqslant \nu < \infty$.

- Let $\Delta(\gamma) \triangleq \{T\colon\; \mathsf{ARL}(T) \geqslant \gamma\}$ be the class of stopping times for which the ARL to false alarm is at least $\gamma > 0$, a given level.

- Pollak's (1985) minimax change-point detection problem is to find $T_{\rm opt} = \arg\inf_{T\in\Delta(\gamma)} \mathsf{SADD}(T)$ for all $\gamma > 0$.

- Let $\mathcal{H}_\nu$, $\nu \in [0,\infty)$, be the hypothesis that the change-point is at a given time instance ν, and $\mathcal{H}_\infty$ denote the hypothesis that the change does not take place at all (i.e., $\nu = \infty$).
- The likelihood ratio (LR) to test $\mathcal{H}_0$ against $\mathcal{H}_\infty$ is

$$\Lambda_t \triangleq \frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}(X_t, t) = \exp\left\{\mu X_t - \mu^2 \frac{t}{2}\right\},$$

so that by Itô formula, $d\,\Lambda_t = \mu\,\Lambda_t\,dX_t$ for $t \geqslant 0$ with $\Lambda_0 = 1$.

- The Generalized Shiryaev–Roberts (GSR) detection statistic is given by

$$\begin{aligned} R_t^r &\triangleq r\,\Lambda_t + \int_0^t \frac{\Lambda_t}{\Lambda_s}\,ds \\ &= r\exp\left\{\mu X_t - \frac{\mu^2 t}{2}\right\} + \int_0^t \exp\left\{\mu(X_t - X_s) - \frac{\mu^2(t-s)}{2}\right\} ds,\;\; t \geqslant 0, \end{aligned}$$

so that $R_0^r = r$, which is referred to as the **headstart**. The corresponding stopping rule is defined as

$$\mathcal{S}_A^r \triangleq \inf\left\{t > 0\colon\, R_t^r \geqslant A\right\}, \quad A > 0.$$

- Alternatively, by Itô formula, we have $dR_t^r = dt + \mu R_t^r dX_t$.
- Hence, when $\nu = \infty$, we have $dX_t = dB_t$, and the above equation is simply

$$dR_t^r = dt + \mu R_t^r dB_t,$$

  recalling again that $R_0^r = r \geqslant 0$.

- However, then $\nu = 0$, we have $dX_t = \mu dt + dB_t$, and therefore

$$dR_t^r = (1 + \mu^2 R_t^r)dt + \mu R_t^r dB_t,$$

  with $R_0^r = r \geqslant 0$.

- Both differentials can be combined into one as follows:

$$dR_t^r = (1 + \theta\mu^2 R_t^r)dt + \mu R_t^r dB_t,$$

  where θ is either 0 (for $\nu = \infty$) or 1 (for $\nu = 0$).

- Note that $(R_t^r)_{t \geqslant 0}$ is a time-homogeneous Markov diffusion for both $\nu = \infty$ and $\nu = 0$.

- Our goal is to find closed-form expressions for

$$\mathbb{P}_\infty(\mathcal{S}_A^r \geqslant t) \text{ and } \mathbb{P}_0(\mathcal{S}_A^r \geqslant t),$$

i.e., for the pre- and post-drift survival functions of the GSR stopping time $\mathcal{S}_A^r$.

- Define

$$p_\theta(y,t|x,s) \triangleq \frac{\partial}{\partial y}\begin{cases}\mathbb{P}_\infty(R_t^r \leqslant y, \mathcal{S}_A^r \geqslant t | R_s^r = x), & \text{if } \theta = 0;\\ \mathbb{P}_0(R_t^r \leqslant y, \mathcal{S}_A^r \geqslant t | R_s^r = x), & \text{if } \theta = 1,\end{cases}$$

where $0 \leqslant s \leqslant t$, i.e., $p_\theta(y,t|x,s)$ is the transition probability density of the time-homogeneous Markov diffusion $(R_t^r)_{t\geqslant 0}$ joint with the event that the respective GSR stopping time $\mathcal{S}_A^r$ does not terminate the diffusion $(R_t^r)_{t\geqslant 0}$ prior to a given time point $t \geqslant 0$.

- Since $(R_t^r)_{t\geqslant 0}$ is a time-homogeneous Markov process, we will deal with $p_\theta(x,t|r) \triangleq p_\theta(x,t|r,0)$.
- At this point note that since

$$\mathbb{P}_\infty(\mathcal{S}_A^r \geqslant t) = \int_0^A p_0(x,t|r)\,dx \text{ and } \mathbb{P}_0(\mathcal{S}_A^r \geqslant t) = \int_0^A p_1(x,t|r)\,dx,$$

where $t \geqslant 0$ and $r \in [0,A]$ with $A > 0$

Kolmogorov forward and backward equations

- In our case, $p_\theta(x,t|r)$ satisfies the Kolmogorov forward equation

$$\frac{\partial}{\partial t}p_\theta(x,t|r) = -\frac{\partial}{\partial x}\Big[(1+\theta\mu^2 x)p_\theta(x,t|r)\Big] + \frac{\mu^2}{2}\frac{\partial^2}{\partial x^2}\Big[x^2 p_\theta(x,t|r)\Big],$$

where $t \geqslant 0$ and $x, r \in [0, A]$.

