The Parametrix Technique for Density Estimates in Stable-driven SDEs

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Introduction

A Stochastic Differential Equation (SDE) is an equation of the form

$$X_t = x + \int_0^t b(u, X_u)du + \int_0^t \sigma(u, X_{u^-})dZ_u, \tag{1}$$

where the coefficients:

- $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, is bounded or Lipschitz,
- $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^q$, Hölder and bounded.

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- $(Z_t)_{t\geq 0}$ a Lévy process in $\mathbb{R}^q$
  - Brownian Motion : the continuous case
  - Stable Process : the pure-jump case

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the solution $(X_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued stochastic process.

$\rightarrow$ We are interested in the density of the solution

Motivation

The solution of an SDE is a Markov process and its generator writes:

- In the Brownian case:

$$L_t(x,\nabla_x)\varphi(x) = \langle b(t,x), \nabla\varphi(x)\rangle + \frac{1}{2}\mathrm{Tr}\Big(\sigma\sigma^*(t,x)\nabla^2\varphi(x)\Big).$$

- In the pure-jump case, denoting ν the Lévy measure of $(Z_t)_{t\geq 0}$,

$$L_t(x,\nabla_x)\varphi(x) = \langle b(t,x), \nabla\varphi(x)\rangle + \int_{\mathbb{R}^q} \varphi(x+\sigma(t,x)z) - \varphi(x) - \frac{\langle \nabla\varphi(x), \sigma(t,x)z\rangle}{1+|z|^2}\nu(dz).$$

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Moreover, when b and σ are non degenerated, the solution of the Cauchy problem:

$$\begin{cases} \partial_t u(t,x) + L_t(x,\nabla_x)u(t,x) = 0 \\ u(T,x) = f(x) \end{cases}$$

Then: $u(t,x) = \mathbb{E}[f(X_T)|X_t = x] = \int_{\mathbb{R}^d} f(y)p(t,T,x,y)dy$.

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Then: $u(t,x) = \mathbb{E}[f(X_T)|X_t = x] = \int_{\mathbb{R}^d} f(y)p(t,T,x,y)dy$.

$\rightarrow$ The density of the solution of the SDE is the Green function.

The non degenerated Brownian case

The objective is to give a two-sided estimate of the density of the solution (when the density exists).

- In the Brownian case, we know that when $\sigma\sigma^*$ is Hölder and uniformly elliptic, ie: $\exists C > 1$, $\forall x, \xi \in \mathbb{R}^d$, $\forall t > 0$,

$$C^{-1}|\xi|^2 \leq \langle \xi, \sigma\sigma^*(t,x)\xi \rangle \leq C|\xi|^2,$$

the density of the solution exists and the following **Aronson estimates** holds: $\forall T > 0$, $\exists C_1, C_2 \geq 1$,

$$\frac{C_1^{-1}}{(s-t)^{d/2}} e^{-C_2 \frac{|x-y|^2}{s-t}} \leq p(t,s,x,y) \leq \frac{C_1}{(s-t)^{d/2}} e^{-C_2^{-1} \frac{|x-y|^2}{s-t}}.$$

$\rightarrow$ We have a Gaussian estimate on the solution of the SDE.

The non degenerated Stable Process

When $(Z_t)_{t\geq 0}$ is a stable process:

$$\mathbb{E}(e^{i\langle p,Z_t\rangle}) = \exp\left(-t\int_{S^{d-1}} |\langle p,\theta\rangle|^{\alpha}\mu(d\theta)\right).$$

- when $\mu(d\theta)$ has a strictly positive and smooth density on the sphere,
- when σ is non degenerated,

we have a similar result $\forall x,y\in\mathbb{R}^d;\ 0\leq t<s\leq T$:

$$C^{-1}\frac{(s-t)^{-d/\alpha}}{\left(1+\frac{|y-x|}{(s-t)^{1/\alpha}}\right)^{d+\alpha}} \leq p(t,s,x,y) \leq C\frac{(s-t)^{-d/\alpha}}{\left(1+\frac{|y-x|}{(s-t)^{1/\alpha}}\right)^{d+\alpha}}.$$

$\rightarrow$ Once again the estimate on the noise is transmitted to the solution of the SDE

Main Goal

The main purpose of our work is to obtain Aronson estimates for other types of noise.

