Painlevé equations and quantum cluster algebras

Marta Mazzocco

Based on Chekhov-M.M. arXiv:1509.07044 and Chekhov-M.M.-Rubtsov arXiv:1511.03851

Painlevé equations

$$\frac{d^2 w}{dz^2} = 6w^2 + z \qquad \frac{d^2 w}{dz^2} = 2w^3 + zw + \alpha$$

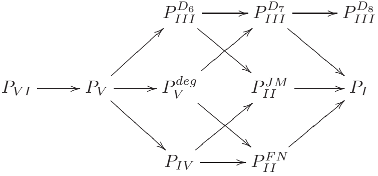
$$\frac{d^2 w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{d^2 w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\frac{d^2 w}{dz^2} = \frac{3w-1}{2w(w-1)}w_z^2 - \frac{1}{z}\frac{dw}{dz} + \frac{\gamma w}{z} + \frac{(w-1)^2}{z^2}\frac{\alpha w^2 + \beta}{w} + \frac{\delta w(w+1)}{w-1}$$

$$\frac{d^2 w}{dz^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)w_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w_z + \\ + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right]$$

Confluences of the Painlevé equations



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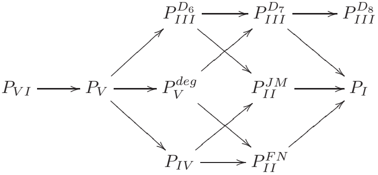

Example

Take $w(z) = \epsilon \tilde{w}(\tilde{z}) + \frac{1}{\epsilon^5}$, $z = \epsilon^2 \tilde{z} - \frac{6}{\epsilon^{10}}$, $\alpha = \frac{4}{\epsilon^{15}}$ then PII

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

becomes:

$$\frac{\mathrm{d}^2 \tilde{w}}{\mathrm{d}\tilde{z}^2} = 6\tilde{w}^2 + \tilde{z} + \epsilon^6(2\tilde{w}^3 + \tilde{z}\tilde{w}),$$

that for $\epsilon \to 0$ is PI.

All Painlevé equations are **isomonodromic deformation equations** (Jimbo-Miwa 1980)

$$\frac{\mathrm{d}B}{\mathrm{d}\lambda} - \frac{\mathrm{d}A}{\mathrm{d}z} = [A, B]$$

$A = A(\lambda; z, w, w_z),\ B = B(\lambda; z, w, w_z) \in \mathfrak{sl}_2.$

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This means that **the monodromy data** of the linear system

$$\frac{\mathrm{d}Y}{\mathrm{d}\lambda} = A(\lambda; z, w, w_z)Y$$

are **locally constant along solutions of the Painlevé equation**.

Monodromy manifolds for the Painlevé equations

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- Each point in the monodromy manifold determines a unique local solution to the Painlevé equation (modulo Okamoto transformations).

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P-eqs	Polynomial φ
PVI	$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$
PV	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$
PV_{deg}	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_1 - 1$
PIV	$x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2(x_2 + x_3) + \omega_2(1 + \omega_1 - \omega_2)$
$PIII$	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_1 - 1$
$PIII^{D_7}$	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 - x_2$
$PIII^{D_8}$	$x_1x_2x_3 + x_1^2 + x_2^2 - x_2$
PII^{JM}	$x_1x_2x_3 - x_1 + \omega_2 x_2 - x_3 - \omega_2 + 1$
PII^{FN}	$x_1x_2x_3 + x_1^2 + \omega_1 x_1 - x_2 - 1$
PI	$x_1x_2x_3 - x_1 - x_2 + 1$

Saito and van der Put

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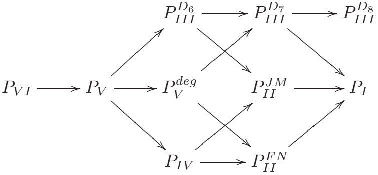
Use the confluence scheme of the Painlevé equations.



