Problems on Random Matrices

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Outline

1. Three problems on Random Matrices
   - Singular Deformation on the Jacobi Ensembles, $\gamma = 2$.
   - Linear statistics of Matrix Ensembles in Classical Background, $\gamma = 1, 4$.
   - Small eigenvalues of **LARGE** Hankel matrices.

Multiple integrals

$$D_n[w,\gamma] := \frac{1}{n!}\int_{[A,B]^n}\prod_{1\le j<k\le n}|x_k - x_j|^{\gamma}\prod_{l=1}^{n} w(x_l)dx_l,$$

Value of γ	Symmetry type	Matrix entries
$\gamma = 1$	Orthogonal	Real
$\gamma = 2$	Unitary	Complex
$\gamma = 4$	Symplectic	Quaternion

Hankel Matrices, determinants, related objects.

Hankel or Moment Matrices of order n, $(\mu_{j+k}[w])_{0 \le j,k \le n-1}$,

$$\mu_j[w] := \int_A^B x^j \, w(x) dx, \quad j = 0, 1, \ldots$$

Assume w, smooth positive function, supported on $[A, B]$.
For $\gamma = 2$, we have **Hankel determinant**,

$$D_n[w, 2] = \det(\mu_{j+k})_{0 \le j,k \le n-1}$$

Deformed Jacobi Weight, with Chen Min.

The **deformed Jacobi weight**: Obtained by multiplying the Jacobi weight $x^{\alpha}(1-x)^{\beta}, \ \ x \in [0,1], \ \ \alpha, \ \beta > -1$, by $\mathrm{e}^{-t/x}$,

$$w(x,t) := x^{\alpha}(1-x)^{\beta}\mathrm{e}^{-t/x}, \quad x \in [0,1], \ t \geq 0,$$

and

$$D_n(t;\alpha,\beta) = \det\left(\int_0^1 x^{j+k}\, x^{\alpha}(1-x)^{\beta}\mathrm{e}^{-t/x}dx\right)_{0\leq j,k\leq n-1}.$$

Remark 1. If $t > 0$, then we may take $\alpha \in \mathbb{R}$.
Remark 2. The factor, $\mathrm{e}^{-t/x}$, introduces an infinitely strong zero on the weight at 0.

Recall some results.

$$H_n(t;\alpha,\beta) := t\frac{d}{dt}\ln\frac{D_n(t;\alpha,\beta)}{D_n(0;\alpha,\beta)},$$

satisfies the $\sigma-$ form of a Painlevé V,

$$(tH_n'')^2 = [n(n+\alpha+\beta) - H_n + (\alpha+t)H_n']^2 + 4H_n'(tH_n' - H_n)(\beta - H_n'),$$

and

$$H_n(t;\alpha\beta) = \int_0^t F(y(\lambda), y'(\lambda), \alpha, \beta, n, \lambda)d\lambda,$$

where

$$y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y'}{t} + \frac{(2n+1+\alpha+\beta)^2(y-1)^2 y}{2t^2} - \frac{(y-1)^2\beta^2}{2t^2 y} + \frac{\alpha y}{t} - \frac{y(y+1)}{2(y-1)}.$$

Remark. $P_V(\frac{(2n+1+\alpha+\beta)^2}{2}, -\frac{\beta^2}{2}, \alpha, -\frac{1}{2})$. **Note n in the equation.**

Double Scaling.

Double scaling: $n \to \infty, \ t \to 0,$ **such that** $s := 2n^2\, t$ **finite.**

Theorem 1

Let

$$y(t, \alpha, \beta, n) = 1 + \frac{1}{n^2}\, f\left(\frac{s}{2n^2}, \alpha, \beta, n\right),$$

then

$$g(s, \alpha, \beta) = \lim_{n \to \infty} f\left(\frac{s}{2n^2}, \alpha, \beta, n\right),$$

satisfies,

$$g'' = \frac{g'^2}{g} - \frac{g'}{s} + \frac{2g^2}{s^2} + \frac{\alpha}{2s} - \frac{1}{4g},$$

a "lesser" PIII.

Double Scaling.

