One-day conference
dedicated to the memory of academician A.A. Gonchar
December 23, 2015, MIAN, Gubkina, 8, Moscow

Symmetrization of condensers and geometrical properties of multivalent functions

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Covering theorem for univalent functions

S is the class of functions f analytic and univalent in the unit disk $U = \{z : |z| < 1\}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The leading example of a function of class S is the Koebe function

$$k(z) = z(1-z)^{-2} = z + 2z^2 + 3z^3 + ...$$

Theorem(Koebe [1907], Bieberbach [1916]). *The range of every function of class S contains the disk $\{w : |w| < 1/4\}$.*

Two examples

$$f_1(z) = \frac{k(z)}{1 - \lambda k(z)} = z + a_2 z^2 + ...;$$

$$f_1(z) \neq f_1(-1) = -\frac{1}{4 + \lambda}.$$

$$f_2(z) = \frac{1 - (1 - z)^p}{p} = z + b_2 z^2 + ...;$$

$$f_2(z) \neq \frac{1}{p} \quad for \;\; z \in U.$$

Multivalent functions

A function f in a domain G in the complex plane $\overline{\mathbb{C}}_z$ is said to be *p-valent* in G, $p = 1, 2, \dots,$ if it takes each complex value w at most p times in G.

Let a function f be meromorphic in the disk $U = \{z : |z| < 1\}$ and distinct from a constant and let $n(w, f)$ denote the number of roots of the equation $f(z) = w$ in U. The function f is said to be *circumferentially mean p-valent* (c.m. p-valent) if for any $\rho > 0$ the inequality

$$\frac{1}{2\pi}\int\limits_0^{2\pi} n(\rho e^{i\varphi}, f)\, d\varphi \leqslant p$$

holds.

Growth, distortion and covering theorem

Theorem(Hayman [1950]). *Suppose that* $f(z) = z^p + a_{p+1}z^{p+1} + \ldots.$ *is holomorphic c.m.p-valent in* U. *Then* $|a_{p+1}| \le 2p$. *Further we have for* $|z| = r$ $(0 < r < 1)$

$$\frac{r^p}{(1+r)^{2p}} \le |f(z)| \le \frac{r^p}{(1-r)^{2p}},$$

$$|f'(z)| \le \frac{p(1+r)}{r(1-r)}|f(z)| \le \frac{pr^{p-1}(1+r)}{(1-r)^{2p+1}}.$$

Finally the equation $f(z) = w$ *has exactly* p *roots in* U *if* $|w| < 4^{-p}$.

- Functions with a zero of order p at the origin

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- Functions with a restriction on the covering of a disk

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- Functions with a restriction on the covering of a disk
- Functions with Montel's normalization

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- Examples of applications

Class $S_p(\tau)$, $0 < \tau < \infty$.

$S_p(\tau)$ is the class of the c.m. p-valent functions of the form

$$f(z) = z^p + a_{p+1} z^{p+1} + \dots, \qquad |z| < t \leqslant 1,$$

such that the total multiplicity of their poles in U does not exceed $p-1$ and the moduli of their nonzero critical values are greater than or equal to τ (a critical value is the value $f(z)$ of f at a point where $f'(z) = 0$).

$$f(z; p, \tau) := \tau \left[T_p \left(\frac{(1-z)^2}{2z} (2\tau)^{1/p} - \cos \frac{\pi}{2p} \right) \right]^{-1} \in S_p(\tau),$$

where $T_p(\zeta) = 2^{p-1}\zeta^p + \dots$ is the Chebyshev polynomial.

Theorem 1. *For each function f in $S_p(\tau)$*

$$|f(z)| \geqslant |f(-|z|; p, \tau)|$$

for any $z \in U$. The equality holds for the function $f(z; p, \tau)$ at points z in the interval $-1 < z \leqslant 0$.