Chewing-gum moves

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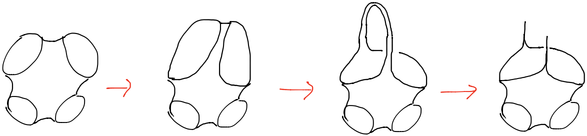
- **Hooking holes:**



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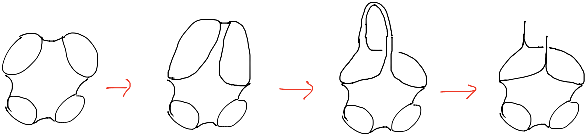
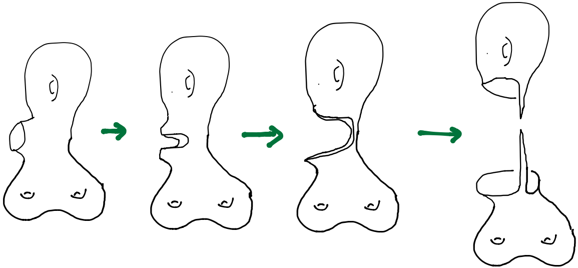


- **Pinching two sides of the same hole:**



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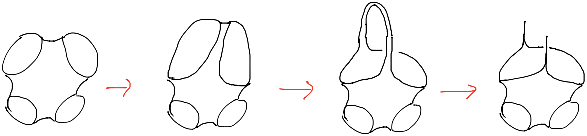

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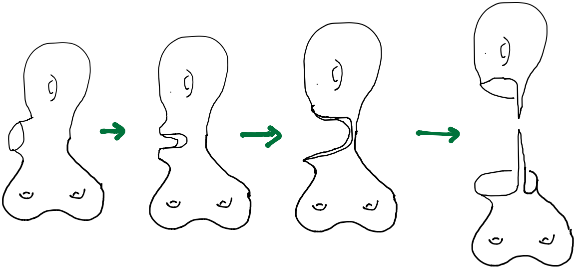
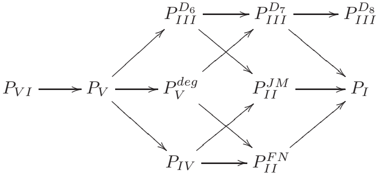
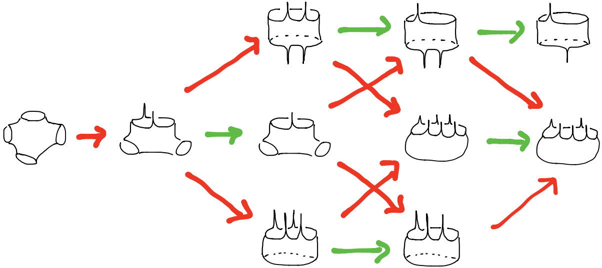


Bordered cusps á la Fomin-Shapiro-Thurston.

Geometric interpretation of confluences





[Chekhov-M.M.-Rubtsov arXiv:1511.03851]

Chewing-gums in Poincaré uniformisation

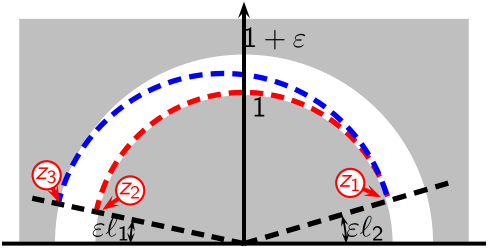
Start with a Riemann surface with holes:

$$\Sigma = \mathbb{H}/\Delta,$$

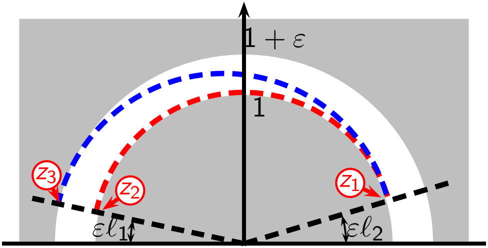
where Δ is a *Fuchsian group*, i.e. a discrete sub-group of $\mathbb{P}SL_2(\mathbb{R})$.
We want to understand

- What happens to the fundamental domain of Δ under the chewing-gum move.
- What happens to the closed geodesics.