- ...and the Kolmogorov backward equation

$$-\frac{\partial}{\partial t}p_\theta(x,t|r) = (1+\theta\mu^2 x)\frac{\partial}{\partial x}\Big[p_\theta(x,t|r)\Big] + \frac{\mu^2 x^2}{2}\frac{\partial^2}{\partial x^2}\Big[p_\theta(x,t|r)\Big],$$

where $t \geqslant 0$ and $x, r \in [0, A]$.

- We shall focus on the forward equation.

Kolmogorov forward equation and boundary conditions

- We shall focus on the Kolmogorov forward equation

$$\frac{\partial}{\partial t} p_\theta(x,t|r) = -\frac{\partial}{\partial x}\Big[(1+\theta\mu^2 x)p_\theta(x,t|r)\Big] + \frac{\mu^2}{2}\frac{\partial^2}{\partial x^2}\Big[x^2 p_\theta(x,t|r)\Big],$$

  where $t \geqslant 0$ and $x, r \in [0, A]$.

- This is a PDE of order one in the temporal variable t and of order two in the spacial variable x. Hence, the PDE is to be complemented with one initial temporal condition and two spacial boundary conditions.

- The initial temporal condition is as follows: $\lim_{t\to 0+} p_\theta(x,t|r) = \delta(x-r)$, where $\delta(z)$ is the Dirac delta function.

- The two boundary conditions are as follows: $p_\theta(A,t|r) = 0$ for all $r = (0, A)$, which is an absorbing boundary condition, and

$$\lim_{x\to 0+}\left\{p_\theta(x,t|r) + \frac{\mu^2}{2}\frac{\partial}{\partial x}\Big[x^2\, p_\theta(x,t|r)\Big]\right\} = 0,\quad r \in (0, A),$$

  which is an entrance boundary condition.

Outline of solution strategy: Separation of variables

- Consider the following operators

$$\mathcal{G}^* \triangleq \frac{1}{2}a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x} \text{ and } \mathcal{G} \triangleq \frac{1}{2}\frac{\partial^2}{\partial x^2}a(x) - \frac{\partial}{\partial x}b(x),$$

where $a(x)$ and $b(x)$ are two given sufficiently smooth functions.

- In terms of these operators, the Kolmogorov backward and forward equations can be written compactly as follows: $\mathcal{G}^* \circ p = -\partial p/\partial t$ and $\mathcal{G} \circ p = \partial p/\partial t$.

- These operators can be expressed as follows

$$\mathcal{G}^* = \frac{1}{\mathfrak{m}(x)}\frac{d}{dx}\frac{1}{\mathfrak{s}(x)}\frac{d}{dx} \text{ and } \mathcal{G} = \frac{d}{dx}\frac{1}{\mathfrak{s}(x)}\frac{d}{dx}\frac{1}{\mathfrak{m}(x)},$$

where

$$\mathfrak{s}(x) \triangleq \exp\left\{-\int \frac{2b(x)}{a(x)}\,dx\right\} \text{ and } \mathfrak{m}(x) \triangleq \frac{2}{a(x)\,\mathfrak{s}(x)}.$$

i.e., $\mathfrak{s}(x)$ is the solution of the ODE $[\mathcal{G}^* \circ \mathfrak{s}](x) = 0$ while $\mathfrak{m}(x)$ satisfies the ODE $[\mathcal{G} \circ \mathfrak{m}](x) = 0$.

- Let us temporarily "lighten" the notation $p_\theta(x,t|r)$ to $p(x,t)$.
- Key idea: try to fit a solution of the form $p(x,t)=\mathfrak{m}(x)\,\psi(x)\,\tau(t)$ for some $\psi(x)$ and $\tau(t)$.
- This substitution converts the PDE into the following form:

$$\frac{\tau'(t)}{\tau(t)}=\frac{1}{\mathfrak{m}(x)\,\psi(x)}\left(-\frac{d}{dx}\Big[\mathfrak{m}(x)\,\psi(x)\Big]+\frac{\mu^2}{2}\frac{d^2}{dx^2}\Big[x^2\,\mathfrak{m}(x)\,\psi(x)\Big]\right),$$

  which has a nontrivial solution only when both sides are equal to the same constant λ.
- This leads to two ODEs

$$\frac{\tau'(t)}{\tau(t)}=\lambda \;\text{ and }\; \frac{1}{\mathfrak{m}(x)\,\psi(x)}\left(-\frac{d}{dx}\Big[\mathfrak{m}(x)\,\psi(x)\Big]+\frac{\mu^2}{2}\frac{d^2}{dx^2}\Big[x^2\mathfrak{m}(x)\,\psi(x)\Big]\right)=\lambda,$$

  so that the whole problem becomes that of finding $\tau(t)$, $\psi(x)$, and all permissible λ's.

