EXTREMAL PROPERTIES

OF HYPERGEOMETRIC POLYNOMIALS

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OUTLINE OF THE TALK

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1. **Orthogonal polynomials**

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2. Concepts of tropical geometry

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3. Amoebas of algebraic hypersurfaces

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4. Hypergeometric functions

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1. Orthogonal polynomials
2. Concepts of tropical geometry
3. Amoebas of algebraic hypersurfaces
4. Hypergeometric functions
5. Main result

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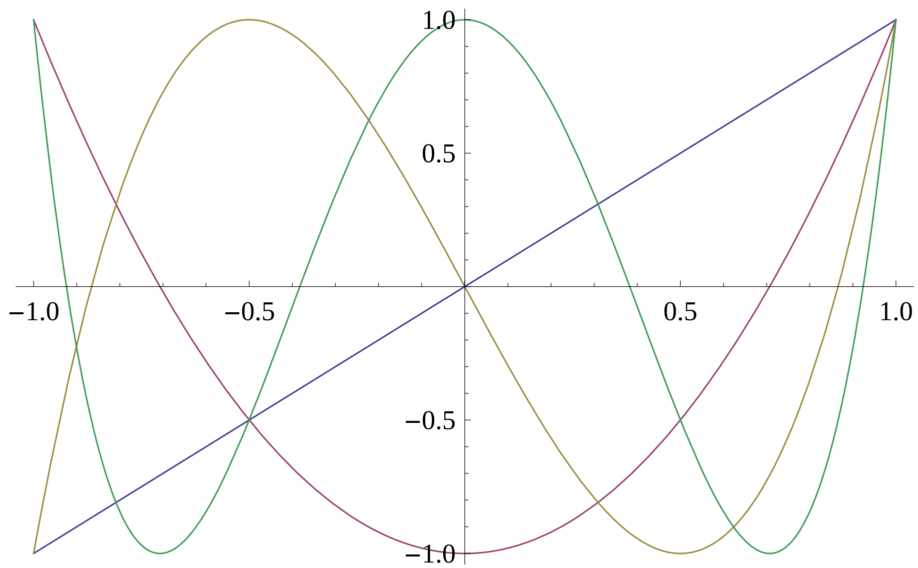

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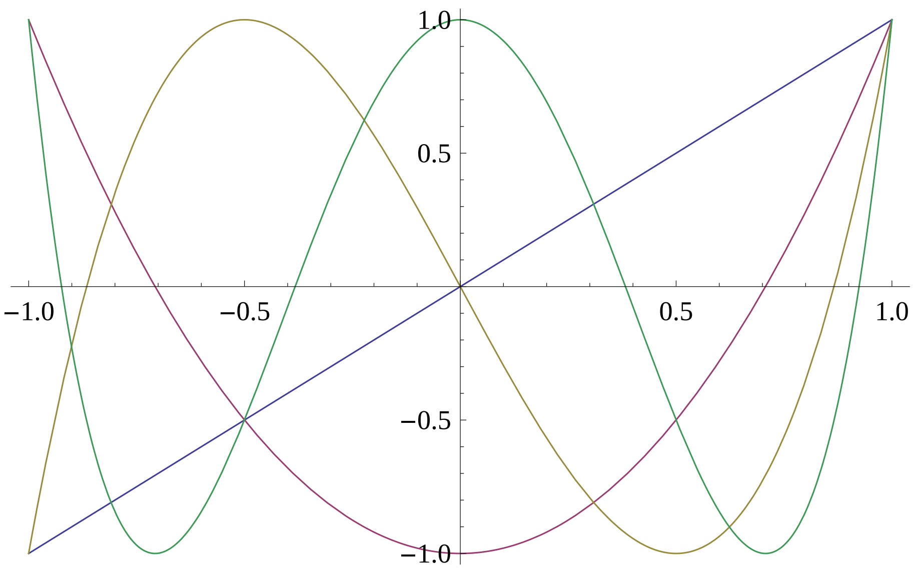

The Chebyshev polynomial of degree n has n different simple roots in $[-1, 1]$. The roots are symmetric with respect to the origin.

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Monic Chebyshev polynomial of degree n has the least deviation from zero on $[-1, 1]$ among all polynomials of degree n.

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... Appell Polynomials, Affine q-Krawtchouk polynomials, Affine root system, Al-Salam-Carlitz polynomials, Al-Salam-Chihara polynomials, Al-Salam-Ismail polynomials, Askey-Wilson polynomials, Associated Legendre polynomials, Bateman polynomials, Bender-Dunne polynomials, Bessel polynomials, Big q-Jacobi polynomials, Big q-Laguerre polynomials, Big q-Legendre polynomials, Biorthogonal polynomial, Brenke-Chihara polynomials, Charlier Polynomials, Chebyshev Polynomials of the Second Kind, Gegenbauer Polynomials, Hahn Polynomials, Hermite Polynomials, Jack Polynomials, Jacobi Polynomials, Krawtchouk Polynomials, Laguerre Polynomials, Legendre Polynomials, Meixner-Pollaczek Polynomials, Pollaczek Polynomials, Spherical Harmonics, Stieltjes-Wigert Polynomials, Schur polynomials, Sieved Jacobi polynomials, Sieved orthogonal polynomials, Sieved Pollaczek polynomials, Sieved ultraspherical polynomials, Stieltjes polynomials, Stieltjes-Wigert polynomials, Szego polynomials, Zernike Polynomials etc etc ...