Chewing gum

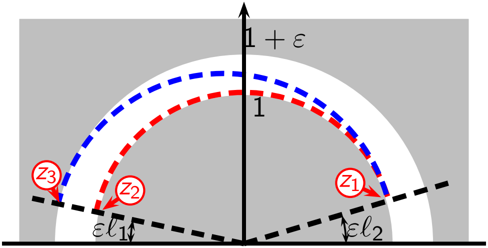


Chewing gum



- $\left(\sinh \frac{d_{\mathbb{H}}(z_1,z_2)}{2}\right)^2 = \frac{|z_1-z_2|^2}{4\Im z_1 \Im z_2}$

Chewing gum



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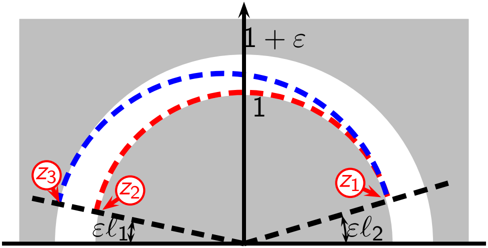
- $e^{d_{\mathbb{H}}(z_1,z_2)} \sim \frac{1}{l_1 l_2 \epsilon^2} + \frac{(l_1+l_2)^2}{l_1 l_2} + \mathcal{O}(\epsilon),$

Chewing gum



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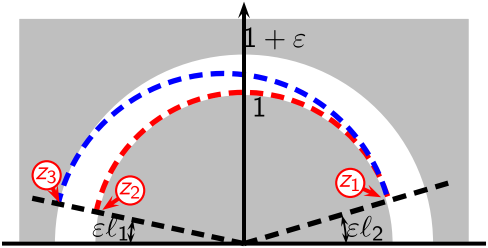

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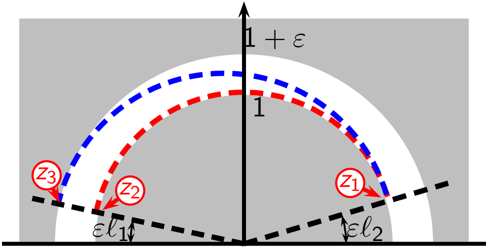
$\Rightarrow$ Rescale all geodesic lengths by ϵ and take the limit $\epsilon \to 0$.

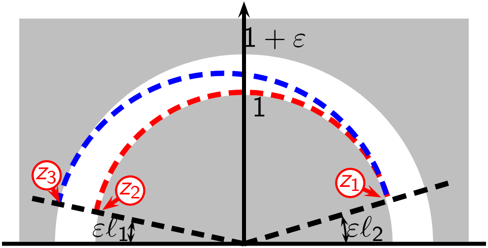
[Chekhov-M.M. arXiv:1509.07044]

Chewing gum



As $\epsilon \to 0$:

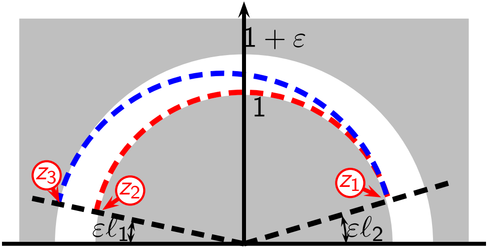
Chewing gum



As $\epsilon \to 0$:

- Vertical segment becomes infinitely distant from the collars.

Chewing gum



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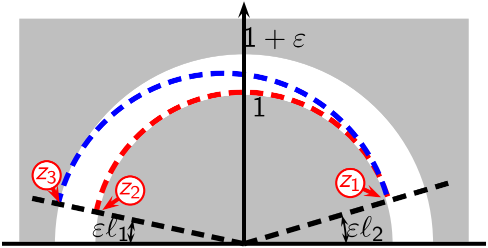

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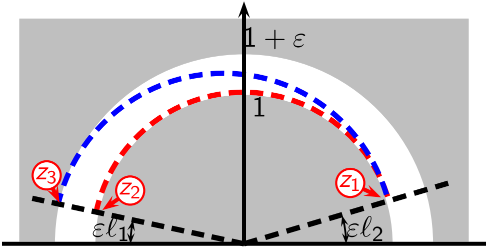

As $\epsilon \to 0$:

- Vertical segment becomes infinitely distant from the collars.
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- Collars become horocycles.
- Lengths of closed geodesics become λ lengths of infinite arcs.