- The tempered stable process: when the Lévy measure of $(Z_t)_{t\geq 0}$ is dominated by:

$$\nu(A) \leq \int_{S^{d-1}} \int_0^{+\infty} \mathbf{1}_A(s\theta) \frac{\bar{q}(s)}{s^{1+\alpha}} ds \mu(d\theta).$$

- the degenerated stable process:

$$\begin{aligned}
dX_t^1 &= \left(a_t^{1,1}X_t^1 + a_t^{1,2}X_t^2 + \cdots + a_t^{1,n-1}X_t^{n-1} + a_t^{1,n}X_t^n\right)dt + \sigma(t, X_{t-})dZ_t \\
dX_t^2 &= \left(a_t^{2,1}X_t^1 + a_t^{2,2}X_t^2 + \cdots + a_t^{2,n-1}X_t^{n-1} + a_t^{2,n}X_t^n\right)dt \\
dX_t^3 &= \left(a_t^{3,2}X_t^2 + \cdots + a_t^{3,n-1}X_t^{n-1} + a_t^{3,n}X_t^n\right)dt \\
&\vdots \\
dX_t^n &= \left(a_t^{n,n-1}X_t^{n-1} + a_t^{n,n}X_t^n\right)dt
\end{aligned}$$

The Parametrix for SDEs

- We consider the SDE:

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- Fix $y \in \mathbb{R}^d$ arbitrary terminal point where we wish to approximate the density of (2).
- We freeze the coefficients of (2) at point y:

$$\tilde{X}_t^y = x + \int_0^t b(u, y)du + \int_0^t \sigma(u, y)dZ_u.$$

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$$\tilde{X}_t^y = x + \int_0^t b(u, y)du + \int_0^t \sigma(u, y)dZ_u. \quad (3)$$

- We refer to $(\tilde{X}_t)_{t \geq 0}$ as the frozen process.
- Formally, those two processes should be close.
- The freezing point y is arbitrary.

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$$(p - \tilde{p})(t, s, x, y)$$

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$$= \int_t^s du \left(\int_{\mathbb{R}^d} \partial_u p(t,u,x,z)\tilde{p}(u,s,z,y) + p(t,u,x,z)\partial_u \tilde{p}(u,s,z,y)dz \right)$$

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$$= \int_t^s du \int_{\mathbb{R}^d} dz \Big(L_u(x,\nabla_z)^* p(t,u,x,z)\tilde{p}(u,s,z,y)$$

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$$= \int_t^s du \int_{\mathbb{R}^d} p(t,u,x,z) \underbrace{\Big(L_u(x,\nabla_z) - L_u(y,\nabla_z) \Big) \tilde{p}(u,s,z,y)}_{=H(u,s,z,y)} dz.$$

Thus

$$p(t,s,x,y) = \tilde{p}(t,s,x,y) + \int_t^s du \int_{\mathbb{R}^d} p(t,u,x,z)H(u,s,z,y)dz$$

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Thus

$$\begin{aligned} p(t,s,x,y) &= \tilde{p}(t,s,x,y) + \int_t^s du \int_{\mathbb{R}^d} p(t,u,x,z)H(u,s,z,y)dz \\ &= \tilde{p}(t,s,x,y) + p \otimes H(t,s,x,y). \end{aligned}$$

Proposition

Let $(P_{t,s})_{0\leq t\leq s}$ *the semigroup associated to* $(X_s^{t,x})_{0\leq t\leq s}$.
$\forall 0 \leq t < s,\ (x,y)\in(\mathbb{R}^d)^2,\ f:\mathbb{R}^d\to\mathbb{R}$, *bounded and measurable,*

$$P_{t,s}f(x) = \mathbb{E}[f(X_s)|X_t = x] = \int_{\mathbb{R}^d}\left(\sum_{r=0}^{+\infty}(\tilde{p}\otimes H^{(r)})(t,s,x,y)\right) f(y)dy,$$

The notation $\otimes$ *stands for the time-space convolution:*

$$\varphi\otimes\psi(t,s,x,y) = \int_t^s du \int_{\mathbb{R}^d} dz\ \varphi(t,u,x,z)\psi(u,s,z,y),$$

The sum of the series then provides a representation of the density of $(X_s^{t,x})_{s\geq 0}$, *we have* $\forall 0\leq t < T,\ (x,y)\in(\mathbb{R}^d)^2$:

$$p(t,s,x,y) = \sum_{r=0}^{+\infty}(\tilde{p}\otimes H^{(r)})(t,s,x,y).$$

Convergence of the series

- First step is to obtain an estimate on the frozen density:

$$\tilde{p}(t,s,x,y) \leq C\bar{p}(t,s,x,y)$$

ex: In the case of the non degenerated rotationally invariant stable process, one has:

$$\bar{p}(t,s,x,y) = C(s-t)^{-d/\alpha}\left(1+\frac{|y-x|}{(s-t)^{\frac{1}{\alpha}}}\right)^{-(d+\alpha)}.$$

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- Next we obtain an estimate on the kernel H.

ex: For the stable case, one has:

$$H(t,s,x,y) \leq C\frac{\delta \wedge |y-x|^{\eta(\alpha\wedge 1)}}{s-t}\bar{p}(t,s,x,y),$$

- Finally, one of the key steps of the procedure is to prove a regularising property for the kernel for some $\omega > 0$:

$$\int_{\mathbb{R}^d} H(t,s,x,y)dx \le C \int_{\mathbb{R}^d} \frac{\delta \wedge |y-x|^{\eta(\alpha\wedge 1)}}{s-t} \bar{p}(t,s,x,y)dx \le C(s-t)^{\omega-1}.$$