The "double scaled" Hankel determinant

If,

$$\Delta(s,\alpha,\beta) := \lim_{n\to\infty} \frac{D_n(s/(2n^2),\alpha,\beta)}{D_n(0,\alpha,\beta)},$$

then

$$\ln \Delta(s,\alpha,\beta) = \int_0^s \left(\frac{(\xi\, g' - g)^2}{4\xi\, g^2} + \frac{4\alpha\xi\, g - \xi^2}{16\xi\, g^2} - \frac{4g + \alpha^2}{4\xi} \right) d\xi.$$

For large s,

$$\Delta(s,\alpha,\beta) \sim \exp\left[c(\alpha) - \frac{9}{8} s^{2/3} + \frac{3\alpha}{2} s^{1/3} + \frac{1-6\alpha^2}{36} \ln s + \frac{\alpha(1-\alpha^2)}{18\, s^{1/3}} + \ldots \right],$$

where

$$c(\alpha) = \ln \frac{G(1+\alpha)}{(2\pi)^{\alpha/2}}, \;\; G(\cdot) \;\; \textit{is} \;\; \text{Barnes G} - \text{function}.$$

Properties of Barnes-G function I.

E. W. Barnes (1899, 1900, 1901, 1904). Properties of Barnes-G function,
(1) $G(z+1) = \Gamma(z)G(z) \quad (z \in C)$;
(2) $G(1) = 1$;
(3) As $n \to \infty$,

$$\log G(z+n+2) = \frac{(n+1+z)\log(2\pi)}{2} - \frac{3z^2}{4} - n - nz - \log \mathcal{A} + \frac{1}{12} + \left(\frac{n^2}{2} + n(z+1) + \frac{z^2}{2} + z + \frac{5}{12}\right)\log n + \mathcal{O}\left(\frac{1}{n}\right),$$

with $\mathcal{A}$ is the Glaisher-Kinkelin constant,

$$\mathcal{A} = \exp\left(\frac{1}{12} - \zeta'(-1)\right).$$

Properties of Barnes-G function II.

Expansions,

$$\log G(1+z) = \frac{(\log(2\pi)-1)\,z}{2} - \frac{(1+\widehat{\gamma})z^2}{2} + \sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{k+1} z^{k+1}, \quad z \to 0,$$

where $\widehat{\gamma}$ is the Euler's constant and $\zeta(\cdot)$ is the Riemann Zeta function.

$$\log G(1+z) = \frac{1}{12} - \log \mathcal{A} + \frac{z}{2}\log(2\pi) + \left(\frac{z^2}{2} - \frac{1}{12}\right)\log z - \frac{3z^2}{4} + \sum_{k=1}^{\infty} \frac{B_{2k+2}}{4k(k+1)z^{2k}}, \qquad z \to \infty,$$

where B_k is the Bernoulli number.

Another Hankel determinant generated by a related weight, supported on $\mathbb{R}^+$,

$$w(x,t) = (x+t)^{-(\lambda+\beta)} x^{\beta}\, x^{r} \mathrm{e}^{-x-t}, \quad t>0,\ \lambda<0,\ \beta,\ r>0,\ x\in(0,\infty).$$

Double scaling: $s = nt$, $n \to \infty$, $t \to 0$, s finite. we find,

$$\ln \hat{\Delta}(s,r) = \int_0^s \left(\frac{(\xi\, g')^2}{4g(g-1)^2} - \frac{((\beta+\lambda)g-\lambda)^2}{4g} - \frac{\xi\, g}{g-1} \right) \frac{d\xi}{\xi},$$

where $g = g(s)$, satisfies another Painlevé V. For $s \to \infty$,

$$\Delta(s,r) = s^{(\beta+\lambda)(\beta+\lambda-2r)/4} \ \exp\left(a(r) - s - 2(\beta+\lambda)s^{1/2} + \mathcal{O}(1/\sqrt{s})\right)$$

The *constant* $a(r)$ is **determined up to a further constant** $b(r)$:

$$a(r) = \ln\left(\frac{G(1+r-\lambda-\beta)}{\Gamma(r)}\right) + b(r).$$

$G(\cdot)$ is the Barnes function, satisfies the functional equation,

$$G(z+1) = \Gamma(z)G(z).$$

Note: $b(r)$ **satisfies the difference equation**

$$b(r+1) - b(r) = \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} - \frac{1}{\mathrm{e}^t - 1}\right) \mathrm{e}^{-r\,t} dt.$$

$$b(r) \sim \kappa - \frac{\ln r}{2} + \frac{1}{720r^2}, \qquad \kappa \quad \text{unknown}.$$

Linear Statistics for $\gamma = 1$, and 4 with Min Chao.