Outline of solution strategy: Separation of variables (cond't)

- After some algebra it can be shown that the ODE

$$\frac{1}{\mathfrak{m}(x)\,\psi(x)}\left(-\frac{d}{dx}\big[\mathfrak{m}(x)\,\psi(x)\big]+\frac{\mu^2}{2}\frac{d^2}{dx^2}\big[x^2\mathfrak{m}(x)\,\psi(x)\big]\right)=\lambda,$$

can be rewritten as

$$\frac{\mu^2}{2}\frac{d}{dx}\big[x^2\,\mathfrak{m}(x)\,\psi'(x)\big]=\lambda\,\mathfrak{m}(x)\,\psi(x),$$

whence it is clear that λ and $\psi(x)$ are effectively the eigenvalues and eigenfunctions of the operator

$$\mathcal{D}\triangleq\frac{\mu^2}{2\,\mathfrak{m}(x)}\frac{d}{dx}\,x^2\,\mathfrak{m}(x)\,\frac{d}{dx}.$$

- To emphasize the dependence of $\psi(x)$ on λ let us use the notation $\psi(x,\lambda)$.

- Since the ODE on $\tau(t)$ is trivial (the solution is $\tau(t) = Ce^{\lambda t}$), the whole problem is to find λ's and $\psi(x,\lambda)$'s.

- Once all eigenpairs of operator $\mathcal{D}$ are found, the solution to the original PDE is constructed as their superposition

$$p(x,t|r=y) = \mathfrak{m}(x)\sum_{k} e^{\lambda_k t}\,\psi(x,\lambda_k)\,\psi(y,\lambda_k),$$

  and the series in the right-hand side is *absolutely* convergent for all $t \geqslant 0$ and all $x, y \in [0,A]\times[0,A]$.

- In addition, the above expansion assumes that the eigenfunctions are of unit length, with the length understood as follows:

$$\|\psi(\cdot,\lambda)\|^2 \triangleq \int_0^A \mathfrak{m}(x)\,\psi^2(x,\lambda)\,dx.$$

The solution

- The main idea is to note that the change-of-variables

$$x \mapsto u = u(x) \triangleq -\int \frac{2}{a(x)}\,dx = \frac{2}{\mu^2 x},\;\; u \mapsto x = x(u) = \frac{2}{\mu^2 u} \text{ and } \frac{dx}{x} = -\frac{du}{u},$$

and the substitution

$$\psi(x) \mapsto \psi(u) \triangleq \frac{v(u)}{\sqrt{\mathfrak{m}(u)}} = \left(\frac{\mu^2}{2}\right)^{\theta+\frac{1}{2}} u^{\theta-1}\, e^{\frac{u}{2}}\, v(u) \propto u^{\theta-1}\, e^{\frac{u}{2}}\, v(u),$$

convert the equation into the form

$$v_{uu}(u) + \left\{-\frac{1}{4} + \frac{1-\theta}{u} + \frac{1/4 - \xi^2/4}{u^2}\right\} v(u) = 0,$$

where

$$\xi \equiv \xi(\lambda) \triangleq \sqrt{1 + \lambda\frac{8}{\mu^2}} \text{ so that } \lambda \equiv \lambda(\xi) = \frac{\mu^2}{8}(\xi^2 - 1).$$

The solution (cond't)

- The obtained equation is a particular case of the Whittaker equation whose two independent solutions are the Whittaker M and W functions.

- Therefore, the general solution to our equation is as follows:

$$\psi(u,\lambda) = u^{\theta-1}\, e^{\frac{u}{2}} \left\{ C_1 M_{1-\theta, \frac{\xi(\lambda)}{2}}(u) + C_2 W_{1-\theta, \frac{\xi(\lambda)}{2}}(u) \right\},$$

  and it remains to "pin down" the two constants by making $\psi(u,\lambda)$ satisfy the boundary conditions.

- To that end, the boundary condition at zero is satisfied only when $C_1 = 0$. Hence, the above expression for the eigenfunction simplifies down to

$$\psi(u,\lambda) = C u^{\theta-1}\, e^{\frac{u}{2}}\, W_{1-\theta, \frac{\xi(\lambda)}{2}}(u).$$

- The constant factor C can be found from the requirement that $\|\psi(\cdot,\lambda)\|^2 = 1$.