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- This property allows to prove convergence of the series in small time:

$$|\tilde{p} \otimes H(t,s,x,y)| \le C \int_t^s d\tau \int_{\mathbb{R}^d} \bar{p}(t,\tau,x,z) \frac{\delta \wedge |y-z|^{\eta(\alpha\wedge 1)}}{s-\tau} \bar{p}(\tau,s,z,y)dz$$

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- The convergence of the series then gives the upper bound for small time:

$$p(s,t,x,y) \leq C\bar{p}(t,s,x,y).$$

- We then extend the upper bound to arbitrary time exploiting the semigroup property:

$$p(t,s,x,y) = \int_{\mathbb{R}^d} dz_1 \cdots \int_{\mathbb{R}^d} dz_k \prod_{i=0}^{k} p(\tau_i,\tau_{i+1},z_i,z_{i+1}) \leq C^k \bar{p}(t,s,x,y).$$

Stability for the density

In this section, we discuss the problem of quantifying the distance between the density of

$$X_s = x + \int_t^s b(u, X_u)du + \int_t^s \sigma(u, X_u)dZ_u$$

and the one of

$$X_s^n = x + \int_t^s b_n(u, X_u^n)du + \int_t^s \sigma_n(u, X_u^n)dZ_u,$$

in terms of the distance between the coefficients (in a certain norm):

$$\Delta_n = |b - b_n| + |\sigma - \sigma_n| \underset{n \to +\infty}{\longrightarrow} 0.$$

→ We use the Parametrix series representation and quantify the distance between each terms

Stability for the density (II)

Starting from the parametrix representations:

$$p(t,s,x,y) = \sum_{k=0}^{+\infty} \tilde{p} \otimes H^{(k)}(t,s,x,y), \quad p_n(t,s,x,y) = \sum_{k=0}^{+\infty} \tilde{p}_n \otimes H_n^{(k)}(t,s,x,y).$$

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Thus, we write:

$$p(t,s,x,y) - p_n(t,s,x,y) = \sum_{k=0}^{+\infty} \left(\tilde{p} \otimes H^{(k)} - \tilde{p}_n \otimes H_n^{(k)} \right)(t,s,x,y).$$

We control each term with an estimate involving $\Delta_n \bar{p}(t,T,x,y)$.

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We control each term with an estimate involving $\Delta_n \bar{p}(t,T,x,y)$.
For the first term, we have:

$$|\tilde{p} - \tilde{p}_n|(t,T,x,y) \leq C \Delta_n \bar{p}(t,T,x,y).$$

Stability for the density (III)

We proceed by induction

$$\tilde{p}\otimes H^{(k+1)}-\tilde{p}_n\otimes H_n^{(k+1)} = \Big(\tilde{p}\otimes H^{(k)}-\tilde{p}_n\otimes H_n^{(k)}\Big)\otimes H+\tilde{p}_n\otimes H_n^{(k)}\otimes\Big(H-H_n\Big).$$

For the next terms, we use the following estimate:

$$|H-H_n|(t,T,x,y) \le C\Delta_n \frac{\delta\wedge|y-x|^{\eta(\alpha\wedge 1)}}{T-t}\bar{p}(t,T,x,y).$$

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Theorem (Konanov *et al.*- 2015, H.-2016)

Fix a finite time horizon $T>0$. For all $t\le T$ and all $x,y\in\mathbb{R}^d$, there exists $C>0$ such that

$$|(p-p_n)(t,T,x,y)| \le C\Delta_n\bar{p}(t,T,x,y).$$

Well-posedness of the Martingale Problem and Parametrix

- Assume $\mathbb{P}^1$ and $\mathbb{P}^2$ are solution to the martingale problem associated to $(L_s)_{s\in[t,T]}$, starting at x at time t:

$$f(s,X_s) - f(t,x) - \int_t^s \Big(\partial_u + L_u\Big) f(u,X_u)du \text{ is a } \mathbb{P}^i \text{ martingale}$$

- For $f : [0,T]\times\mathbb{R}^{nd} \to \mathbb{R}$, measurable and bounded, we define:

$$S^i f = \mathbb{E}^i\left(\int_t^T f(s,X_s)ds\right),$$

for $(X_s)_{s\in[t,T]}$ the canonical process associated with $(\mathbb{P}^i)_{i\in\{1,2\}}$.

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- Setting

$$S^\Delta f=S^1f-S^2f,$$

the aim is to show that $S^\Delta f=0$, for f in a large enough class.