A. Aptekarev, M. Derevyagin, and W. van Assche (2015): Multiple orthogonal polynomials can be associated with two-dimensional discrete Schrödinger operators (arxiv:1410.1332v2)

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N. Cotfas (2006 – present): Systems of orthogonal polynomials defined by hypergeometric type equations with application to quantum mechanics (e.g. arXiv:math-ph/0602037)

TROPICAL GEOMETRY

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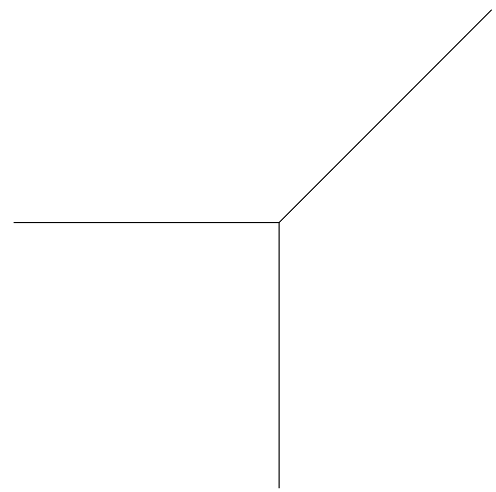
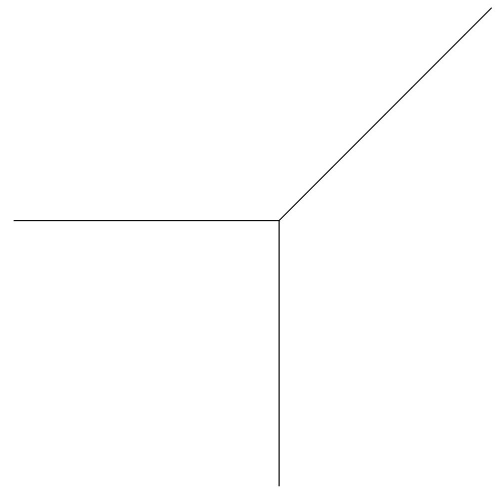


A tropical line

TROPICAL GEOMETRY



A tropical line

A *tropical polynomial* is a concave, continuous, piecewise linear function.

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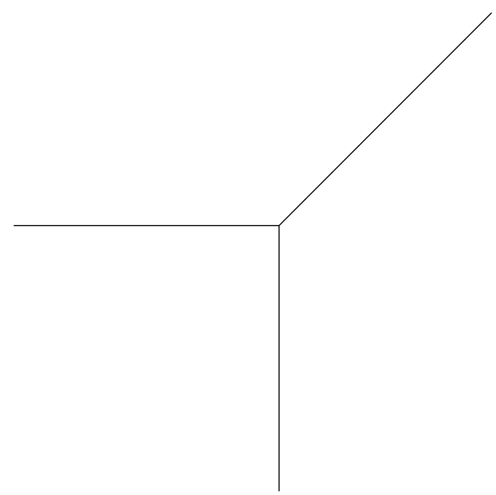
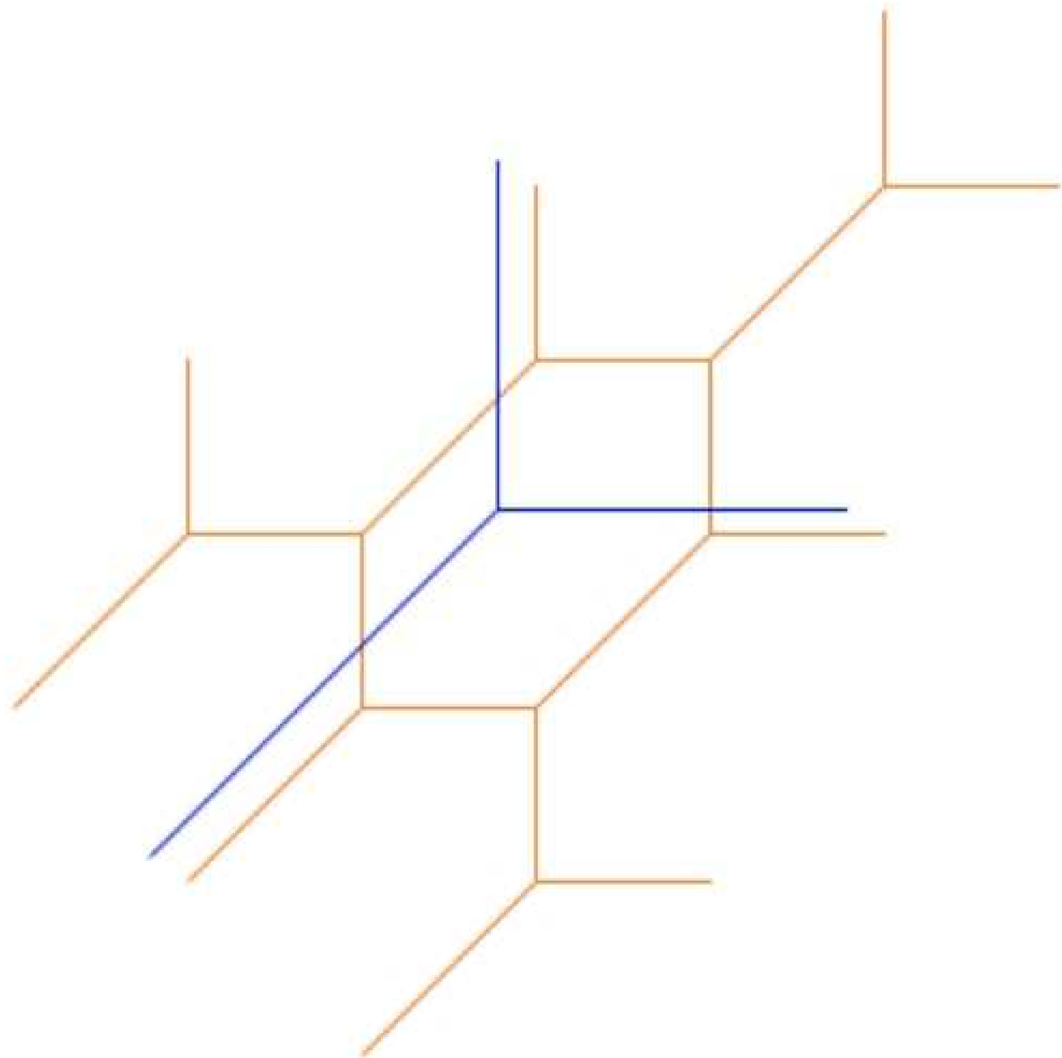