Corollary 1. *If $f \in S_p(\tau)$, then the radius ρ_f of the maximal disc centred at the origin which is covered by f with multiplicity p satisfies*

$$\rho_f \geqslant \frac{\tau}{\left|T_p(\cos\frac{\pi}{2p} + 2(2\tau)^{1/p})\right|}.$$

The equality holds for $f(z) \equiv f(z; p, \tau)$.

Corollary 2. *If* $f(z) = z^p + a_{p+1}z^{p+1} + \dots$ *is a function in the class* $S_p(\tau)$*, then*

$$|a_{p+1}| \leqslant 2p\left[1 + (2\tau)^{-1/p}\cos\frac{\pi}{2p}\right].$$

The equality holds for $f(z) \equiv f(z; p, \tau)$*.*

Theorem 2. *If* f *is a function in the class* $S_p(\tau)$*, then for* $0 < |z| < 1$*,*

$$\tau|z|\frac{1-|z|}{1+|z|}\left|\frac{f'(z)}{f^2(z)}\right| \leqslant \left|T_p'\left(T_p^{-1}\left(\frac{\tau}{|f(z)|}\right)\right)\right|\left(T_p^{-1}\left(\frac{\tau}{|f(z)|}\right) + \cos\frac{\pi}{2p}\right),$$

where the value $T_p^{-1}(\tau/|f(z)|) \geqslant \cos(\pi/(2p))$*. The equality holds for function* $f(z; p, \tau)$ *for any* z*,* $0 < z < \tilde{z}$*, where* $\tilde{z}$*,* $0 < \tilde{z} < 1$*, is the root of the equation*

$$(1-z)^2(2\tau)^{1/p} = 4z\cos\frac{\pi}{2p}.$$

Theorem 3. *If a function*

$$f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \dots$$

belongs to the class $S_p(\tau)$, *then*

$$\left| a_{p+2} - \frac{p+1}{2p} a_{p+1}^2 \right| \leqslant p \left(1 + \frac{1}{(2\tau)^{2/p}} \right). \qquad (1)$$

Formula (1) *becomes an equality, for example, for the function*

$$f(z) = \frac{\tau}{T_p((1 - z^2)(2\tau)^{1/p}/(2z) - \cos(\pi/(2p)))}.$$

$D_p(\lambda)$ is the class of the holomorphic c.m. p-valent functions f such that for each $\rho \geqslant \lambda$ the set $\{z \in U : f'(z) \neq 0, |f(z)| = \rho\}$ contains no a closed curves. In other words: the Riemann surface $\mathscr{R}(f)$ contains no a closed k-valent disk, $k \leqslant p$, branching over the disk $|w| \leqslant \lambda$.

$D_p(0)$ is the class of the holomorphic c.m. p-valent functions without zeros in U, $D_p(0) \subset D_p(\lambda)$, $\lambda \geqslant 0$.

Theorem 4. *If f is a function in the class $D_p(\lambda)$, $\lambda > 0$, then the following sharp estimate holds for each z in the disc U:*

$$(1-|z|^2)|f'(z)| \leqslant 4\lambda \left[1 + T_p^{-1}\left(\left|\frac{f(z)}{\lambda}\right|\right)\right] T_p'\left(T_p^{-1}\left(\left|\frac{f(z)}{\lambda}\right|\right)\right),$$

where the value $T_p^{-1}(|f(z)/\lambda|) \geqslant \cos(\pi/(2p))$. The equality holds for

$$f(z) = \lambda T_p\left(c\left(\frac{1+z}{1-z}\right)^2 - 1\right)$$

on the interval $0 < z < 1$, where $c \geqslant 1 + \cos(\pi/(2p))$ is arbitrary.

Corollary 3. *If* $f(z) = a_0 + a_1 z + \dots$ *is a function in the class* $D_p(\lambda)$, $\lambda > 0$, *then*

$$|a_1| \leqslant 4\lambda \left[1 + T_p^{-1}\left(\left|\frac{a_0}{\lambda}\right|\right)\right] T_p'\left(T_p^{-1}\left(\left|\frac{a_0}{\lambda}\right|\right)\right).$$

The equality holds for the functions from Theorem 4.