In Random Matrix Theory one often encounters a statistical quantity of interest

$$G_N^{(\gamma)}(f) := \mathbb{E}_\gamma\left[\exp\left(-\lambda\sum_{j=1}^{N}F(x_j)\right)\right], \quad f(x) = \mathrm{e}^{-\lambda\,F(x)},$$

—the (exponential) generating function of the random variable $\sum_{j=1}^{N} F(x_j)$. Here $\mathbb{E}_\gamma$ denotes expectation,

$$\int_{\mathbb{R}^n}(\cdots\cdots)\prod_{1\le j<k\le N}|x_k - x_j|^\gamma\prod_{l=1}^{N}w(x_l)dx_l.$$

We proved that, for GSE,

$$[G_N^{(4)}(f)]^2 = \det\left(I + S_N f - \frac{1}{2} S_N \epsilon\, f' + (N+1/2)^{1/2}\ \epsilon\phi_{2N+1} \otimes \phi_{2N} f\right)$$

in the form of $\det(I + T_N)$. The operator ϵ has kernel,

$$\epsilon(x,y) = \frac{1}{2}\mathrm{sgn}(x-y),$$

$$\phi_k(x) = c_j\, H_j(x)\mathrm{e}^{-x^2/2},\ \ H_j(x)\ \text{Hermite polynomials}$$

$$S_N(x,y) = \sum_{j=0}^{N-1} \phi_j(x)\phi_j(y).$$

Replace $F(x)$ by $F(\sqrt{2N}x)$, assuming f belongs to Schwarz class, we find, as $N \to \infty$,

$$\mu_N^{(GSE)} = \frac{1}{2}\mu_N^{(GUE)} - \frac{(-1)^N}{4(2\pi N)^{1/2}} \int_{-\infty}^{\infty} F(x)\cos x\, dx + \mathcal{O}(N^{-1}), \qquad \text{mean},$$

$$V_N^{(GSE)} = \frac{1}{2} V_N^{(GUE)} + \frac{A}{N^{1/2}} + \frac{(-1)^N}{N^{1/2}} B + \mathcal{O}(N^{-1}), \qquad \text{variance}$$

here

$$A := \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{\sin(x-y)}{x-y} \mathrm{Si}(x-y) F'(y) dxdy,$$

$$B := -\frac{1}{4(2\pi)^{1/2}} \left[\int_{\mathbb{R}} F^2(x) \cos x \, dx - \frac{2}{\pi} \int_{\mathbb{R}^2} \frac{\sin(x-y)}{x-y} \cos \; x \, F(x) F(y) dxdy \right]$$

where

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t} dt.$$

LARGER formulas for the mean and variance with Laguerre or Gamma background.

We need

$$\phi_j^{\alpha-1}(x) = c_j^{\alpha-1} L_j^{(\alpha-1)}(x)\, x^{\alpha/2}\, \mathrm{e}^{-x/2},\ j = 0, 1, .., \ \ x \in \mathbb{R}^+,$$

$$\widetilde{\phi}_j(x) = c_j^{\alpha-1}\, L_j^{(\alpha-1)}(x)\, x^{-1+\alpha/2}\, \mathrm{e}^{-x/2},\ j = 0, 1, ..,$$

And find, with $F(x)$ **REPLACED BY** $F(\sqrt{8Nx})$,

$$\mu_N^{(LSE,\alpha)} = \frac{1}{2}\mu_N^{LUE,\alpha-1} - \frac{1}{4}\int_0^\infty \left[\int_0^x J_{\alpha-1}(y)dy\right] J_{\alpha-1}(x)F(x)dx + \frac{A}{N} + \mathcal{O}(N^{-2}), \ \ \textbf{mean},$$

where A is a linear functional in F', and

$$
\begin{aligned}
V_N^{LSE,\alpha} =&\frac{1}{2} V_N^{LUE,\alpha-1} - \frac{1}{4}\int_0^\infty \mathbf{J}_{\alpha-1}(x) J_{\alpha-1}(x)\, F^2(x) dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^2} B^{(\alpha-1)}(x,y) J_{\alpha-1}(x) \mathbf{J}_{\alpha-1}(y) F(x) F(y) dx dy \\
&- \frac{1}{8}\int_{\mathbb{R}^2} \mathbf{J}_{\alpha-1}(x) J_{\alpha-1}(x) F(x) \mathbf{J}_{\alpha-1}(y) J_{\alpha-1}(y) F(y) dx dy \\
&+ \frac{B}{N} + \mathrm{O}(N^{-2}).\ ]; \qquad \text{variance}
\end{aligned}
$$

where

$$\mathbf{J}_\alpha(x) = \int_0^x J_\alpha(y) dy.$$

Small Eigenvalues of Large Hankel Matrices, with Niall Emmart and Charles C. Weems

A test of a parallel computational algorithm for finding the smallest eigenvalue of a family of Hankel matrices that are ill-conditioned. Asymptotic formulas obtained from potential theory.

$$(A_{i,j})_{0\leq i,j\leq N-1} = (\mu_{i+j})_{0\leq i,j\leq N-1},$$

where the moments are defined for $\beta > 0$, by

$$\mu_j = \int_0^\infty x^j \, \mathrm{e}^{-x^\beta} dx = \frac{1}{\beta}\Gamma\left(\frac{1+j}{\beta}\right), \quad j = 0, 1, 2, \ldots$$