A tropical line

A *tropical polynomial* is a concave, continuous, piecewise linear function.

The set of points where a tropical polynomial is non-differentiable is called its associated *tropical algebraic hypersurface.*



Tropical line intersecting tropical cubic transversally

Pandanus Tectorius

Monstera

Monstera leaf: a closer look

AMOEBAS OF ALGEBRAIC HYPERSURFACES

DEFINITION. Let f be a Laurent polynomial

$$f = \sum_{\alpha \in S} c_{\alpha_1,\ldots,\alpha_n}\ x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

Its *amoeba* $\mathcal{A}_f$ is defined to be the image of the hypersurface $\{f = 0\}$ under the mapping

$$\mathrm{Log} : (x_1, \ldots, x_n) \mapsto (\log|x_1|, \ldots, \log|x_n|).$$

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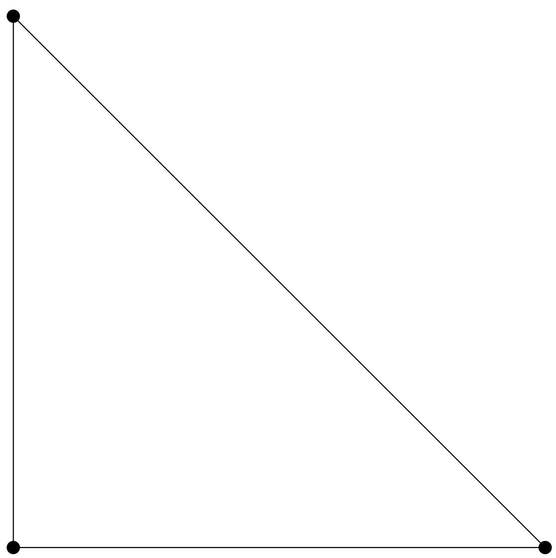
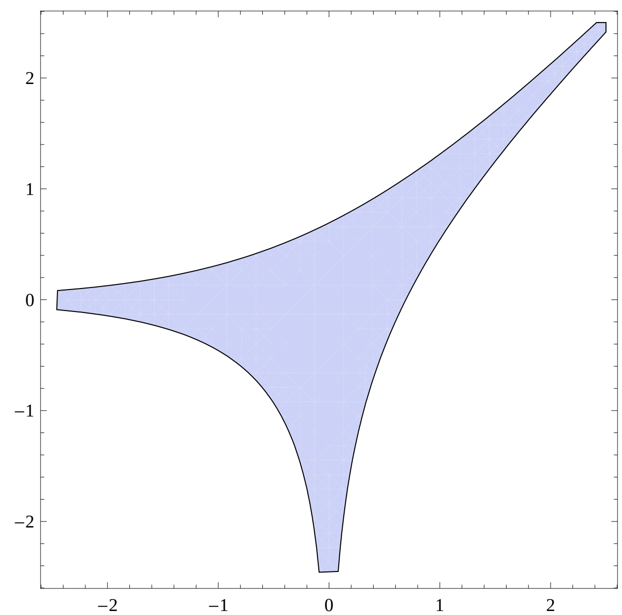
The *Newton polytope* of f is defined to be the convex hull of S.