- With regard to ensuring that $\|\psi(\cdot,\lambda)\| = 1$ for each particular eigenvalue λ, observe that

$$\|\psi(\cdot,\lambda)\|^2 \triangleq \int_0^A \mathfrak{m}(x)\,\psi^2(x,\lambda)\,dx = C^2\left(\frac{2}{\mu^2}\right)^{2\theta}\int_{\frac{2}{\mu^2 A}}^{+\infty} W^2_{1-\theta,\frac{\xi(\lambda)}{2}}(u)\,\frac{du}{u^2},$$

whence it follows that to "pin down" C so as to have $\|\psi(\cdot,\lambda)\| = 1$ we are to compute the integral

$$\int_{\frac{2}{\mu^2 A}}^{+\infty} W^2_{1-\theta,\frac{\xi(\lambda)}{2}}(u)\,\frac{du}{u^2}$$

for each particular eigenvalue λ.

- This task can be undertaken using the indefinite integral identity

$$\int W_{a,b_1}(z)\,W_{a,b_2}(z)\,\frac{dz}{z^2} = \frac{1}{b_2^2 - b_1^2}\left\{W_{a,b_1}(z)\,\frac{\partial}{\partial z}W_{a,b_2}(z) - W_{a,b_2}(z)\,\frac{\partial}{\partial z}W_{a,b_1}(z)\right\},$$

for $b_1 \neq b_2$.

- After quite a bit of calculus it can be shown that if the normalizing factor C is chosen as follows

$$C^2 \equiv C^2_{\lambda,\theta,A} = \left(\frac{\mu^2}{2}\right)^{2\theta} \xi(\lambda) \Big/ \left\{ \left[\frac{\partial}{\partial b} W_{1-\theta,b}\left(\frac{2}{\mu^2 A}\right)\right]\Bigg|_{b=\frac{\xi(\lambda)}{2}} \left[\frac{\partial}{\partial u} W_{1-\theta,\frac{\xi(\lambda)}{2}}(u)\right]\Bigg|_{u=\frac{2}{\mu^2 A}} \right\},$$

then $\|\psi(\cdot,\lambda)\|^2 = 1$.

- The only problem now is to actually find the eigenvalues λ. This can be done using the absorbing boundary condition $\psi(A,\lambda) = 0$.

The solution: Finding the eigenvalues

- From the absorbing boundary condition $\psi(A, \lambda) = 0$ we have

$$e^{\frac{1}{\mu^2 A}} \left(\frac{2}{\mu^2 A}\right)^{\theta-1} W_{1-\theta, \frac{\xi(\lambda)}{2}}\left(\frac{2}{\mu^2 A}\right) = 0,$$

which is equivalent to

$$W_{1-\theta, \frac{\xi(\lambda)}{2}}\left(\frac{2}{\mu^2 A}\right) = 0,$$

and the only way solve this equation is numerically.

- The question however is does it have any solutions and how many?
- The answer depends on the second index of the Whittaker function. In our case, the second index can either be purely real or purely complex.
- When the second index is purely real, then there may be at most one eigenvalue if $\theta = 0$ or no eigenvalue if $\theta = 1$.
- When the second index is purely imaginary, then the number of eigenvalues is countably many for either value of θ.

- We are now in a position to put all of the above together and write down the sought-after density, $p_\theta(x,t|r)$, in a closed form.
- Specifically, we obtain:

$$p_\theta(x,t|r=y) = \frac{\mu^2}{2}\, e^{\frac{1}{\mu^2 y}-\frac{1}{\mu^2 x}}\, e^{-\frac{\mu^2 t}{8}} \left(\frac{y}{x}\right)^{1-\theta} \times$$
$$\times \left\{ (1-\theta)\, e^{\frac{\mu^2 t}{8}\alpha_{0,A}^2}\, \tilde{C}_{0,0,A}^2\, W_{1,\frac{\alpha_{0,A}}{2}}\left(\frac{2}{\mu^2 x}\right) W_{1,\frac{\alpha_{0,A}}{2}}\left(\frac{2}{\mu^2 y}\right) + \right.$$
$$+ \sum_{n=1}^{\infty} e^{-\frac{\mu^2 t}{8}\beta_{n,\theta,A}^2} \times$$
$$\left. \times\, \tilde{C}_{n,\theta,A}^2\, W_{1-\theta,\frac{\imath\beta_{n,\theta,A}}{2}}\left(\frac{2}{\mu^2 x}\right) W_{1-\theta,\frac{\imath\beta_{n,\theta,A}}{2}}\left(\frac{2}{\mu^2 y}\right) \right\},$$

where $x, y \in [0, A]$ and $t \geqslant 0$, and recall that θ is either 0 ($\nu = \infty$) or 1 ($\nu = 0$), the detection threshold $A > 0$ is given.