Geodesic lengths

Theorem

The geodesic length functions form an algebra with multiplication:

$$G_\gamma G_{\tilde\gamma} = G_{\gamma\tilde\gamma} + G_{\gamma\tilde\gamma^{-1}}.$$

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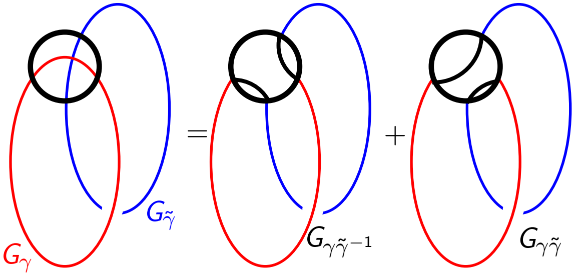
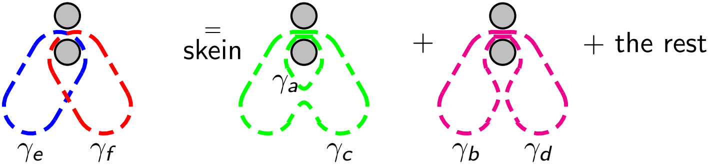
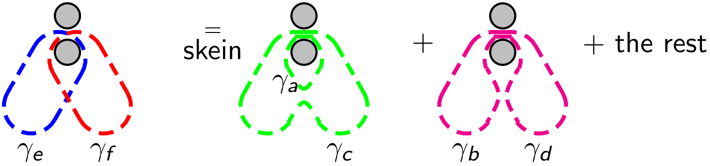



γ_e
γ_f
$\overset{=}{\text{skein}}$
γ_a
γ_c
$+$
γ_b
γ_d
$+$ the rest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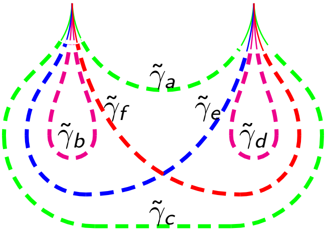
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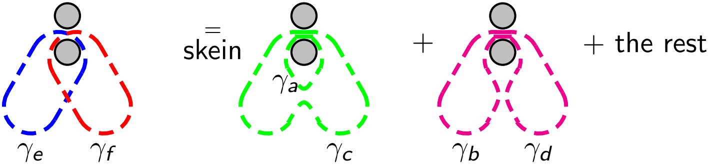
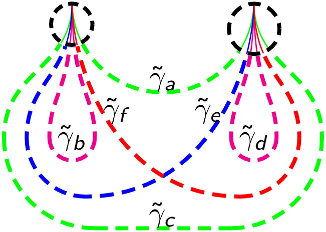
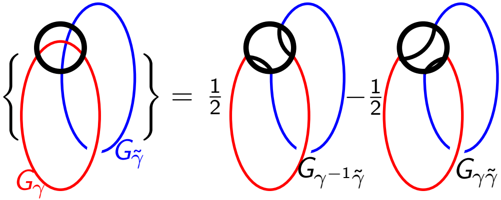




$$G_{\tilde\gamma_e} G_{\tilde\gamma_f} = G_{\tilde\gamma_a} G_{\tilde\gamma_c} + G_{\tilde\gamma_b} G_{\tilde\gamma_d}$$

Poisson structure

$$\{G_\gamma, G_{\tilde\gamma}\} = \frac{1}{2} G_{\gamma\tilde\gamma} - \frac{1}{2} G_{\gamma\tilde\gamma^{-1}}.$$



Poisson brackets after chewing-gum

Definition

A cusped lamination is a lamination made of arcs that can only meet at the cusps.