Letting λ approach 0 we obtain Hayman's inequality

$$|a_1| \leqslant 4p|a_0|,$$

which holds for each holomorphic c.m. p-valent function f without zeros in U. The equality holds for the function

$$f(z) = a_0[(1+z)/(1-z)]^{2p}.$$

Theorem 5. *Let a function f belongs to the class $D_p(\lambda)$, $\lambda > 0$. In this case, for any two points z_1 and z_2 in the disk U, the following inequality holds*:

$$(1-|z_1|^2)|f'(z_1)|(1-|z_2|^2)|f'(z_2)|\left|\frac{z_1-z_2}{1-\overline{z}_1 z_2}\right|^2 \leqslant$$

$$\leqslant \lambda^2 T_p'\left(T_p^{-1}\left(\left|\frac{f(z_1)}{\lambda}\right|\right)\right) T_p'\left(T_p^{-1}\left(\left|\frac{f(z_2)}{\lambda}\right|\right)\right) \times$$

$$\times \left[T_p^{-1}\left(\left|\frac{f(z_1)}{\lambda}\right|\right) + T_p^{-1}\left(\left|\frac{f(z_2)}{\lambda}\right|\right)\right]^2. \qquad (2)$$

Formula (2) becomes an equality, for example, for functions of the form

$$f(z) = \lambda T_p \left(\frac{az}{1 - z^2} \right), \quad z \in U,$$

for all positive a such that $|T_p(ai/2)| \geqslant 1$ *and for any points* z_1, z_2 *such that*

$$z_1 = -z_2 = x, \;\; 1 > x \geqslant -\frac{a}{\cos(\pi/(2p))} + \sqrt{\frac{a^2}{4\cos^2(\pi/(2p)) + 1}}.$$

Corollary 4. *Suppose that a holomorphic and p-valent function f does not vanish in this disk U. Then the inequality*

$$(1-|z_1|^2)|f'(z_1)|(1-|z_2|^2)|f'(z_2)|\left|\frac{z_1-z_2}{1-\overline{z}_1 z_2}\right|^2 \leqslant$$

$$\leqslant p^2|f(z_1)f(z_2)|\left[\left|\frac{f(z_1)}{f(z_2)}\right|^{1/(2p)}+\left|\frac{f(z_2)}{f(z_1)}\right|^{1/(2p)}\right]^2$$

holds for any two points z_1 and z_2 in U. The formula becomes an equality, for example, for the function $f(z)=[(1+z)/(1-z)]^{2p}$, and points z_k, $[(1+z_k)/(1-z_k)]^2=(-1)^k it,\ \ t>0,\ \ k=1,2.$

Theorem 6. *If $f(z) = a_0 + a_1 z + \dots$ is a function in the class $D_p(\lambda)$, $\lambda > 0$, then for each z, $z \in U$,*

$$|f(z)| \leqslant \lambda T_p \left[T_p^{-1}\left(\left|\frac{a_0}{\lambda}\right|\right) + \frac{4|z|}{(1-|z|)^2}\left(T_p^{-1}\left(\left|\frac{a_0}{\lambda}\right|\right) + 1\right)\right],$$

with equality as in Theorem 4.

As might be expected, letting $\lambda \to 0$ we arrive at Hayman's result

$$|f(z)| \leqslant |a_0| \left(\frac{1+|z|}{1-|z|}\right)^{2p}.$$

Montel's normalization

$M_p(\omega)$ is the class of the holomorphic c.m. p-valent functions in the disk U normalized by $f(0) = 0$, $f(\omega) = \omega$ $(0 < \omega < 1)$.

Let $\mathscr{R}(f)$ be the Riemann surface of the function inverse f, $f \in M_p(\omega)$.