- To that end, we prove that

$$\|S^\Delta\|:=\sup_{|f|_\infty\le 1}|S^\Delta f|=0.$$

- Since $\mathbb{P}^1$ and $\mathbb{P}^2$ are solution to the martingale problem, we have: $\forall f \in \mathcal{C}_0^{1,1}([0,T) \times \mathbb{R}^{nd}, \mathbb{R})$:

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- We then have

$$f(t,x) + \mathbb{E}^i \left(\int_t^T \Big(\partial_s + L_s \Big) f(s, X_s) ds \right) = 0.$$

- Thus, for all $f \in \mathcal{C}_0^{1,1}([0,T) \times \mathbb{R}^{nd}, \mathbb{R})$,

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$$S^\Delta \Big((\partial_. + L_.) f \Big) = 0.$$

- We make appear the frozen process:

$$\underbrace{S^\Delta \Big((\partial_. + \tilde{L}_.) f \Big)}_{\text{Estimate on } \tilde{p}} + \underbrace{S^\Delta \Big((L_. - \tilde{L}_.) f \Big)}_{\text{Estimate on } H} = 0.$$

- Since $\mathbb{P}^1$ and $\mathbb{P}^2$ are solution to the martingale problem, we have: $\forall f \in \mathcal{C}_0^{1,1}([0,T) \times \mathbb{R}^{nd}, \mathbb{R})$:
- We then have

$$f(t,x) + \mathbb{E}^i\left(\int_t^T \Big(\partial_s + L_s\Big) f(s, X_s) ds\right) = 0.$$

- Thus, for all $f \in \mathcal{C}_0^{1,1}([0,T) \times \mathbb{R}^{nd}, \mathbb{R})$,

$$S^{\Delta}\Big((\partial_\cdot + L_\cdot) f\Big) = 0.$$

- We make appear the frozen process:

$$\underbrace{S^{\Delta}\Big((\partial_\cdot + \tilde{L}_\cdot) f\Big)}_{\text{Estimate on } \tilde{p}} + \underbrace{S^{\Delta}\Big((L_\cdot - \tilde{L}_\cdot) f\Big)}_{\text{Estimate on } H} = 0.$$

- For h an arbitrary test function, we define

$$\Psi_\varepsilon(t,x) = \int_t^T ds \int_{\mathbb{R}^{nd}} \tilde{p}^{s+\varepsilon,y}(t, s+\varepsilon, x, y) h(s,y) dy.$$

The Standard Stable Process

- The stable process is a Lévy process presenting the auto-similarity property:

$$Z_{ct} \overset{(d)}{=} c^{1/\alpha} Z_t.$$

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- We refer to μ as the spectral measure. Besides, when

$$\forall p\in\mathbb{R}^d,\ \exists C>1,\quad C^{-1}|p|^\alpha \le \int_{S^{d-1}} |\langle p,\xi\rangle|^\alpha \mu(d\xi) \le C|p|^\alpha,$$

we see that Z_t has density for all $t>0$.

$\rightarrow$ The spectral measure has an important role on the density of Z

Stable-driven SDE

Watanabe [Wat07] shows that if there exists a compact $K \subset S^{d-1}$ such that: $\forall \theta \in K \subset S^{d-1}$, $\forall r \leq 1/2$

$$C^{-1} r^{\gamma-1} \leq \mu\left(B(\theta, r) \cap K\right) \leq C r^{\gamma-1},$$

then, $\forall x \in \mathbb{R}^d$ such that $x/|x| \in K$:

$$C^{-1} \frac{t^{-d/\alpha}}{\left(1 + \frac{|x|}{t^{1/\alpha}}\right)^{\alpha+\gamma}} \leq p_Z(t,x) \leq C \frac{t^{-d/\alpha}}{\left(1 + \frac{|x|}{t^{1/\alpha}}\right)^{\alpha+\gamma}}.$$

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- When the spectral measure is equivalent to the Lebesgue measure, we have $\gamma = d$.

$\rightarrow$ We recover the estimate on the rotationally invariant stable process

- In general, we see that the spectral measure influences the tails the stable process.

- When $|x| \leq Ct^{1/\alpha}$, the estimate is equivalent to:

$$C^{-1}t^{-d/\alpha} \leq p_Z(t,x) \leq Ct^{-d/\alpha}.$$

  - This is the diagonal regime.
  - It is an estimate reflecting the auto-similarity of the stable process.

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- When $|x| \geq Ct^{1/\alpha}$, we have:

$$C^{-1}\frac{t^{1+\frac{\gamma-d}{\alpha}}}{|x|^{\alpha+\gamma}} \leq p_Z(t,x) \leq C\frac{t^{1+\frac{\gamma-d}{\alpha}}}{|x|^{\alpha+\gamma}}.$$

  - This is the off-diagonal regime
  - It is an estimate reflecting the heavy tails of the stable process in large deviation regime.
  - For this bound to be integrable, we have the constrain $\alpha + \gamma > d$.