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A cusped lamination is complete if all geodesic functions and all λ–lengths of arcs in the Riemann surface are Laurent polynomials of the λ–lengths of the arcs in the cusped lamination.

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Theorem

Given a Riemann surface of any genus, any number of holes and at least one cusp, there always exists a complete cusped lamination [Chekhov-M.M. ArXiv:1509.07044].

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The Poisson algebra of the λ-lengths of a complete cusped lamination is a cluster algebra [ArXiv:1509.07044].

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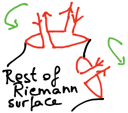

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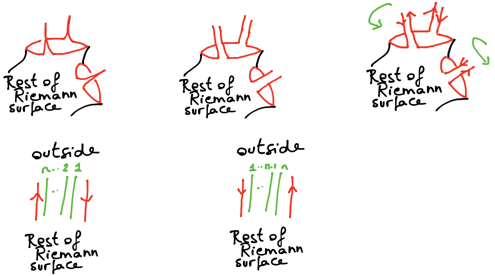

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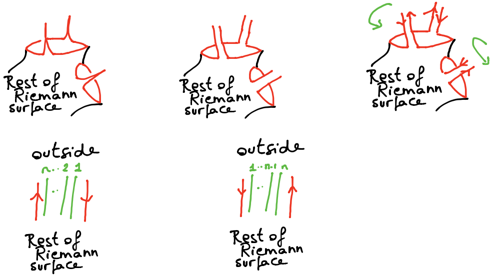
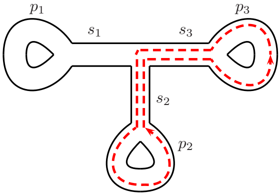


$$\{g_{s_i,t_j}, g_{p_r,q_l}\} = g_{s_i,t_j} g_{p_r,q_l} \mathcal{I}_{s_i,t_j,p_r,q_l}$$

$$\mathcal{I}_{s_i,t_j,p_r,q_l} = \frac{\epsilon_{i-r}\delta_{s,p}+\epsilon_{j-r}\delta_{t,p}+\epsilon_{i-l}\delta_{s,q}+\epsilon_{j-l}\delta_{t,q}}{4}$$

Shear coordinates in the Teichmüller space

Fatgraph:



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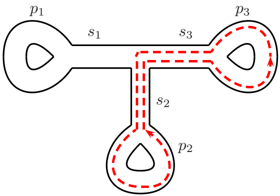
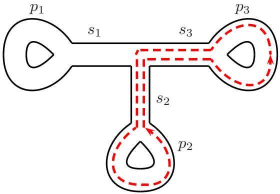


Decompose each hyperbolic element in Right, Left and Edge matrices Fock, Thurston

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

$$X_y := \begin{pmatrix} 0 & -\exp\left(\frac{y}{2}\right) \\ \exp\left(-\frac{y}{2}\right) & 0 \end{pmatrix}.$$



The three geodesic lengths: $x_i = \mathrm{Tr}(\gamma_{jk})$

$$x_1 = e^{s_2+s_3} + e^{-s_2-s_3} + e^{-s_2+s_3} + (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{s_3} + (e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}})e^{-s_2}$$
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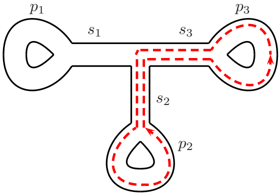

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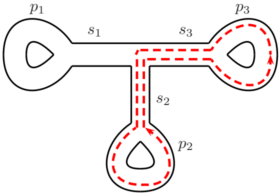