Condition number

$$\mathrm{cond}(A_N) = \frac{\lambda_N[A_N]}{\lambda_1[A_N]}$$

Lower bound on $\mathrm{cond}(A_N)$

$$\beta = 7/4, \quad 3.32 \times\ 10^{5125}, \quad N = 1500$$

$$\beta = 1, \quad 8.45 \times\ 10^{9187}, \quad N = 1500$$

Asymptotic estimate from **Classical Potential Theory**

For $\beta = 7/4$, inverse of the smallest eigenvalue, $\lambda_1(N)$, for large N.

$$\frac{1}{\lambda_1(N)} \sim \frac{1}{8\pi^{5/4}} \frac{c^{1/4}}{\sqrt{A_0}} \mathrm{e}^{\sec(7\pi/4)} N^{-5/7} \exp\left[\frac{2N^{5/7}}{\sqrt{\pi\, c}}\left(A_0 - \frac{A_1}{c} N^{-4/7}\right)\right],$$

$$c = 4\left[\frac{(\Gamma(7/4))^2}{\Gamma(7/2)}\right]^{4/7}, \quad A_0 = \frac{14\sqrt{\pi}}{5}, \quad A_1 = \frac{7\sqrt{\pi}}{3}.$$

$\beta = 7/4$: Numerical VS. Theoretical Result.

Table 1, $\beta = 7/4$: Numerical VS. Theoretical Result.

N	Numberical $\lambda_1(N)$	Theoretical $\lambda_1(N)$	Error
100	1.6976×10^{-45}	1.7424×10^{-45}	2.64%
300	1.4844×10^{-102}	1.5074×10^{-102}	1.55%
500	6.7121×10^{-149}	6.7929×10^{-149}	1.20%
1000	3.6209×10^{-246}	3.6514×10^{-246}	0.84%
1500	6.4232×10^{-330}	6.4667×10^{-330}	0.68%
2000	6.2011×10^{-406}	6.2369×10^{-406}	0.58%
2500	1.1483×10^{-476}	1.1542×10^{-476}	0.51%

$\beta = 1$: Numerical VS. Theoretical Result.

For $\beta = 1$, Szegö's classic,

$$\lambda_1(N) \sim 2^{7/2}\, \pi^{3/2}\, \mathrm{e}\, N^{1/4}\, \mathrm{e}^{-4\sqrt{N}}.$$

Table 2, $\beta = 1$: Numerical VS. Theoretical Result.

N	Numberical $\lambda_1(N)$	Theoretical $\lambda_1(N)$	Error
100	2.1079×10^{-15}	2.3006×10^{-15}	9.14%
300	5.5215×10^{-28}	5.8083×10^{-28}	5.19%
500	1.1138×10^{-36}	1.1585×10^{-36}	4.01%
1000	1.0892×10^{-52}	1.1200×10^{-52}	2.83%
1500	5.4593×10^{-65}	5.5852×10^{-65}	2.31%

$\beta = 1/2$: Numerical VS. Theoretical Result.

For $\beta = 1/2$, $\lambda_1(N) \sim 8\pi \dfrac{\sqrt{\ln(4\pi\, N\, \mathrm{e})}}{(4\pi\, N\, \mathrm{e})^{2/\pi}}$. Conjecture.

Table 3, $\beta = 1/2$: Numerical VS. Theoretical Result.

N	Numberical $\lambda_1(N)$	Theoretical $\lambda_1(N)$	Error
100	0.27397	0.40360	47.32%
300	0.15837	0.21365	34.91%
500	0.12047	0.15855	31.61%
1000	0.082087	0.10555	28.58%
1500	0.066295	0.08313	25.39%

Table 4: Constant $\beta = 7/4$, Total Time

Cores	$N = 1000$	$N = 1500$	$N = 2000$	$N = 2500$
80	0 : 43 : 57	4 : 04 : 31	13 : 57 : 35	Run not performed
240	0 : 17 : 33	1 : 24 : 53	4 : 45 : 25	Run not performed
440	0 : 13 : 26	0 : 53 : 06	2 : 47 : 48	7:33:36

Table 5: Constant $N = 1500$, Total Time

Cores	$\beta = 1/3$	$\beta = 1/2$	$\beta = 1$	$\beta = 7/4$
80	8 : 29 : 45	6 : 38 : 32	3 : 59 : 40	4 : 04 : 31
240	4 : 21 : 12	3 : 38 : 29	2 : 23 : 07	1 : 24 : 53

Thank you!