Theorem 7. *For every function f from the class $M_p(\omega)$, $p \geqslant 2$, the Riemann surface $\mathscr{R}(f)$ contains an open k-valent disk, $k \leqslant p$, branching over the disk*

$$|w| < \rho(p, \omega) := \frac{\omega}{T_p\left[\frac{4\omega + (1+\omega)^2 \cos(\pi/(2p))}{(1-\omega)^2}\right]}.$$

The constant $\rho(p, \omega)$ is the best possible.

Two examples

For fixed ω, r $(0 < r < 1)$ and $p \geq 2$

$$f_1(z) = \omega \frac{T_p\left[\frac{2z(1+r)^2}{r(1-z)^2}\cos\frac{\pi}{2p} + \cos\frac{\pi}{2p}\right]}{T_p\left[\frac{2\omega(1+r)^2}{r(1-\omega)^2}\cos\frac{\pi}{2p} + \cos\frac{\pi}{2p}\right]};$$

$f_1(z) \in M_p(\omega)$ and $f_1(-r) = 0$.

For fixed ω, r, p and $\lambda > 0$

$$f_2(z) \equiv f(z;\omega,p,\lambda) = \lambda T_p\left[\frac{z(1+\omega)^2}{\omega(1+z)^2}\left(T_p^{-1}\left(\frac{\omega}{\lambda}\right) + \cos\frac{\pi}{2p}\right) - \cos\frac{\pi}{2p}\right];$$

$|f(-r)| \to \infty$ when $\lambda \to \infty$.

Montel's normalization

Theorem 8. *If a function f belongs to the class $M_p(\omega)$ and $\mathscr{R}(f)$ contains no a closed k-valent disk, $k < p$, over the disk $|w| \leqslant \lambda, \lambda > 0$, then, for any $z \in (-1, 0)$,*

$$|f(z)| \leqslant |f(z; \omega, p, \lambda)|,$$

where

$$f(z; \omega, p, \lambda) = \lambda T_p \left[\frac{z(1+\omega)^2}{\omega(1+z)^2} \left(T_p^{-1}\left(\frac{\omega}{\lambda}\right) + \cos\frac{\pi}{2p} \right) - \cos\frac{\pi}{2p} \right].$$

Theorem 9. *With the hypotheses of Theorem 8 we have*

$$|f'(0)| \leq \frac{\lambda p(1+\omega)^2}{\omega \sin(\pi/(2p))} \left(T_p^{-1}(\omega/\lambda) + \cos\frac{\pi}{2p} \right),$$

where $T_p^{-1}(\omega/\lambda) \geq \cos(\pi/(2p))$.

Equality is attained only for $f(z) = f(z; \omega, p, \lambda)$.

Theorem 10. *Suppose that all critical values of the polynomial*

$$P(z) = c_0 + c_1 z + \ldots + c_n z^n, \;\; c_n \neq 0, \;\; n \geqslant 2,$$

belong to the disk $|w| \leqslant 1$. *Then for any point* z *the following inequality holds*:

$$|P'(z)| \leqslant 2^{\frac{1-n}{n}} |c_n|^{\frac{1}{n}} T_n'(T_n^{-1}(|P(z)|)),$$

where $T_n^{-1}(|P(z)|)$ *is situated on the ray* $[\cos(\pi/(2n)), +\infty]$. *Equality is attained, for instance, for* $P = T_n$ *and any real* z, $|z| \geqslant \cos(\pi/(2n))$.

Markov-type inequality

$$M = \sup\{|P(z)| : z \in E\},$$
$$M_c = \sup\{|P(z)| : P'(z) = 0\}.$$

Theorem 11. *Let E be a bounded set in the complex plane, and P a polynomial of degree n. Then*

$$\operatorname{cap} E \sup_E |P'| \leq \left(\frac{2M}{M_c}\right)^{\frac{1-n}{n}} T_n'\left(T_n^{-1}\left(\frac{M}{M_c}\right)\right) M.$$

Equality is attained, for instance, for $P = M_c T_n$ and $E = \{z : |P(z)| \leq M\}$, where M_c and M are an arbitrary positive numbers.