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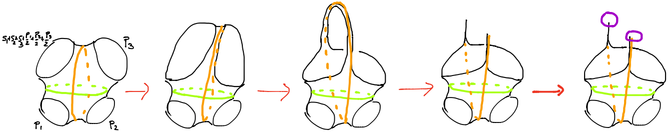
The confluence from PVI to PV is realised by

$$p_3 \to p_3 - 2\log[\epsilon], \qquad \epsilon \to 0$$



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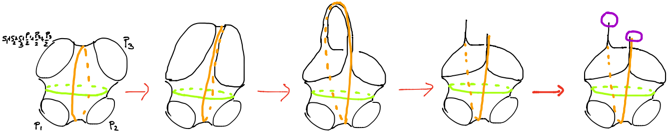

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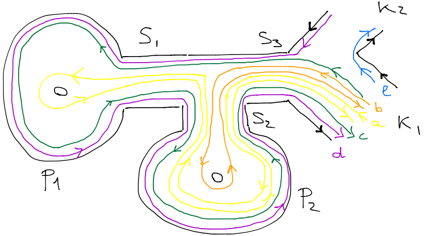
[Chekhov-M.M.-Rubtsov arXiv:1511.03851]

Lamination for PV

PV



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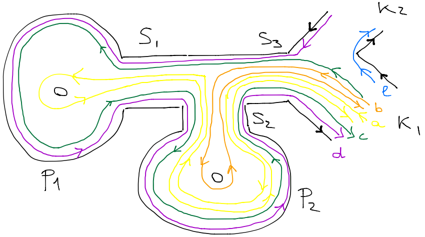

$$\{g_{s_i,t_j}, g_{p_r,q_l}\} = g_{s_i,t_j} g_{p_r,q_l} \frac{\epsilon_{i-r}\delta_{s,p}+\epsilon_{j-r}\delta_{t,p}+\epsilon_{i-l}\delta_{s,q}+\epsilon_{j-l}\delta_{t,q}}{4}$$

$$\begin{aligned} \{b,d\} &= \{g_{1_3,1_4}, g_{2_1,1_8}\} \\ &= g_{1_3,1_4} g_{2_1,1_8} \frac{\epsilon_{3-1}\delta_{1,2}+\epsilon_{4-1}\delta_{1,2}+\epsilon_{3-8}\delta_{1,1}+\epsilon_{4-8}\delta_{1,1}}{4} \\ &= -bd\frac{1}{2} \end{aligned}$$

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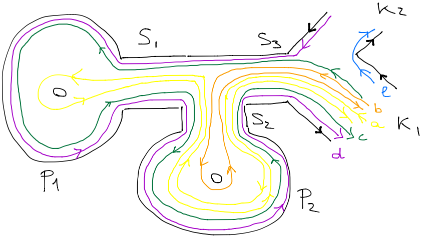

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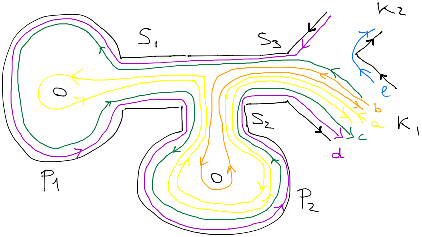

$$\gamma_b = X(k_1)RX(s_3)RX(s_2)RX(p_2)RX(s_2)LX(s_3)LX(k_1)$$

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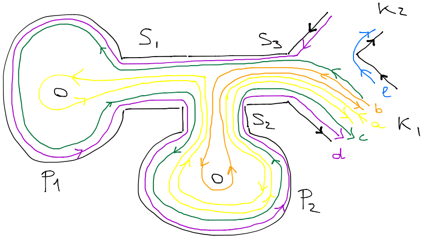

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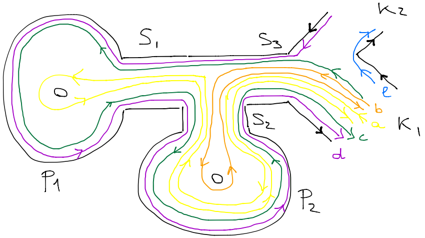

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$$b = \exp(k_1 + s_3 + s_2 + \frac{p_2}{2}).$$

General result

Theorem

All λ-lengths in the complete cusped lamination are monomials in the exponentiated shear coordinates.