EXAMPLE. The amoeba of a complex line.

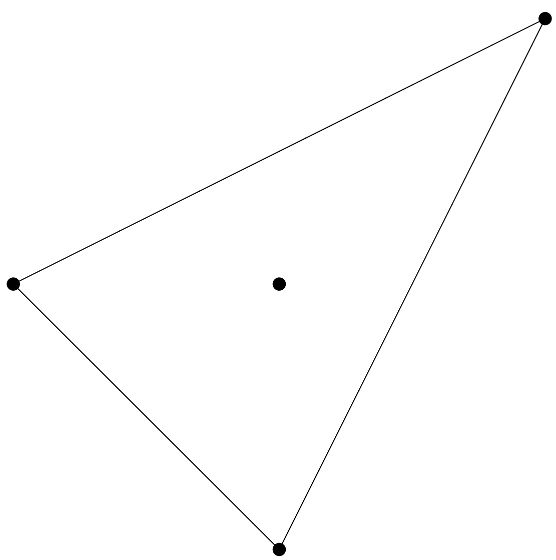
$$p(x, y) = 1 + x + y$$

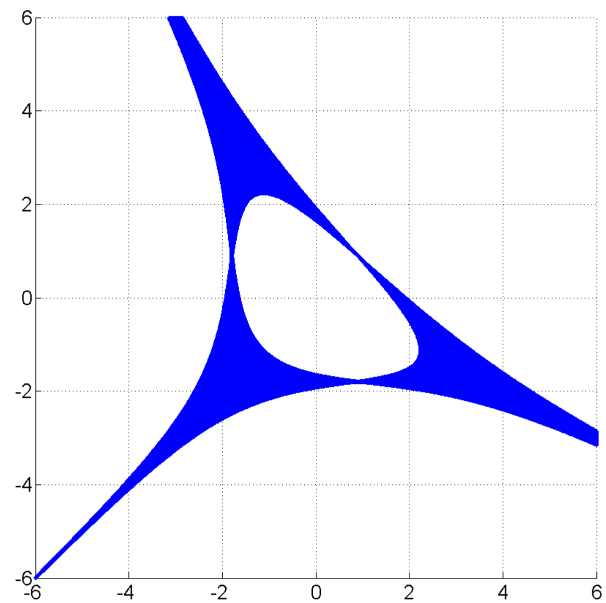




EXAMPLE.

$$p(x, y) = x + y + 6xy + x^2y^2$$





THEOREM. (Forsberg, Passare, Tsikh, 2000)

$$\#\,\text{vertices of } \mathcal{N}_f \;\leq\; \#\,{}^c\!\mathcal{A}_f \;\leq\; \#\,\mathcal{N}_f \cap \mathbf{Z}^n.$$

An amoeba is called *optimal* if the upper bound is attained.

HYPERGEOMETRIC FUNCTIONS

The Gauß hypergeometric series:

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}, $$

where $(\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$ is the Pochhammer symbol.

Special instances:

$$ z \cdot {}_2F_1(1, 1; 2; -z) = \ln(1 + z), $$

$$ {}_2F_1(a, 1; 1; z) = (1 - z)^{-a}, $$

$$ z \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \arcsin z; $$

algebraic functions

$$ 1 - z \cdot {}_2F_1\left(\frac{1}{2}, 1; 2; 4z\right) = \frac{1 + \sqrt{1 - 4z}}{2}, $$

the Legendre functions

$$ {}_2F_1(a, 1 - a; c; z) = \Gamma(c)\, z^{\frac{1-c}{2}} (1 - z)^{\frac{c-1}{2}} P_{-a}^{1-c}(1 - 2z), $$

complete elliptic integrals

$$\frac{\pi}{2} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z^2\right) = K(z),$$

$$\frac{\pi}{2} \cdot {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; z^2\right) = E(z),$$

the Chebyshev and Gegenbauer polynomials as well as elliptic modular functions.

OTHER CLASSICAL HYPERGEOMETRIC OBJECTS:

The Gauß hypergeometric equation:

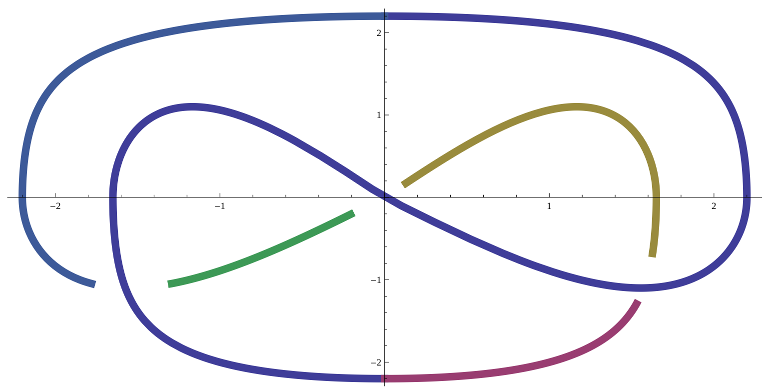
$$x(x-1)y''(x) + ((\alpha+\beta+1)x-\gamma)y'(x) + \alpha\beta y(x) = 0. \qquad (1)$$