Comparison with Eremenko's inequality

$$\operatorname{cap} E \sup_E |P'| \le \left(\frac{2M}{M_c}\right)^{\frac{1-n}{n}} T_n'\left(T_n^{-1}\left(\frac{M}{M_c}\right)\right) M.$$

Theorem (Eremenko [2007]). *Let E be a continuum in the complex plane, and P a polynomial of degree n. Then*

$$\operatorname{cap} E \sup_E |P'| \le 2^{1/n-1} n^2 \sup_E |P|.$$

$$(\sup_E |P| = M)$$

Lower bound for moduli of critical values

Theorem 12. *For any polynomial* $P(z) = c_1 z + ... + c_n z^n$, $c_1 \neq 0$, $c_n \neq 0$, *there exist a critical value* $P(\zeta)$ $(P'(\zeta) = 0)$ *such that*

$$|P(\zeta)| \geq 2 \left(\frac{1}{n} \sin \frac{\pi}{2n} \right)^{\frac{n}{n-1}} \left| \frac{c_1^n}{c_n} \right|^{\frac{1}{n-1}}.$$

Equality is attained for the polynomial $T_n(z - \cos(\pi/(2n)))$ *whose critical values are unimodal.*

Upper bound for moduli of critical values

Theorem 13. *For any polynomial* $P(z) = c_1 z + ... + c_n z^n$, $c_1 \neq 0$, $c_n \neq 0$, *there exist a critical value* $P(\zeta)$ *such that*

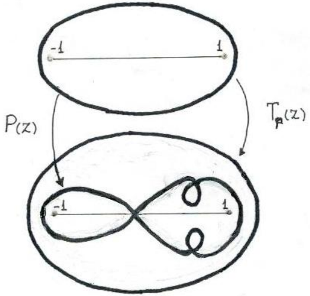
$$|P(\zeta)| \leq (n-1) \left(\frac{1}{n}\right)^{\frac{n}{n-1}} \left|\frac{c_1^n}{c_n}\right|^{\frac{1}{n-1}}.$$

Equality holds for $P(z) = c_1 z + c_n z^n$.

Chebyshev polynomial

Recall that the Chebyshev polynomial of the first kind $T_p(z) = 2^{p-1}z^p + \ldots$ can be defined in terms of conformal maps as the composite of the inverse Zhukovskii function, a power function, and Zhukovskii function:

$$T_p(z) = \frac{1}{2}\left((z + \sqrt{z^2 - 1})^p + (z - \sqrt{z^2 - 1})^p\right), \quad z \in \mathbb{C}.$$

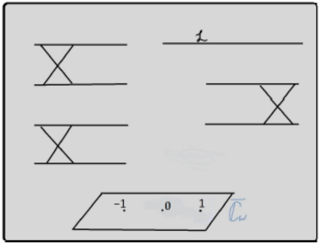


Riemann surfaces

Throughout we consider compact Riemann surfaces with boundary. We also treat such a surfaces as 'glued' from planar domains, so that for points on a surface we have naturally defined projections and local parameters. Let $\gamma(\rho) = \{w : |w| = \rho\}$, $0 \leqslant \rho \leqslant \infty$. Then $\mathfrak{R}_p$, $p \geqslant 1$, will denote the class of Riemann surfaces $\mathscr{R}$ over the complex w-sphere which satisfy the following conditions:
1) taking account of multiplicities, the total linear measure of any system of arcs on $\mathscr{R}$ which lies over an arbitrary circle $\gamma(\rho)$, $0 < \rho < \infty$, has the estimate $2\pi p\rho$;
2) for $1 \leqslant \rho < \infty$, any closed Jordan curve on $\mathscr{R}$ over a circle $\gamma(\rho)$ which does not pass through ramification points of $\mathscr{R}$ covers this circle with multiplicity p.

