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Best-choice game with incomplete information

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TV show "The Voice"

Best-choice game $\Gamma_{K,N}$, where K experts, N candidates

k experts observe a sequence of i.i.d. random variables (x_i, y_i), $i = 1 \ldots, n$, which represent the quality of incoming objects.

The players observe only the first component x_i (vocal).
y_i (attractiveness) is hidden.

Each player has a possibility to select only one contestant.

Each expert has to maximise the resultant quality $x_i + y_i$ of the selected candidate.

In the game-theoretic approach the goal of the player is to select the candidate with the resultant quality which is higher than the resultant qualities of the selected candidates by other players.

On the first step the experts observe the value x_1 from the collection of qualities (x_1, y_1) of the first candidate. Each player can choose the contestant or reject it.

If one of players makes a choice other players continue the selection.

If k players select a candidate, the contestant chooses each coach with equal probabilities $1/k$.

If the contestant is rejected by all players he or she leaves the scene.

After the players made a choice they compare the selected values $x_{i_j} + y_{i_j}, j = 1, ..., k$.

A player with the maximal value of these qualities is winner in this game.

Let $n = 2$, i.e. there are two candidates for a position.

On the first step the experts observe a value x_1.

Determine the selection rules of experts via threshold value u, so, the expert selects a candidate if $x_1 \geq u$.

Suppose that $K - 1$ palyers use the strategy v. The profile of strategies is v^{K-1}.

We find the best responce of player I u, and demand that u is equal to v.

Assume that $u \geq v$.

The payoff of first player is

$$H_{K,2}(u, v^{K-1}) = E\frac{1}{K}(I_{\{x_1 \leqslant v\}} + I_{\{x_1 \geqslant u\}}) +$$

$$+\frac{1}{K-1}P\{x_2 + y_2 \geqslant x_1 + y_1 / v < x_1 < u\} =$$

$$= \frac{1}{K-1}\left(\frac{u^4}{12} - \frac{u^3}{6} - \frac{u^2}{4} - \frac{u}{6} - \frac{v^4}{12} + \frac{v^3}{6} + \frac{v^2}{4} + \frac{v}{6} + \frac{u-v}{K}\right) + \frac{1}{K},$$

$H_{K,2}(u, v^{K-1})$ is concave on u and convex on v.

It yields the existence and uniqueness of the equilibrium.

Symmetry gives $u^* = v^*$ and

$$\frac{u^3}{3} - \frac{u^2}{2} - \frac{u}{2} - \frac{1}{6} + \frac{1}{K} = 0.$$

The value of game is $1/K$.

For $K = 2$ $u_{2,2} = u^* = v^* = 0.5$,

For $K = 3$ $u_{2,3} = u^* = v^* \approx 0.272$,

For $K = 4$ $u_{2,4} = u^* = v^* \approx 0.147$.

For $K \geq 6$ optimal thresholds are equal to zero.

There are N candidates and two experts. A chosen candidate selects an expert with probability $1/2$.

From symmetry $H_{2,N} = 1/2$ and optimal threshods are equal. .

Assume that expert II made a choice with resulting candidate's quality z..

Expert I continues the selection. This game is $\Gamma_{1,n}(z)$.

Expected payoff of I is $H_{1,n}(z)$.

Current candidate (x_1, y_1).

Let $u_{1,n}$ —threshold. .

If $x_1 < u_{1,n}$, expert I rejects the candidate and go to next step with expected payoff $H_{1,n-1}(z)$..

If $x_1 \geqslant u_{1,n}$, I accept the candidate and wins iff $x_1 + y_1 \geqslant z$.

Optimality equation

$$H_{1,n}(z) = E\{H_{1,n-1}(z) \cdot I_{\{x_1 < u_{1,n}\}} + P\{x_1 + y_1 \geqslant z / x_1 \geqslant u_{1,n}\}.$$

If $z < 1$

$$H_{1,n}(z) = H_{1,n-1}(z) \cdot u_{1,n} + \int_{u_{1,n}}^{z} (1 - (z - x_1))\, dx_1 + \int_{z}^{1} dx_1.$$

If $z \geqslant 1$ $(u_{1,n} \leqslant 1)$

$$H_{1,n}(z) = H_{1,n-1}(z) \cdot u_{1,n} + \int_{u_{1,n}}^{1} (1 - (z - x_1))\, dx_1.$$

Optimal threshold $u_{1,n}$:

$$P\{u_{1,n} + y_1 \geqslant z\} = H_{1,n-1}(z),$$

It yields

$$u_{1,n} = z - (1 - H_{1,n-1}(z)).$$

$$H_{1,n}(z) = \begin{cases} H_{1,n-1}(z) \cdot u_{1,n} + \int\limits_{u_{1,n}}^{z} (1 - (z - x_1))\, dx_1 + 1 - z, \; z < 1, \\ H_{1,n-1}(z) \cdot u_{1,n} + \int\limits_{u_{1,n}}^{1} (1 - (z - x_1))\, dx_1, \; z \geqslant 1, \end{cases} \quad (1)$$