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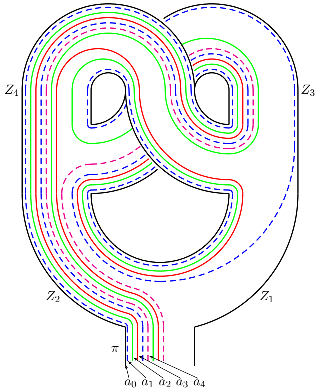
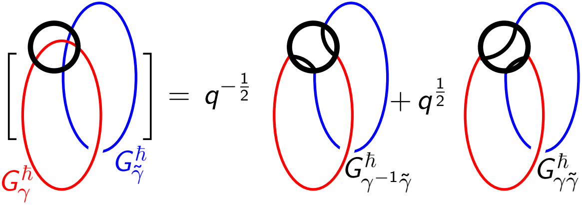


Quantisation

For standard geodesic lengths $G_\gamma \to G^\hbar_\gamma$ [Chekhov-Fock '99]:



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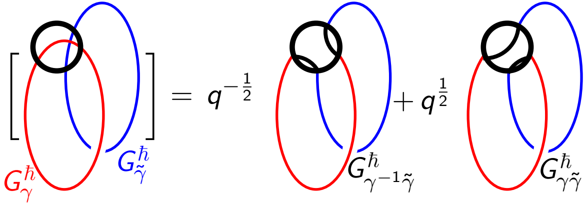

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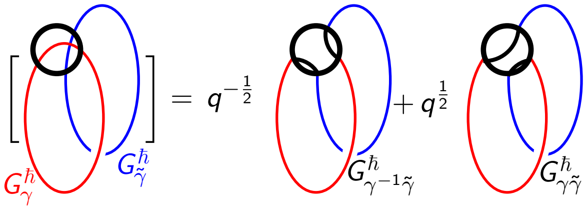

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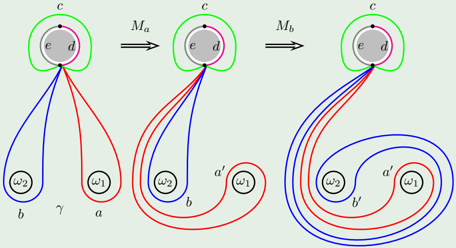
This identifies the geometric basis of the quantum cluster algebras introduced by Berenstein and Zelevinsky.

Mutations

Example

Riemann sphere with three holes, and two cusps on one of the holes. Frozen variables: d, e. Exchangeable variables: a, b, c.



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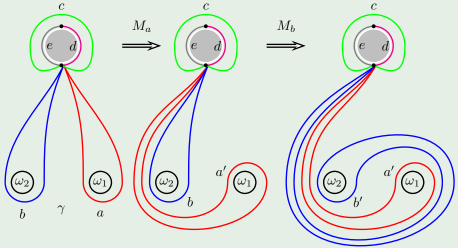

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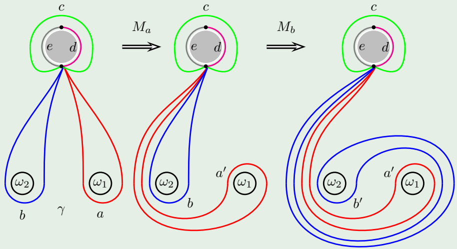

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$a = g_{1_5,1_6}$, $b = g_{1_3,1_4}$, $d = g_{1_8,2_2}$, $\{a, b\} = ab$, $\{a, d\} = -\frac{ad}{2}$ (frozen variables are NOT central).

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Open problem: close the Poisson algebra of geodesic functions on any Riemann surface $\Sigma_{g,s}$ of genus g with $s > 0$ holes.

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Decorated character variety [Chekhov-M.M.-Rubtsov arXiv:1511.03851]

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$$\begin{aligned} \mathrm{tr}_K :\quad & SL_2(\mathbb{C}) \to \mathbb{C} \\ & M \mapsto \mathrm{tr}(MK), \qquad \text{where} \quad K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Conclusion

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Many thanks for your attention!!!