A special case of a surface in $\mathfrak{R}_p$, which is important for us, is the Riemann surface $\mathscr{R}(T_p)$ of the inverse function of the Chebyshev polynomial T_p. Let us describe this surface for $p \geqslant 2$. The hyperbolas with foci at $z = \pm 1$ which pass through the critical points $z = \cos(k\pi/p)$, $k = 1, \ldots, p-1$, of the Chebyshev polynomial partition the z-plane into p paiwise disjoint domains. Let $B_1, \ldots, B_p$ be these domains from right to left. The polynomial T_p maps B_1 conformally and univalently onto the domain D_1 equal to the w-plane cut along the ray $L^- := [-\infty, -1]$. The same polynomial takes the domains $B_2, \ldots, B_{p-1}$ to $D_2, \ldots, D_{p-1}$, copies of the w-plane cut along the rays L^- and $L^+ := [1, +\infty]$. Finally, B_p is mapped onto D_p, the w-plane cut along the ray L^- if p is even or along L^+ if p is odd.

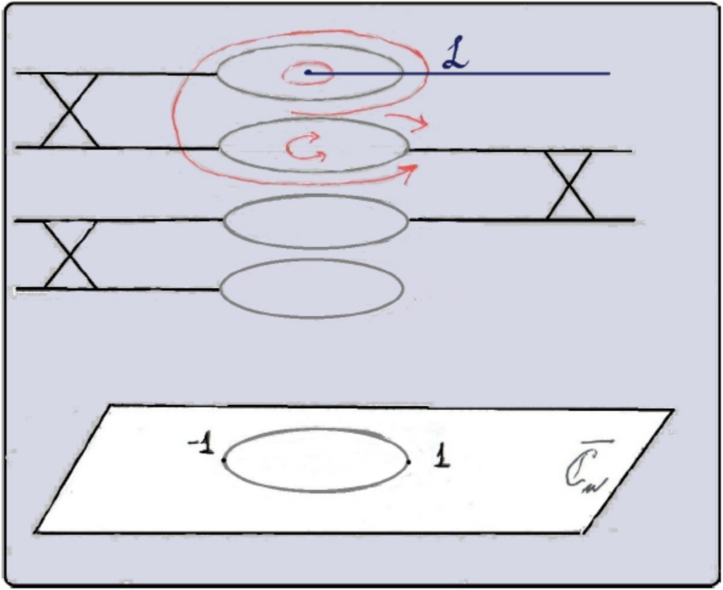
We construct the Riemann surface $\mathscr{R}(T_p)$ by gluing together the domain D_k, $k = 1, \ldots, p$, as follows: D_1 is glued crosswise to D_2 along the sides of the cuts made along the ray L^-. Domain D_2 is glued to D_3 along the sides of the cuts made along the ray L^+, and so on. Domain D_{p-1} is glued to D_p through L^- if p is even and though L^+ is p is odd. The domain D_k viewed as a subset of the $\mathscr{R}(T_p)$ will be denoted by $\mathscr{D}_k$. Let $\mathscr{L}$ denote the 'ray' over $[0, +\infty]$ on $\mathscr{D}_1$.