A singular point a of a (linear) differential equation is called a ***regular singularity*** if any its solution has moderate growth in an arbitrary convex cone with the apex at a. Any linear homogeneous second order differential equation with three regular singularities in the Riemann sphere can be transformed to (1).

The hypergeometric integral:

$$\int_C t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt,$$

where the contour C is the Pochhammer cycle separating $0, 1, \infty$ in the Riemann sphere:



SOLVING ALGEBRAIC EQUATIONS

THROUGH HYPERGEOMETRIC SERIES

A root of the quintic

$$x^5 - x + z = 0$$

admits the representation

$$x = z \cdot {}_4F_3\left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; \; \frac{5^5}{4^4} \cdot z\right).$$

Any germ of the reduced algebraic equation with independent coefficients

$$y^m + x_1 y^{m_1} + \ldots + x_n y^{m_n} - 1 = 0$$

can be represented as the sum of the multiple series

$$y(x) =$$

$$\sum_{\nu_1,\ldots,\nu_n \geq 0} \frac{(-1)^{|\nu|}}{m^{|\nu|}} \frac{\prod_{\mu=1}^{|\nu|-1}(m_1\nu_1 + \ldots + m_n\nu_n - m\mu + 1)}{\nu_1! \ldots \nu_n!} x_1^{\nu_1} \ldots x_n^{\nu_n}.$$

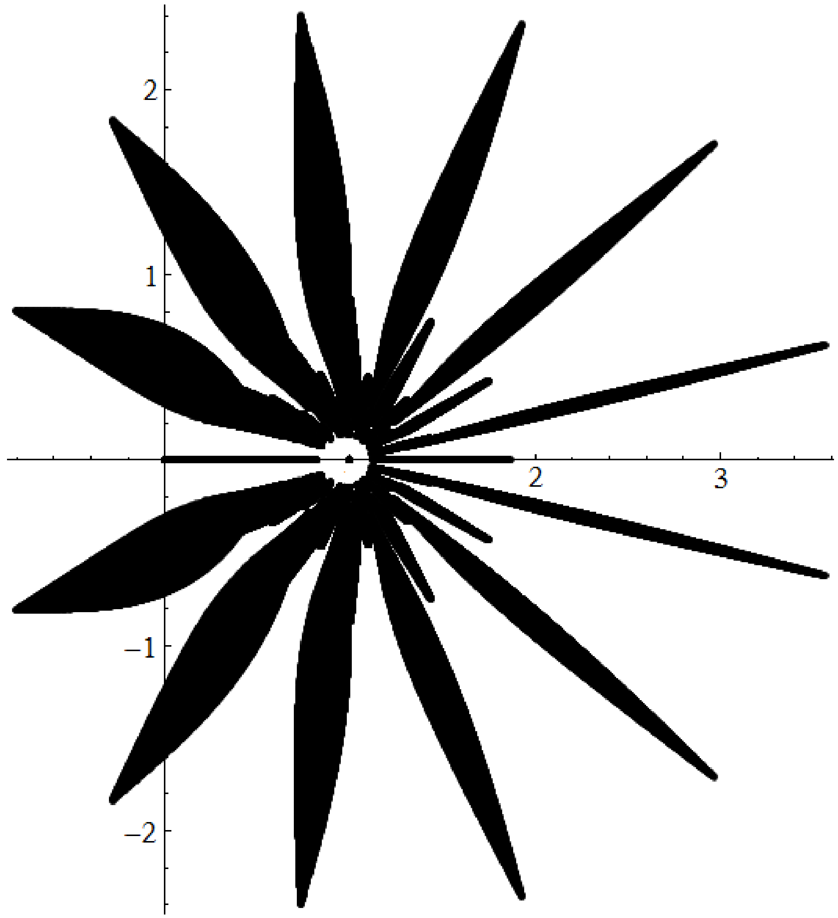
Here $|\nu| = \nu_1 + \ldots + \nu_n$, while the empty product is defined to be 1.

QUESTION:

WHAT IS THE GEOMETRY OF THE ZERO LOCUS

OF A HYPERGEOMETRIC POLYNOMIAL?

ZEROS OF UNIVARIATE HYPERGEOMETRIC POLYNOMIALS



The hypergeometric aster: zeros of the polynomials ${}_2F_1\,(-12, b; c; x)$ with $b, c \in \left\{\frac{k}{1000} : k = 100, \dots, 4000\right\}$

HOW TO DEFINE

A MULTIVARIATE HYPERGEOMETRIC POLYNOMIAL?