The Tempered Stable Process

Consider now the case where the Lévy measure ν only satisfy the domination:

$$\nu(A) \leq \int_{S^{d-1}} \int_0^{+\infty} \mathbf{1}_A(s\theta) \frac{\bar{q}(s)}{s^{1+\alpha}} ds \mu(d\theta).$$

(Upper bound) Assume that the function $\bar{q} : \mathbb{R}_+ \to \mathbb{R}_+$, is such that: $\bar{q}(s) \leq C\bar{q}(2s)$. If there exists $\gamma \in [1, d]$ such that $\forall \theta \in S^{d-1}$, $\forall r \leq 1/2$:

$$\mu\left(B(\theta, r) \cap S^{d-1}\right) \leq C r^{\gamma-1},$$

then, Sztonyk [Szt10] proves the upper bound:

$$p_Z(t,x) \leq C \frac{t^{-d/\alpha}}{\left(1 + \frac{|x|}{t^{1/\alpha}}\right)^{\alpha+\gamma}} \bar{q}(|x|).$$

(Lower bound) Assume in addition that there exists $A_{low} \subset \mathbb{R}^d$, a decreasing function $\underline{q}$, and $\gamma \in [1, d]$ such that $\forall x \in A_{low}$, $\forall r > 0$, we have:

$$\nu\Big(B(x,r)\Big) \geq Cr^{\gamma}\frac{\underline{q}(|x|)}{|x|^{\alpha+\gamma}}, \quad \nu\Big(B(0,r)^c\Big) \leq Cr^{-\alpha}, \ \forall 0 < r < 1,$$

then we have the lower bound:

$$C^{-1}\frac{t^{-d/\alpha}}{\left(1+\frac{|x|}{t^{1/\alpha}}\right)^{\alpha+\gamma}}\underline{q}(|x|) \leq p_Z(t,x). \tag{4}$$

- Can we transfer these estimates to the solution of the SDE?
- Kolokoltsov [Kol00] proves those estimates in the stable case, when the spectral measure has a smooth strictly positive density.

Presentation of the results in the tempered case

We consider the SDE driven by such process $(Z_t)_{t\geq 0}$:

$$X_t = x + \int_0^t b(u, X_u)du + \int_0^t \sigma(u, X_{u^-})dZ_u,$$

with $b = 0$ when $\alpha \leq 1$.

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with $b = 0$ when $\alpha \leq 1$.
For all $A \in \mathcal{B}$, we define

$$\nu_t(x, A) = \nu\big(\{z \in \mathbb{R}^d;\ \sigma(t,x)z \in A\}\big), \tag{5}$$

and we assume that those measures satisfies:

$$|\nu_t(x,A) - \nu_t(x',A)| \leq C\delta \wedge |x-x'|^{\eta(\alpha\wedge 1)} \int_{S^{d-1}} \int_0^{+\infty} \mathbf{1}_A(s\theta)\frac{\bar{q}(s)}{s^{1+\alpha}} ds\mu(d\theta).$$

with σ Hölder continuous and bounded, uniformly elliptic: $\exists C > 1$, such that $x, \xi \in \mathbb{R}^d$, $t \geq 0$,

$$C^{-1}|\xi|^2 \leq \langle \xi, \sigma(t,x)\xi \rangle \leq C|\xi|^2.$$

For fixed $T > 0$ and $y \in \mathbb{R}^d$, we define:

$$\tilde{X}_s^{T,y} = x + \int_t^s b(u, \theta_{u,T}(y))du + \int_t^s \sigma(u, \theta_{u,T}(y))dZ_u$$

where

- When the drift b is bounded, θ is the identity map $\theta_{u,T}(y) = y$
- When b is Lipschitz continuous, $\theta_{u,T}(y)$ is the backward transport by the solution of the deterministic ODE assiciated:

$$\frac{d}{du}\theta_{u,T}(y) = b\big(u, \theta_{u,T}(y)\big), \ \theta_{T,T}(y) = y, \ \forall 0 \leq u \leq T.$$

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Consequently, for the frozen process, we have the estimates proven by Sztonyk:

$$\tilde{p}(t,s,x,y) \leq C \frac{(s-t)^{-d/\alpha}}{\left(1 + \frac{|\theta_{\mathbf{t},\mathbf{s}}(y)-x|}{(s-t)^{\mathbf{1}/\alpha}}\right)^{\alpha+\gamma}} \bar{q}(|\theta_{t,s}(y) - x|).$$

- What upper bound for the kernel H?

$$\int_0^{+\infty}\int_{S^{d-1}}\Big(\tilde{p}(t,s,x+\rho\theta,y)-\tilde{p}(t,s,x,y)\Big)\mathbf{1}_{\{\rho\geq(s-t)^{1/\alpha}\}}\frac{\bar{q}(\rho)d\rho}{\rho^{1+\alpha}}\mu(d\theta)$$

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- We have the following upper bound:

$$|H(t,s,x,y)|\leq C\left((s-t)^{-1/\alpha}\mathbf{1}_{\{\alpha>1\}}+\frac{\delta\wedge|x-\theta_{t,s}(y)|^{\eta(\alpha\wedge1)}}{s-t}\right)\times\frac{(s-t)^{-d/\alpha}}{\left(1+\frac{|\theta_{t,s}(y)-x|}{(s-t)^{1/\alpha}}\right)^{\alpha+\gamma}}Q(|\theta_{t,s}(y)-x|).$$

- When b is bounded
  - if μ has a density, $\gamma=d$ and $Q(s)=\bar{q}(s)$,
  - else, $Q(s)=\min(1,s^{\gamma-1})\bar{q}(s)$,
- When b is Lipschitz continuous,
  - if μ has a density, $\gamma=d$ and $Q(s)=\min(1,s)\bar{q}(s)$,
  - else, $Q(s)=\min(1,s,s^{\gamma-1})\bar{q}(s)$.