The answer (cond't)

- In the previous formula, the constant $\alpha_{0,A} \in [0,1]$ is the only zero (should it exist) of the equation

$$W_{1,\frac{\alpha_{0,A}}{2}}\left(\frac{2}{\mu^2 A}\right) = 0.$$

- The constant $\tilde{C}^2_{0,0,A}$ is as follows

$$\tilde{C}^2_{0,0,A} = \alpha_{0,A} \Bigg/ \left\{ \left[\frac{\partial}{\partial b} W_{1,b}\left(\frac{2}{\mu^2 A}\right)\right]\Bigg|_{b=\frac{\alpha_{0,A}}{2}} \left[\frac{\partial}{\partial u} W_{1,\frac{\alpha_{0,A}}{2}}(u)\right]\Bigg|_{u=\frac{2}{\mu^2 A}} \right\}.$$

- The series $\{\beta_{n,\theta,A}\}_{n\geqslant 1}$ is formed of the (countably many) solutions $\beta_{\theta,A} \geqslant 0$ of the equation

$$W_{1-\theta,\frac{\imath\beta_{\theta,A}}{2}}\left(\frac{2}{\mu^2 A}\right) = 0,$$

- Finally, the constants $\tilde{C}^2_{n,\theta,A}$ are as follows:

$$\tilde{C}^2_{n,\theta,A} = \imath\beta_{n,\theta,A} \Bigg/ \left\{ \left[\frac{\partial}{\partial b} W_{1-\theta,b}\left(\frac{2}{\mu^2 A}\right)\right]\Bigg|_{b=\frac{\imath\beta_{n,\theta,A}}{2}} \left[\frac{\partial}{\partial u} W_{1-\theta,\frac{\imath\beta_{n,\theta,A}}{2}}(u)\right]\Bigg|_{u=\frac{2}{\mu^2 A}} \right\}.$$

- For the $\mathbb{P}_\infty$-survival function we have:

$$\mathbb{P}_\infty(\mathcal{S}_A^{r=y} \geqslant t) = 4\,\frac{y}{A}\,e^{\frac{1}{\mu^2 y}-\frac{1}{\mu^2 A}}\,e^{-\frac{\mu^2 t}{8}}\times$$

$$\times\left\{ e^{\frac{\mu^2 t}{8}\alpha_{0,A}^2}\,\frac{\alpha_{0,A}}{1-\alpha_{0,A}^2}\,\frac{W_{1,\frac{\alpha_{0,A}}{2}}\left(\frac{2}{\mu^2 y}\right)}{\left[\frac{\partial}{\partial b}W_{1,b}\left(\frac{2}{\mu^2 A}\right)\right]\Big|_{b=\frac{\alpha_{0,A}}{2}}}+\right.$$

$$+\sum_{n=1}^{\infty} e^{-\frac{\mu^2 t}{8}\beta_{n,0,A}^2}\,\frac{\imath\beta_{n,0,A}}{1+\beta_{n,0,A}^2}\times$$

$$\left.\times\;W_{1,\frac{\imath\beta_{n,0,A}}{2}}\left(\frac{2}{\mu^2 y}\right)\Bigg/\left[\frac{\partial}{\partial b}W_{1,b}\left(\frac{2}{\mu^2 A}\right)\right]\Big|_{b=\frac{\imath\beta_{n,0,A}}{2}}\right\},$$

where $\alpha_{0,A}$ and $\{\beta_{n,0,A}\}_{n\geqslant 1}$ are as before.

- Remark: The series in the right-hand side of the foregoing formula is divergent for $t=0$.

The answer (cond't)

- For the $\mathbb{P}_0$-survival function we have:

$$\mathbb{P}_0(\mathcal{S}_A^{r=y} \geqslant t) = 4\, e^{\frac{1}{\mu^2 y} - \frac{1}{\mu^2 A}} e^{-\frac{\mu^2 t}{8}} \times$$
$$\times \sum_{n=1}^{\infty} e^{-\frac{\mu^2 t}{8}\beta_{n,1,A}^2} \frac{\imath\beta_{n,1,A}}{1+\beta_{n,1,A}^2} \times$$
$$\times\; W_{0,\frac{\imath\beta_{n,1,A}}{2}}\left(\frac{2}{\mu^2 y}\right) \Bigg/ \left[\frac{\partial}{\partial b} W_{0,b}\left(\frac{2}{\mu^2 A}\right)\right]\Bigg|_{b=\frac{\imath\beta_{n,1,A}}{2}},$$