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6. Many of hypergeometric polynomials enjoy some extremal properties.

DEFINITION. By a *multivariate hypergeometric polynomial* supported in a convex integer polytope $P \in R^n$, $n \geq 2$ we will mean the polynomial

$$\sum_{s \in P \cap Z^n} \psi_P(s) x^s \tag{2}$$

with $\psi_P(s)$ defined by

$$\psi_P(s) := \frac{1}{\prod_{j=1}^{q} \Gamma(1 - \langle B_j, s \rangle - c_j)}. \tag{3}$$

Here $\langle B_j, s \rangle + c_j = 0$, $j = 1, \ldots, q$ are the equations of the hyperplanes containing the faces of P with B_j being the outer normal to P at the respective face. Since P is an integer polytope, we may without loss of generality assume the components of the vector B_j to be integer and relatively prime.

EXAMPLE 1. Define $\psi_P(s)$ through

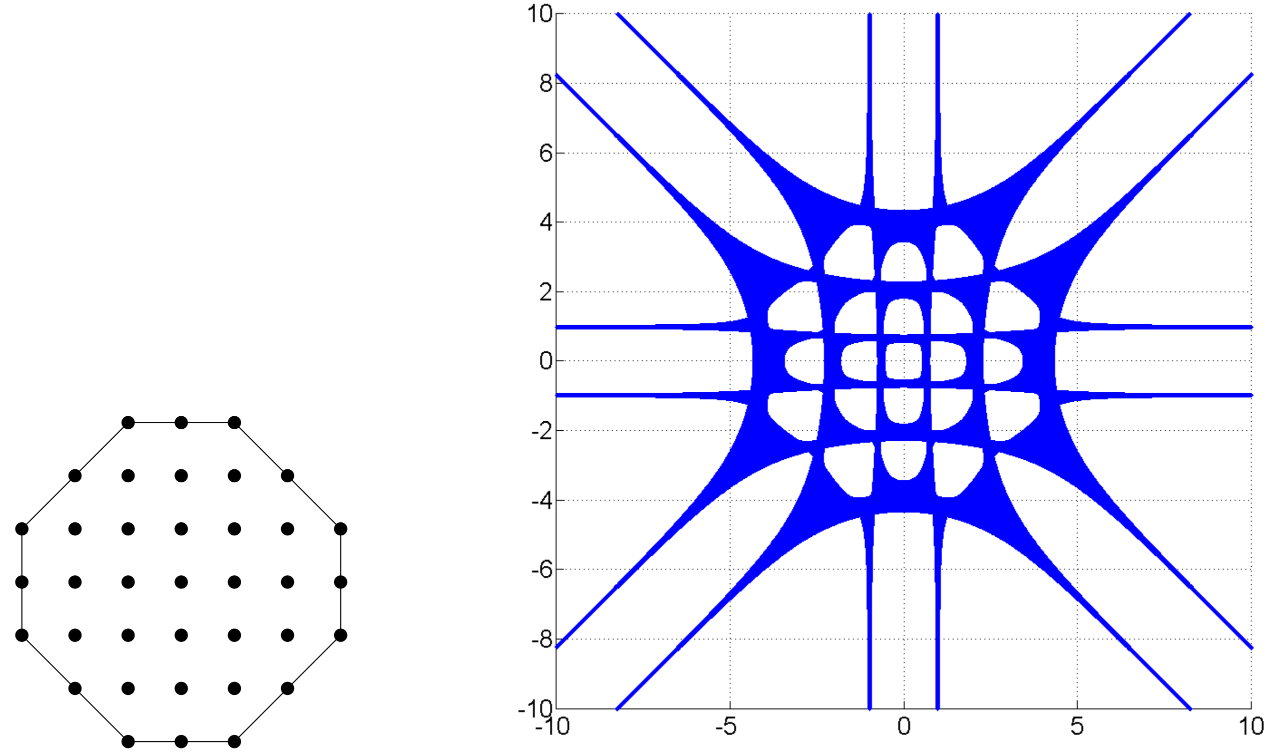
$$\psi_P(s) =$$

$$\Gamma(s-6)\Gamma(s+t-10)\Gamma(t-6)\Gamma(-s+t-4)\Gamma(-s)\Gamma(-s-t+2)\Gamma(-t)\Gamma(s-t-4).$$

The corresponding hypergeometric polynomial is given by

$$p_1(x,y) =$$

$$21x^2+64x^3+21x^4+126xy+2016x^2y+4704x^3y+2016x^4y+126x^5y+ 21y^2 + 2016xy^2 + 22050x^2y^2 + 47040x^3y^2 + 22050x^4y^2 + 2016x^5y^2 + 21x^6y^2 + 64y^3 + 4704xy^3 + 47040x^2y^3 + 98000x^3y^3 + 47040x^4y^3 + 4704x^5y^3 + 64x^6y^3 + 21y^4 + 2016xy^4 + 22050x^2y^4 + 47040x^3y^4 + 22050x^4y^4 + 2016x^5y^4 + 21x^6y^4 + 126xy^5 + 2016x^2y^5 + 4704x^3y^5 + 2016x^4y^5 + 126x^5y^5 + 21x^2y^6 + 64x^3y^6 + 21x^4y^6.$$