$\rightarrow$ We degrade the tempering function to recover the good concentration.

Theorem (**Upper bound**)

There exists a unique solution to the SDE. That solution has a density with respect to the Lebesgue measure $t > 0$*, and* $x, y \in \mathbb{R}^d$*:*

$$\mathbb{P}(X_s \in dy | X_t = x) = p(t, s, x, y)dy.$$

Assume that the function Q *define previously is decreasing: there exists* $C_1 \geq 1$ *depending on the maturity* T *such that:*
$\forall 0 \leq t \leq T,\ \forall (x, y) \in \mathbb{R}^d,$

$$p(t, s, x, y) \leq C_1 \frac{(s-t)^{-d/\alpha}}{\left(1 + \frac{|y-\theta_{s,t}(x)|}{(s-t)^{1/\alpha}}\right)^{\alpha+\gamma}} Q(|y - \theta_{s,t}(x)|).$$

Theorem (Lower bound)

If there is $A_{low} \subset \mathbb{R}^d$ such that $\forall x \in A_{low}$:

$$\forall r > 0,\ \nu\Big(B(x,r)\Big) \geq C r^{\gamma} \frac{q(|x|)}{|x|^{\alpha+\gamma}},$$

if for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\sigma(t,x)A_{low} \subset A_{low}$, and if

$$B\big(\theta_{t,T}(y) - x, C(T-t)^{1/\alpha}\big) \subset A_{low},$$

then there is $C_2 > 1$ such that

$$C_2^{-1} \frac{(s-t)^{-d/\alpha}}{\left(1 + \frac{|y-\theta_{s,t}(x)|}{(s-t)^{1/\alpha}}\right)^{\alpha+\gamma}} q(|y-\theta_{s,t}(x)|) \leq p(t,s,x,y).$$

The Degenerate Case

We now consider the degenerate setting:

$$\begin{aligned}
dX_t^1 &= \Big(a_t^{1,1}X_t^1 + a_t^{1,2}X_t^2 + \cdots + a_t^{1,n-1}X_t^{n-1} + a_t^{1,n}X_t^n\Big)dt + \sigma(t, X_{t-})dZ_t \\
dX_t^2 &= \Big(a_t^{2,1}X_t^1 + a_t^{2,2}X_t^2 + \cdots + a_t^{2,n-1}X_t^{n-1} + a_t^{2,n}X_t^n\Big)dt \\
dX_t^3 &= \Big(a_t^{3,2}X_t^2 + \cdots + a_t^{3,n-1}X_t^{n-1} + a_t^{3,n}X_t^n\Big)dt \\
&\ \vdots \\
dX_t^n &= \Big(a_t^{n,n-1}X_t^{n-1} + a_t^{n,n}X_t^n\Big)dt
\end{aligned}$$

with initial condition $X_0 = x \in \mathbb{R}^{nd}$, where:

- $Z \in \mathbb{R}^d$ is an α stable, symmetric (possibly tempered).
- $\sigma : \mathbb{R}_+ \times \mathbb{R}^{nd} \to \mathbb{R}^d \otimes \mathbb{R}^d$, is Hölder continuous, uniformly elliptic and bounded.
- $a^{i,j} : \mathbb{R}_+ \to \mathbb{R}^d \otimes \mathbb{R}^d$, satisfying Hörmander condition, bounded
- For $x \in \mathbb{R}^{nd}$, we denote $x = (x^1, \cdots, x^n)$, with $x^i \in \mathbb{R}^d$.

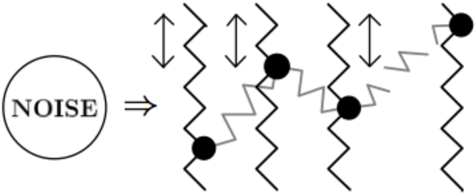
Motivation

There degeneracy appears for instance:

- For $n = 2$, in the case of the asian option pricing for jump diffusions:

$$\begin{aligned} X_t^1 &= x^1 + \int_0^t a_s^1 X_s^1 ds + \int_0^t \sigma(s, X_{s^-}) dZ_s \\ X_t^2 &= x^2 + \int_0^t a_t^2 X_s^1 ds \end{aligned}$$

- In Physics, the chain of perturbed oscillators:



The degenerated Brownian case

- When $\alpha = 2$, Delarue and Menozzi [DM10] considered the chain:

$$\begin{aligned} dX_t^1 &= F_1(t, X_t^1, \cdots, X_t^n)dt + \sigma(t, X_t^1, \cdots X_t^n)dZ_t \\ dX_t^2 &= F_2(t, X_t^1, \cdots, X_t^n)dt \\ dX_t^3 &= F_3(t, X_t^2, \cdots, X_t^n)dt \\ &\vdots \\ dX_t^n &= F_n(t, X_t^{n-1}, X_t^n)dt, \;\; X_0 = x. \end{aligned}$$

- In the Brownian case, Delarue and Menozzi [DM10] obtains the multi-scale Gaussian estimate:

$$C^{-1}(s-t)^{-n^2\frac{d}{2}} \exp\left(-C\left|(\mathbb{T}^2_{s-t})^{-1}(y-\theta_{s,t}(x))\right|^2\right)$$

$$\leq p(t,s,x,y) \leq$$

$$C(s-t)^{-n^2\frac{d}{2}} \exp\left(-C^{-1}\left|(\mathbb{T}^2_{s-t})^{-1}(y-\theta_{s,t}(x))\right|^2\right).$$

- Let us consider the simple case: $dX_t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} X_t dt + \begin{pmatrix} dZ_t \\ 0 \end{pmatrix}$.
- This equation integrates in:

$$\begin{aligned} X_s^1 &= x_1 + Z_s, \\ X_s^2 &= x_2 + s x_1 + \int_0^s Z_t dt. \end{aligned}$$

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- We see that the first component has scale $s^{1/\alpha}$, and the second $s^{1+1/\alpha}$.
- We can put the two component at the same scale by multiplying by the matrix:

$$\mathbb{T}_s^\alpha = \begin{pmatrix} s^{\frac{1}{\alpha}} I_d & 0 \\ 0 & s^{1+\frac{1}{\alpha}} I_d \end{pmatrix}.$$

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→ In the degenerated case, the deviations are to be considered with respect to the transport of the initial condition, re-normalized by the intrinsic time scale:

$$\left|(\mathbb{T}_s^\alpha)^{-1}(y - R_s x)\right| \asymp \frac{|y^1 - R_s^1 x|}{s^{\frac{1}{\alpha}}} + \frac{|y^2 - R_s^2 x|}{s^{1+\frac{1}{\alpha}}}.$$

The Frozen Process

The Frozen process has to present the features exposed before.

- Fix T a time horizon
- Fix $y \in \mathbb{R}^{nd}$ a terminal point.
- Let us denote $R_{s,T}(y)$ the resolvent associated to $\frac{d}{ds} R_{s,T} = A_s R_{s,T}$, with $R_{T,T} = I_{nd}$ in $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$.
- We define:

$$d\tilde{X}_s^{T,y} = A_s \tilde{X}_s^{T,y} ds + B\sigma(s, R_{s,T}(y))dZ_s, \ \tilde{X}_0^{T,y} = x,$$

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\end{aligned}
$$

Proposition

For fixed $t \leq s \leq T$, there exists $(S_u)_{u \geq 0}$, an $\mathbb{R}^{nd}$-valued stable process such that:

$$\tilde{X}_s^{t,x,T,y} \overset{(\mathrm{d})}{=} R_{s,t}x + \mathbb{T}_{s-t}^{\alpha} S_1.$$

We compute the Fourier transform of $\tilde{X}$:

$$\mathbb{E}(e^{i\langle p,\tilde{X}_s\rangle}) = e^{i\langle p,R_{s,t}x\rangle} \exp\left(-\int_t^s du \int_{S^{d-1}} |\langle p^1 + (s-u)p^2, \sigma_u \varsigma\rangle|^\alpha \mu(d\varsigma)\right).$$

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Change variables:

$$\begin{aligned} & \int_t^s du \int_{S^{d-1}} |\langle p^1 + (s-u)p^2, \sigma_u \varsigma\rangle|^\alpha \mu(d\varsigma) \\ = \ & \int_0^1 dv \int_{S^{d-1}} |\langle (s-t)^{1/\alpha} p^1 + v(s-t)^{1+1/\alpha} p^2, \bar{\sigma}_v \varsigma\rangle|^\alpha \mu(d\varsigma) \end{aligned}$$

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= &\int_0^1 dv \int_{S^{d-1}} \left|\left\langle \mathbb{T}^\alpha_{s-t} p, \begin{pmatrix} \bar{\sigma}_v \varsigma \\ v\bar{\sigma}_v \varsigma \end{pmatrix} \right\rangle\right|^\alpha \mu(d\varsigma)
\end{aligned}$$