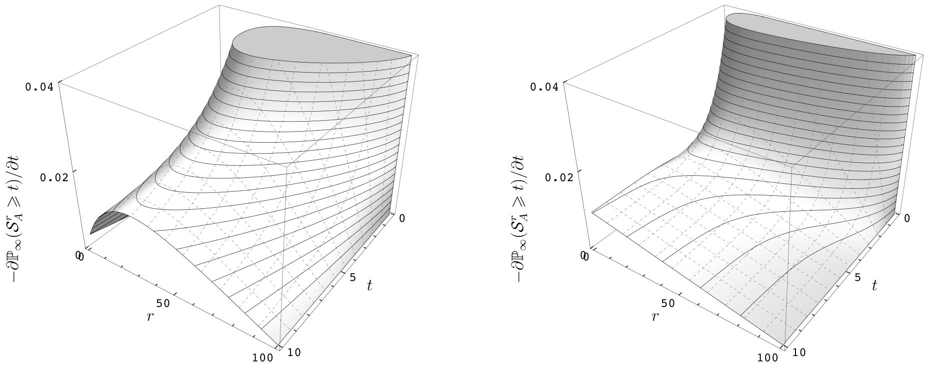
  where again $\{\beta_{n,1,A}\}_{n\geqslant 1}$ are as before.

- Remark: The series in the right-hand side of the foregoing formula is divergent for $t = 0$.

Numerical study

- We would like to put the obtained formulae to work now and analyze the behavior of the GSR stopping time under different settings.
- The specific questions we would like to consider:
  - The effect of the headstart $r \geqslant 0$;
  - The effect of the magnitude of the drift ($\mu \neq 0$);
  - The effect of the detection threshold $A > 0$.
- We will consider the following parameter values:
  - $\mu = 0.5$ and $\mu = 1.5$ (since the pdf is symmetrical in μ, it suffices to consider only positive values of μ);
  - $t \in [0, 10]$;
  - $A = 10^2$ and $A = 10^3$;
  - $r \in [0, A]$.
- We will plot the two survival functions, viz. $\mathbb{P}_\infty(\mathcal{S}_A^r \geqslant t)$ and $\mathbb{P}_0(\mathcal{S}_A^r \geqslant t)$, and also their respective densities, viz. $-\partial\mathbb{P}_\infty(\mathcal{S}_A^r \geqslant t)/\partial t$ and $-\partial\mathbb{P}_0(\mathcal{S}_A^r \geqslant t)/\partial t$.
- All the calculations were performed in Mathematica. The infinite series were truncated to the first 500 terms. The eigenvalues were evaluated to within 400 decimal places.

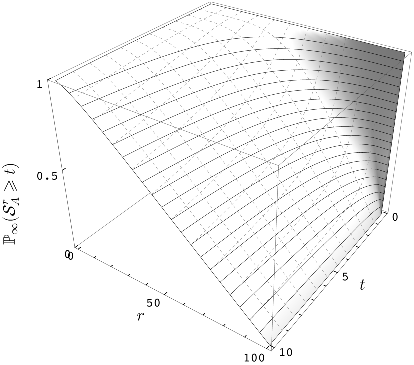
Numerical results: Pre-drift survival function density ($A = 10^2$)



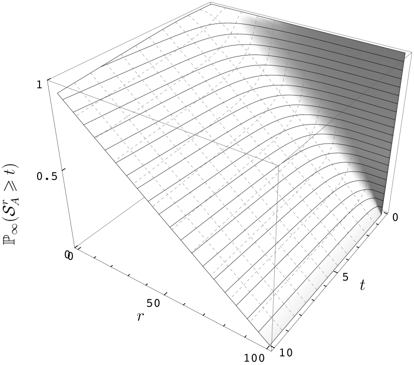
(a) $\mu = 0.5$.

(b) $\mu = 1.5$.

Figure 2. Pre-change survival function density $-\partial\mathbb{P}_\infty(\mathcal{S}_A^r \geqslant t)/\partial t$ as a function of $t \in [0, 10]$ and $r \in [0, A]$ for $A = 10^2$ and $\mu = \{0.5, 1.5\}$.



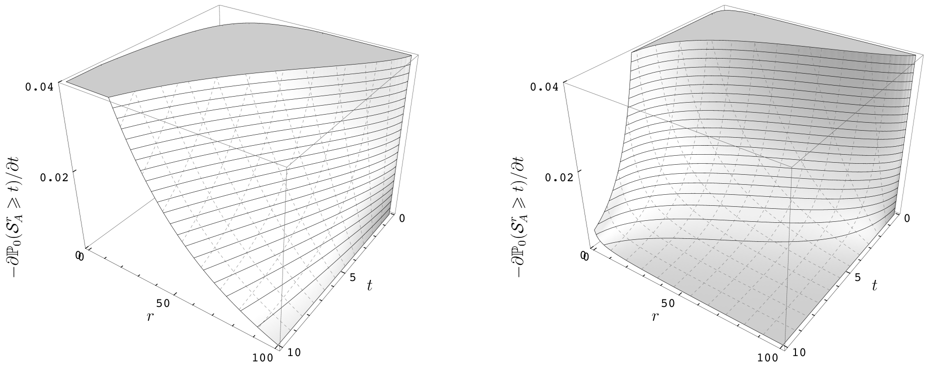
(a) $\mu = 0.5$.