If $n = 0$

$$H_{1,0}(z) = 0, \quad 0 \leq z \leq 2.$$

If $n = 1$ then $u_{1,1} = 0$,

$$H_{1,1}(z) = \begin{cases} 1 - \frac{z^2}{2}, \; z < 1, \\ \frac{(z-2)^2}{2}, \; z \geqslant 1. \end{cases}$$

If $n = 2$ then $u_{1,2} = H_{1,1}(z) - (1 - z)$, and

$$u_{1,2} = u_{1,2}(z) = \begin{cases} z - \frac{z^2}{2}, \; z < 1, \\ \frac{(z-1)^2}{2} + \frac{1}{2}, \; z \geqslant 1. \end{cases}$$

$$H_{1,2}(z) = \begin{cases} \frac{1}{8}(z^4 - 4z^3 + 8), \; z < 1, \\ \frac{1}{8}(z-2)^2(z^2 + 4), \; z \geqslant 1, \end{cases}$$

Simplifying

$$H_{1,n}(z) = \begin{cases} \frac{1}{2} \cdot (1 - H_{1,n-1}(z))^2 + H_{1,n-1}(z)z + 1 - z, \; z < 1, \\ \frac{1}{2} \cdot (1 - H_{1,n-1}(z))^2 + H_{1,n-1}(z)z + \frac{1}{2} \cdot ((2 - z)^2 - 1), \; z \geqslant 1, \end{cases} \tag{2}$$

Game $\Gamma_{2,N}$:

Let experts use the thresholds: u and v, $u \geqslant v$.

Payoff function:

$$H_{2,N}(u,v) = \int_0^v \frac{1}{2} \cdot dx_1 + \int_u^1 \frac{1}{2} \cdot dx_1 +$$

$$+ \int_v^u \Bigg[\int_0^{1-x_1} \Bigg[H_{1,N-1}(x_1+y_1) \cdot u_{1,N} + \int_{u_{1,N}}^{x_1+y_1} (1-(x_1+y_1-x_2))\, dx_2 +$$

$$+1-(x_1+y_1) \Bigg] dy_1 + \int_{1-x_1}^1 \Bigg[H_{1,N-1}(x_1+y_1) \cdot u_{1,N} +$$

$$+ \int_{u_{1,N}}^1 (1-(x_1+y_1-x_2))\, dx_2 \Bigg] dy_1 \Bigg] dx_1.$$

$H_{2,N}(u,v)$ is concave in u and convex in v. Optimal thresholds:

$$\int_0^{1-u^*}\left[H_{1,N-1}(u+y_1)\cdot u_{1,N}+\int_{u_{\mathbf{1},N}}^{u+y_{\mathbf{1}}}(1-(u+y_1-x_2))\,dx_2+1-(u+y_1)\right]dy_1+$$

$$+\int_{1-u^*}^{1}\left[H_{1,N-1}(u+y_1)\cdot u_{1,N}+\int_{u_{\mathbf{1},N}}^{1}(1-(u+y_1-x_2))\,dx_2\right]dy_1=\frac{1}{2}.$$

Best-choice game $\Gamma_{2,N}$ with two experts

n	2	3	4	5	6
$u^* = v^*$	0.5	0.637	0.711	0.757	0.790

General case: N candidates and K experts.

Stage: n candidates and k experts, and z is current maximal quality, $0 \leq z \leq 2$. This game is $\Gamma_{k,n}(z)$, its value is $H_{k,n}(z)$. Notice $H_{K,N} = H_{K,N}(0)$.

Symmetry: optimal strategies are equal and $H_{k,n}(z) \leq 1/k$.

Payoff function in game:

$$H_{k,n}(z) = Val\{H_{k,n}(u, v^{k-1}|z)\},$$

where $(v \leq u)$

$$H_{k,n}(u, v^{k-1}|z) = \int_0^v H_{k,n-1}(z)dx_1 + \int_v^u dx_1 \int_0^1 H_{k-1,n-1}(z \vee x_1 + y_1)dy_1 +$$

$$+ \int_u^1 dx_1 \left[\frac{1}{k} \int_{(z-x_1)\vee 0}^1 (1-(k-1)H_{k-1,n-1}(x_1+y_1))\, dy_1 + \right.$$

$$\left. + \left(1 - \frac{1}{k}\right) \int_0^1 H_{k-1,n-1}(z \vee x_1 + y_1)dy_1 \right],$$

where $a \vee b = \max(a, b)$.

Best-choice game $\Gamma_{K,N}$

$K = 3$	$K = 2$
$u_{3,1} = 0$	$u_{2,1} = 0$
$u_{3,2} = 0.272$	$u_{2,2} = 0.5$
$u_{3,3} = 0.465$	$u_{2,3} = 0.637$
$u_{3,4} = 0.591$	$u_{2,4} = 0.711$

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