Symmetrization of an open set

Let $\mathscr{B}$ be an open set in $\mathscr{R} \in \mathfrak{R}_p$. Then symmetrization Sym transforms $\mathscr{B}$ into a subset $\mathrm{Sym}\,\mathscr{B}$ of $\mathscr{R}(T_p)$ with the following properties. Fix some ρ, $0 \leqslant \rho \leqslant \infty$. If no points in $\mathscr{B}$ lie over the circle $\gamma(\rho)$, then no points in $\mathrm{Sym}\,\mathscr{B}$ lie over it either. If $\mathscr{B}$ covers $\gamma(\rho)$, $1 \leqslant \rho \leqslant \infty$, with multiplicity p, the $\mathrm{Sym}\,\mathscr{B}$ also covers $\gamma(\rho)$ with multiplicity p. If $\mathscr{B}$ covers $\gamma(\rho)$, $0 \leqslant \rho < 1$, with multiplicity $l \leqslant p$, then the part of $\mathrm{Sym}\,\mathscr{B}$ over $\gamma(\rho)$consist of l circles lying on the sheets $\mathscr{D}_1, \ldots, \mathscr{D}_l$. In the other cases, for $1 \leqslant \rho < \infty$ the part of $\mathrm{Sym}\,\mathscr{B}$ lying over $\gamma(\rho)$ is an open arc on $\mathscr{R}(T_p)$ with midpoint on the ray $\mathscr{L}$ and with linear measure equal to the measure of $\mathscr{B}(\rho) := \{W \in \mathscr{B} : |\mathrm{pr}\,W| = \rho\}$. For $0 < \rho < 1$ the part of $\mathrm{Sym}\,\mathscr{B}$ over $\gamma(\rho)$ is a union of m circle $\Gamma_1, \ldots, \Gamma_m$, $0 \leqslant m \leqslant p-1$, and an open arc Γ_{m+1} such that $\Gamma_k = \Gamma_k(\mathscr{B}, \rho) \subset \mathscr{D}_k$, $k = 1, \ldots, m+1$; the total linear measure of these curves is equal to the measure of $\mathscr{B}(\rho)$, and the midpoint of Γ_{m+1} lies over $(-1)^m \rho$. If the measure of $\mathscr{B}(\rho)$ is less than $2\pi\rho$, then necessarily $m = 0$, and there are no full circle.

Symmetrization of an open set



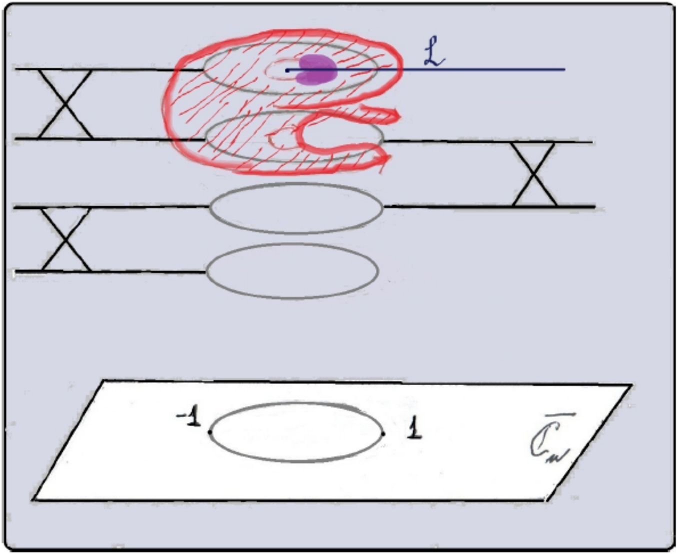
A *condenser* on the surface $\mathscr{R}$ is an ordered pair of sets $\mathscr{C} = (\mathscr{B}, \mathscr{E})$, where $\mathscr{R}$ is an open subset of $\mathscr{R}$ and $\mathscr{E}$ is a compact subset of $\mathscr{B}$. We call $\mathscr{B} \backslash \mathscr{E}$ the field of the condenser. The *capacity* $\mathrm{cap}\mathscr{C}$ of the condenser $\mathscr{C}$ is defined by

$$\mathrm{cap}\,\mathscr{C} = \inf \int\limits_{\mathscr{B}} | \bigtriangledown \mathscr{V} |^2 d\sigma,$$

where the infimum is taken over all real-valued functions $\mathscr{V}$ which have compact support in $\mathscr{B}$, are equal to 1 on $\mathscr{E}$ and are locally Lipschitz in $\mathscr{B}$.

$$\mathrm{Sym}\,\mathscr{C} = (\mathrm{Sym}\,\mathscr{B}, \mathrm{Sym}\,\mathscr{E}).$$