(b) $\mu = 1.5$.

Figure 3. Pre-change survival function $\mathbb{P}_\infty(\mathcal{S}_A^r \geqslant t)$ as a function of $t \in [0, 10]$ and $r \in [0, A]$ for $A = 10^2$ and $\mu = \{0.5, 1.5\}$.

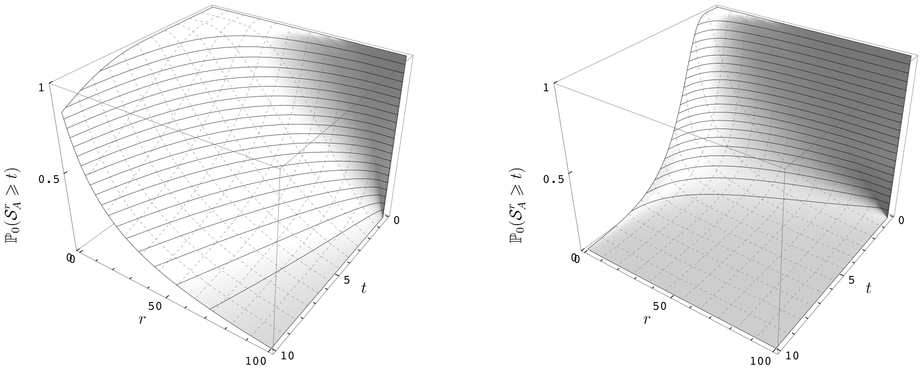
Numerical results: Post-drift survival function density ($A = 10^2$)



(a) $\mu = 0.5$. (b) $\mu = 1.5$.

Figure 4. Post-change survival function density $-\partial\mathbb{P}_0(\mathcal{S}_A^r \geqslant t)/\partial t$ as a function of $t \in [0, 10]$ and $r \in [0, A]$ for $A = 10^2$ and $\mu = \{0.5, 1.5\}$.

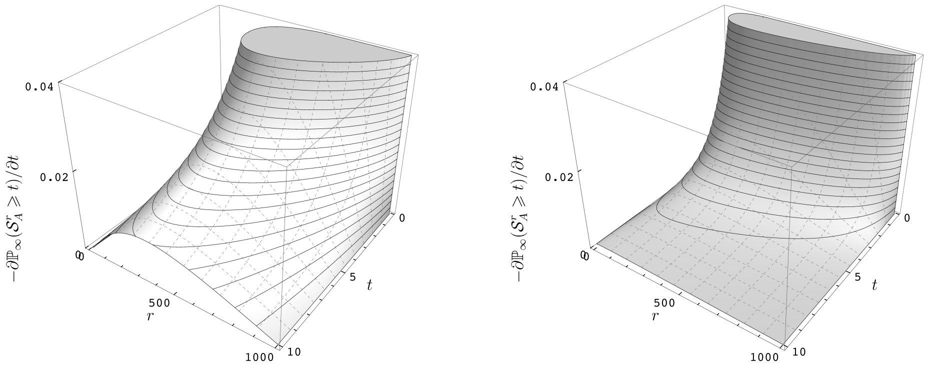
Numerical results: Post-drift survival function ($A = 10^2$)



(a) $\mu = 0.5$. (b) $\mu = 1.5$.

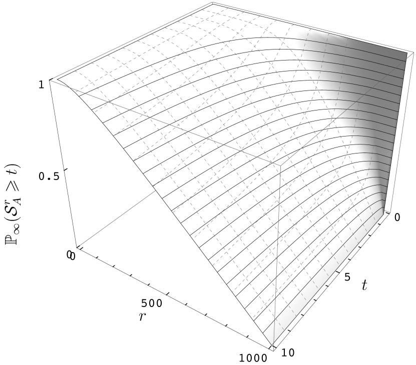
Figure 5. Post-change survival function $\mathbb{P}_0(\mathcal{S}_A^r \geqslant t)$ as a function of $t \in [0, 10]$ and $r \in [0, A]$ for $A = 10^2$ and $\mu = \{0.5, 1.5\}$.

Numerical results: Pre-drift survival function density ($A = 10^3$)

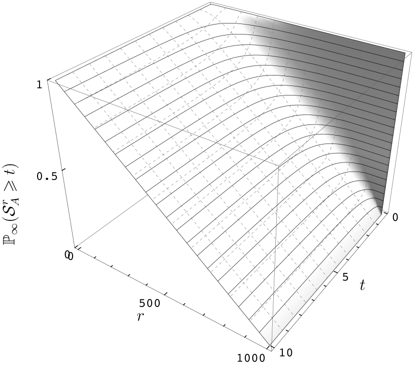


(a) $\mu = 0.5$. (b) $\mu = 1.5$.