The Newton polygon of $p_1(x, y)$ and its amoeba

EXAMPLE 2. Define $\psi_P(s)$ through

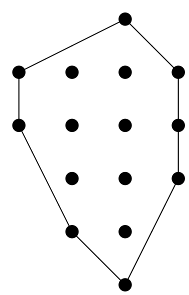
$$\psi_P(s) =$$

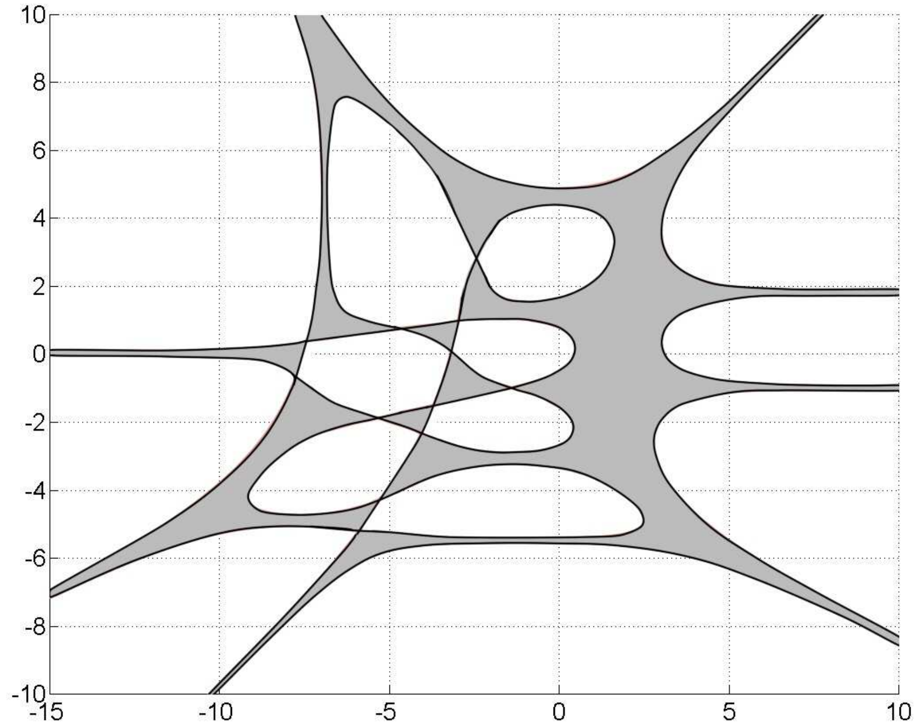
$$\Gamma(s+t-3)\Gamma(-s+2t-6)\Gamma(-3s-2t-5)\Gamma(3s-t-3)\Gamma(2s+t-5).$$

The corresponding hypergeometric polynomial has the form

$$p_2(x,y) =$$

$$528x^2-396xy-138600x^2y+55440xy^2+3104640x^2y^2+475200x^3y^2-165y^3-369600xy^3-8316000x^2y^3-1330560x^3y^3+160y^4+166320xy^4+2217600x^2y^4+201600x^3y^4-20160x^2y^5.$$





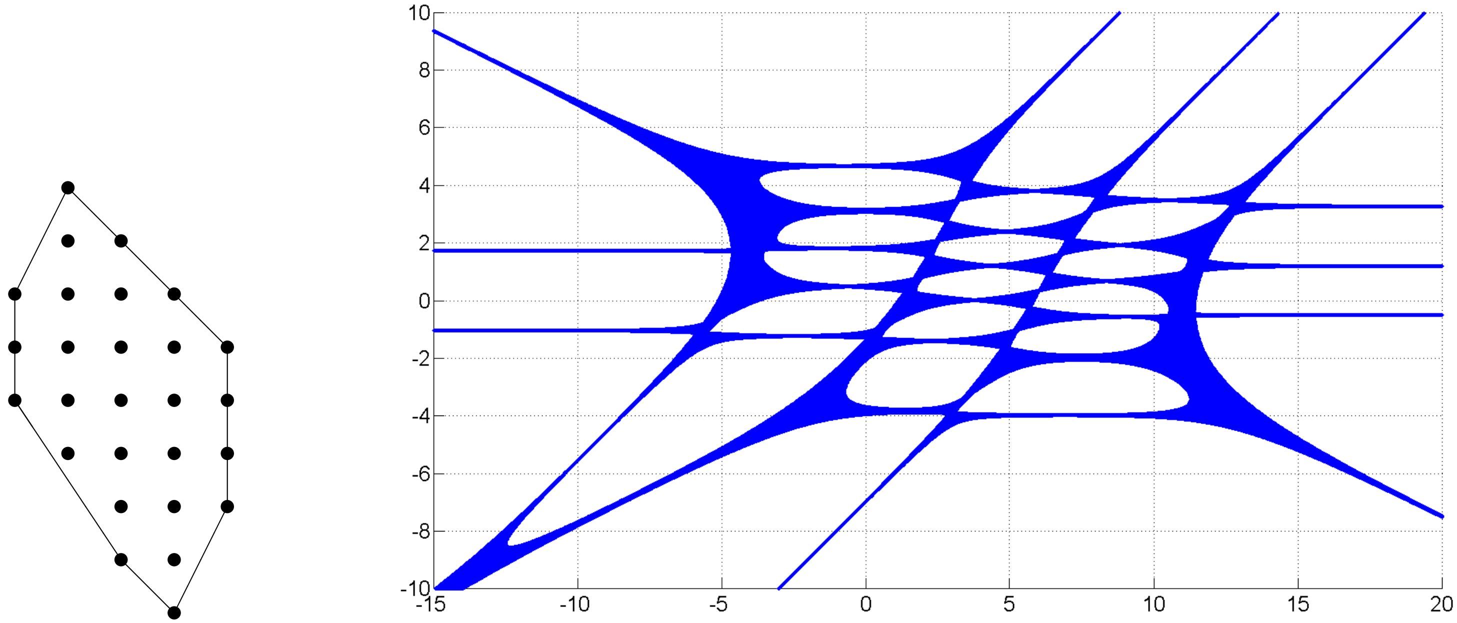
The Newton polygon of $p_2(x, y)$ and its amoeba