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=\ &\int_0^1 dv\int_{S^{d-1}}|\langle (s-t)^{1/\alpha}p^1+v(s-t)^{1+1/\alpha}p^2,\bar{\sigma}_v\varsigma\rangle|^\alpha\mu(d\varsigma)\\
=\ &\int_0^1 dv\int_{S^{d-1}}\left|\left\langle \mathbb{T}^\alpha_{s-t}p,\begin{pmatrix}\bar{\sigma}_v\varsigma\\ v\bar{\sigma}_v\varsigma\end{pmatrix}\right\rangle\right|^\alpha\mu(d\varsigma)\\
=\ &\int_{S^{2d-1}}|\langle \mathbb{T}^\alpha_{s-t}p,\eta\rangle|^\alpha\mu_S(d\eta).
\end{aligned}$$

$$\mathbb{E}(e^{i\langle p,\tilde{X}_s^{t,x,T,y}\rangle}) = e^{i\langle p,R_{s,t}x\rangle}\exp\left(-\int_{S^{2d-1}}|\langle \mathbb{T}^\alpha_{s-t}p,\eta\rangle|^\alpha\mu_S(d\eta)\right).$$

Thus, we have the following upper bound:

$$\tilde{p}^{T,y}(t,s,x,z) \leq C \frac{\det(\mathbb{T}^{\alpha}_{s-t})^{-1}}{\left(1 + |(\mathbb{T}^{\alpha}_{s-t})^{-1}(y - R_{s,t}x)|\right)^{d+\alpha+1}}.$$

→ We have the following restrictions: $\alpha > (n-1)d - 1$.

- $d = 1, n = 2$ for $\alpha \in (0,2)$.
- $d = 1, n = 3$ for $\alpha \in (1,2)$.
- $d = 2, n = 2$ for $\alpha \in (1,2)$.

Thus, we have the following upper bound:

$$\tilde{p}^{T,y}(t,s,x,z) \leq C \frac{\det(\mathbb{T}^{\alpha}_{s-t})^{-1}}{\left(1+|(\mathbb{T}^{\alpha}_{s-t})^{-1}(y-R_{s,t}x)|\right)^{d+\alpha+1}}.$$

→ We have the following restrictions: $\alpha > (n-1)d-1$.

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What estimate on H?

$$\int_{(T-t)^{1/\alpha}}^{+\infty} \int_{S^{d-1}} \left(\tilde{p}^{T,y}(t,T,x+B\rho\theta,y) - \tilde{p}^{T,y}(t,T,x,y)\right) \frac{d\rho}{\rho^{1+\alpha}} \mu(d\theta)$$

→ Problematic case: $\rho\theta \in B((x-R_{t,T}y)^1, \varepsilon|(x-R_{t,T}y)^1|)$.

- Temperation and fast variable dependency $\sigma(t,x)=\sigma(t,x^2)$.

Theorem (H.-Menozzi, 2014)

- *There exists a unique solution to the SDE which admits a density.*
- *When $n = 1$ and $d = 2$, we have the upper bound: $\exists C \geq 1$, such that $\forall 0 \leq t < s \leq T$, $\forall (x, y) \in (\mathbb{R}^2)^2$,*

$$p(t,s,x,y) \leq C\bar{p}(t,s,x,y)\left(1 + \log(K \vee |(\mathbb{T}^{\alpha}_{s-t})^{-1}(y - R_{s,t}(x))|\right),$$

$$\bar{p}(t,s,x,y) = \frac{(s-t)^{-\left(1+\frac{2}{\alpha}\right)}}{\left(K + \frac{|(y-R_{s,t}x)^1|}{(s-t)^{\frac{1}{\alpha}}} + \frac{|(y-R_{s,t}x)^2|}{(s-t)^{1+\frac{1}{\alpha}}}\right)^{2+\alpha}} \times Q\left(|(y - R_{s,t}x)^1| + \frac{|(y-R_{s,t}x)^2|}{(s-t)}\right).$$

- *We also have the following diagonal lower bound in small time: $\forall 0 \leq t < s \leq T$, $\forall (x, y) \in (\mathbb{R}^2)^2$ tel que $|(\mathbb{T}^{\alpha}_{s-t})^{-1}(y - R_{s,t}(x))| \leq K$,*

$$p(t,s,x,y) \geq C^{-1}\det(\mathbb{T}^{\alpha}_{s-t})^{-1}.$$

Thank You!

F. Delarue and S. Menozzi.
Density estimates for a random noise propagating through a chain of differential equations.
Journal of Functional Analysis, 259(6):1577–1630, September 2010.

V. Kolokoltsov.
Symmetric stable laws and stable-like diffusion.
Proceedings of the London Mathematical Society, 80(03):725–768, 2000.

H. Mc Kean and I. Singer.
Curvature and eigen values of the Laplacian.
J. Differential Geometry, pages 43 – 69, 1967.

P. Sztonyk.
Estimates of tempered stable densities.
J. Theoret. Probab., 23(1):127–147, 2010.

T. Watanabe.
Asymptotic estimates of multi-dimensional stable densities and their applications.
Transactions of the American Mathematical Society, 359(6):2851–2879, 2007.