Figure 6. Pre-change survival function density $-\partial\mathbb{P}_\infty(\mathcal{S}_A^r \geqslant t)/\partial t$ as a function of $t \in [0, 10]$ and $r \in [0, A]$ for $A = 10^3$ and $\mu = \{0.5, 1.5\}$.



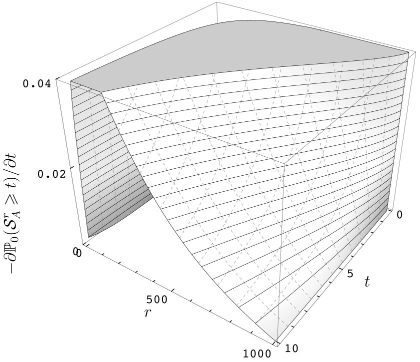
(a) $\mu = 0.5$.



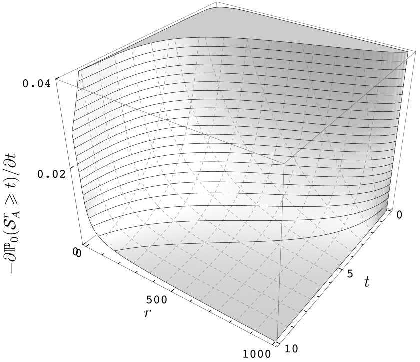
(b) $\mu = 1.5$.

Figure 7. Pre-change survival function $\mathbb{P}_\infty(\mathcal{S}_A^r \geqslant t)$ as a function of $t \in [0, 10]$ and $r \in [0, A]$ for $A = 10^3$ and $\mu = \{0.5, 1.5\}$.

Numerical results: Post-drift survival function density ($A = 10^3$)



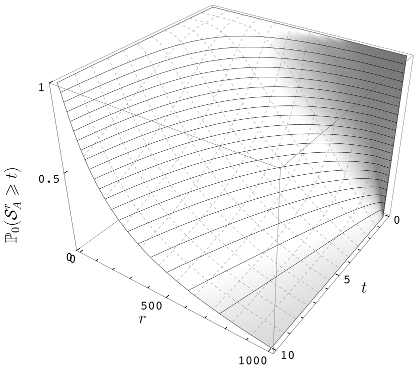
(a) $\mu = 0.5$.



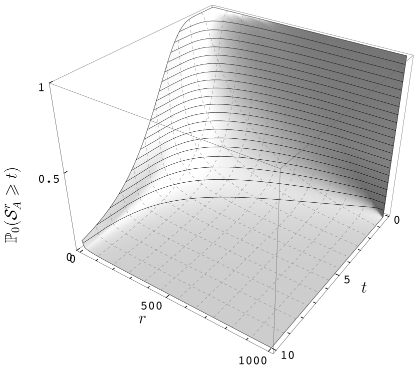
(b) $\mu = 1.5$.

Figure 8. Post-change survival function density $-\partial\mathbb{P}_0(\mathcal{S}_A^r \geqslant t)/\partial t$ as a function of $t \in [0, 10]$ and $r \in [0, A]$ for $A = 10^3$ and $\mu = \{0.5, 1.5\}$.

Numerical results: Post-drift survival function ($A = 10^3$)



(a) $\mu = 0.5$.



(b) $\mu = 1.5$.

Figure 9. Post-change survival function $\mathbb{P}_0(\mathcal{S}_A^r \geqslant t)$ as a function of $t \in [0, 10]$ and $r \in [0, A]$ for $A = 10^3$ and $\mu = \{0.5, 1.5\}$.

Where to next?

- The formulae found in this work can be used to obtained closed-form expressions for a number of other characteristics of the GSR procedure.

- First, it is straightforward to find (in a closed-form) the pre-drift quasi-stationary distribution of the GSR statistic. The cdf of this distribution is defined as $Q_A(x) \triangleq \lim_{t\to+\infty} \mathbb{P}_\infty(R_t^r \leqslant x | \mathcal{S}_A^r \geqslant t)$. The existence of this distribution has been shown by Pollak and Siegmund (1985).

- More importantly, it is possible to find (in a closed-form) the detection delay $\mathrm{ADD}_\nu(\mathcal{S}_A^r) \triangleq \mathbb{E}_\nu[\mathcal{S}_A^r - \nu | \mathcal{S}_A^r > \nu]$ for any $0 \leqslant \nu < \infty$, any $r \in [0, A]$ and any $A > 0$.

- Consequently, it is also possible to express the worst-possible delay $\mathrm{SADD}(\mathcal{S}_A^r) \triangleq \sup_{0\leqslant\nu<\infty} \mathrm{ADD}_\nu(\mathcal{S}_A^r)$.

- Finally, the insight provided by these exact formulae valid for the Brownian motion model is essential to understand what to expect in the discrete-time case.

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