EXAMPLE 3. Define $\psi_P(s)$ through

$$\psi_P(s) = \Gamma(s+t-4)\Gamma(-4s+t-16)\Gamma(-3s-2t-5)\Gamma(3s-t-3)\Gamma(2s+t-5).$$

The hypergeometric polynomial defined by this data has the form

$$p_3(x,y) = -456456x^3 + 488864376x^2y - 28756728x^3y + 25420947552x^2y^2 - 244432188x^3y^2 + 3003x^4y^2 - 119841609888xy^3 + 127104737760x^2y^3 - 465585120x^3y^3 + 6006x^4y^3 + 1396755360y^4 - 508418951040xy^4 + 139815211536x^2y^4 - 232792560x^3y^4 + 1729x^4y^4 + 4190266080y^5 - 355893265728xy^5 + 41611670100x^2y^5 - 29628144x^3y^5 + 57x^4y^5 + 698377680y^6 - 58663725120xy^6 + 3328933608x^2y^6 - 705432x^3y^6 - 2327925600xy^7 + 55023696x^2y^7 - 16930368xy^8.$$



The Newton polygon of $p_3(x, y)$ and its amoeba

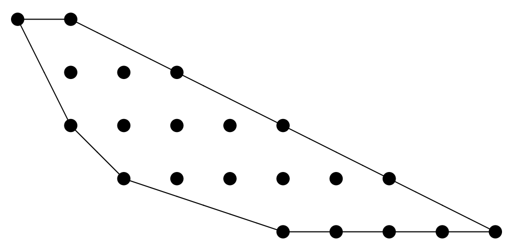
EXAMPLE 4. Define the coefficient of the hypergeometric polynomial to be

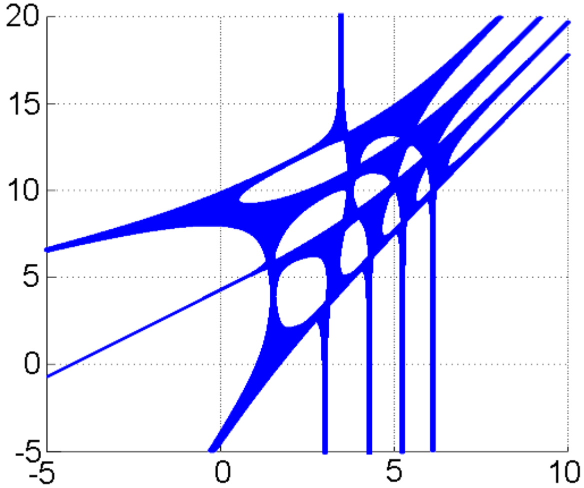
$$\Gamma\left(s+2t-5\right)\Gamma\left(-2s-t-4\right)\Gamma\left(-s-5t+1\right)$$

The corresponding hypergeometric polynomial is given by

$$p_4(x,y) =$$

$$2421619200x^5+172972800x^6+2882880x^7+14560x^8+20x^9+174356582400x^2y+ 48432384000x^3y+2421619200x^4y+34594560x^5y+160160x^6y+208x^7y+ 2421619200xy^2 + 691891200x^2y^2 + 21621600x^3y^2 + 160160x^4y^2 + 286x^5y^2 + 524160xy^3 + 14560x^2y^3 + 56x^3y^3 + 32y^4 + xy^4.$$





The Newton polygon of $p_4(x, y)$ and its amoeba

THEOREM. Hypergeometric polynomials are optimal.