Symmetrization of condenser

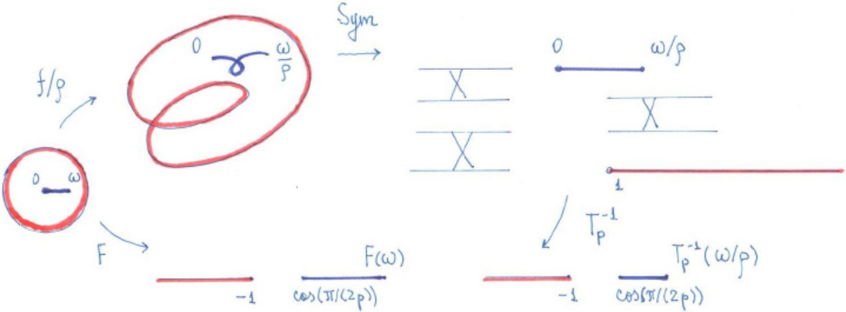


Theorem 14. *For each condenser $\mathscr{C}$ on a surface $\mathscr{R}$ in the class $\mathfrak{R}_p$*

$$\operatorname{cap}\mathscr{C} \geqslant \operatorname{cap}\operatorname{Sym}\mathscr{C}. \tag{3}$$

In addition, if $\mathscr{C} = (\mathscr{B}, \mathscr{E})$ has a connected field and the potential function of $\mathscr{C}$ exists, then equality in (3) *holds only in the following cases*:
(i) *the field of $\mathscr{C}$ coincides with the field of $\operatorname{Sym}\mathscr{C}$ up to rotations about the origin*;
(ii) *for some s, t and $l, 0 < s < t < \infty$, $1 < l \leqslant p$, the field $\mathscr{C}$ covers a circular annulus $s < |w| < t$ with multiplicity l so that the inverse image of each boundary circle of the annulus is either formed entirely by boundary points of $\mathscr{B}$ or is formed entirely by boundary points of $\mathscr{E}$.*

The proof of Theorem 7



$F(\omega) \geq T_p^{-1}(\omega/\rho)$.

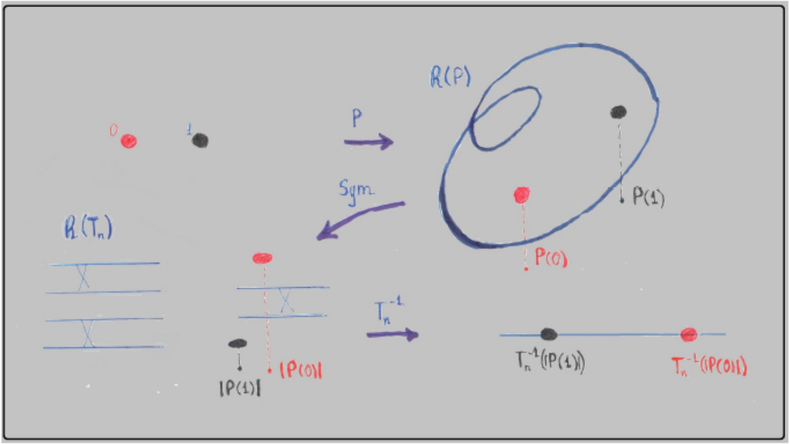
Denote by $\mathscr{P}_n$ the set of all polynomials of degree $n \geqslant 2$ the moduli of whose critical values are at most 1; and let T_n^{-1} be the continuous branch of the inverse function of T_n defined on the ray $[0, +\infty]$ and taking this ray onto the ray $[\cos(\pi/(2n)), +\infty]$.

Theorem 15. *If a polynomial P belongs to $\mathscr{P}_n$ then*

$$|P'(0)P'(1)| \leqslant$$
$$\leqslant T_n'(T_n^{-1}(|P(0)|))T_n'(T_n^{-1}(|P(1)|))[T_n^{-1}(|P(0)|) + T_n^{-1}(|P(1)|)]^2.$$

The equality is attained, for example, if $P(z) = T_n(\alpha z + \beta)$, for all real α and β satisfying the conditions $\alpha + \beta \geqslant \cos(\pi/(2n))$ and $\beta \leqslant -\cos(\pi/(2n))$.

Proof




Thank you for attention!