Open intersection numbers, MKP integrable hierarchy and W-constraints

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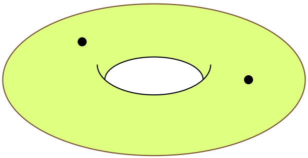


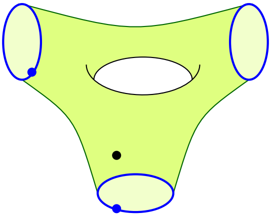
$h = 1, l = 2$

The Kontsevich-Witten description of intersection theory on the moduli spaces $\mathcal{M}_{h,l}$

$$\int_{\overline{\mathcal{M}}_{h;l}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \dots \psi_l^{\alpha_l}$$

- **Kontsevich matrix model**
- **KdV tau-function**
- **Virasoro constraints**

Intersection numbers on moduli spaces of open Riemann surfaces



$h = 1, b = 3, k = 2, l = 1$

$$\text{“}\int_{\overline{\mathcal{M}}_{h,b,k,l}} \psi_1^{\alpha_1} \dots \psi_l^{\alpha_l} \phi_{l+1}^{\beta_1} \dots \phi_{l+k}^{\beta_k}\text{”}$$

Recently [R. Pandharipande, J. Solomon and R. Tessler; A. Buryak ’14] described (conjectured) intersection theory on $\mathcal{M}_{2h+b-1,k,l}$, that is the moduli spaces of Riemann surface with h handles, b boundaries, k marked points on the boundary and l interior marked points

- **Matrix model? Tau-function? Virasoro (W)-constraints?**

The Kontsevich–Penner matrix integral

$$\tau_n = \det(\Lambda)^n \mathcal{C}^{-1} \int_{M\times M} [d\Phi] \exp\left(-\mathrm{Tr}\left(\frac{\Phi^3}{3!} - \frac{\Lambda^2\Phi}{2} + n\log\Phi\right)\right)$$

Tau-function of the MKP hierarchy, describes both **closed** and **open** intersection numbers.

[A.A. ’14]

n	0	1
Intersection numbers	Closed	Open
Integrable hierarchy	KdV	KP
Algebra of constraints	Heisenberg+ Virasoro	Virasoro + $W^{(3)}$
Specified by	String	String+Dilaton
Cut-and-join operator	$e^{W_{KW}}\cdot 1$	“$e^{W_1+W_2/2}$” $\cdot 1$

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[A.A. '14]

Parameter n counts the number of boundaries [B. Safnuk '16]

n	0	1	arbitrary
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Specified by	String	String+Dilaton	String+Dilaton
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KONTSEVICH–WITTEN TAU-FUNCTION

Let $\overline{\mathcal{M}}_{h;l}$ be the Deligne–Mumford compactification of the moduli space of genus h complex curves X with l marked points $x_1, \ldots, x_l$. The generating function of the intersection numbers of ψ-classes

$$\int_{\overline{\mathcal{M}}_{h;l}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \ldots \psi_l^{\alpha_l} = \langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_l} \rangle_h$$

$$\mathcal{F}_{KW}(\mathbf{T}, \hbar) = \sum_{h=0}^{\infty} \hbar^{2h-2} \left\langle \exp\left(\hbar \sum_{m=0}^{\infty} T_m \tau_m \right) \right\rangle_h$$

is the Kontsevich–Witten tau-function of the KdV hierarchy

$$\tau_{KW}(\mathbf{T}, \hbar) = \exp\left(\mathcal{F}_{KW}(\mathbf{T}, \hbar)\right)$$

[E. Witten '91; M. Kontsevich '92]

Below we use the variables $t_{2k+1} = T_k/(2k+1)!!$, times of the KP hierarchy.

The Kontsevich–Witten tau-function is a formal series in odd times t_{2k+1} with rational coefficients. In the Miwa parametrization

$$t_k = \frac{1}{k}\mathrm{Tr}\,\Lambda^{-k}$$

it is equal to the asymptotic expansion of the **Kontsevich matrix integral** over the $M \times M$ Hermitian matrix Φ:

$$\tau_{KW}(\mathbf{t}, \hbar) = \mathcal{C}^{-1} \int [d\Phi] \exp\left(-\frac{1}{\hbar}\mathrm{Tr}\left(\frac{\Phi^3}{3!} + \frac{\Lambda\Phi^2}{2}\right)\right)$$

All t_k can be considered as independent variables as the size of the matrices M tends to infinity and in this limit the integral yields the Kontsevich–Witten tau-function.

It is easy to show that this matrix integral defines a tau-function of the KdV integrable hierarchy.

VIRASORO CONSTRAINTS FOR τ_{KW}

Consider a bosonic current on the curve $y^2 = x$ with odd boundary conditions

$$\widehat{J}_o(x) = \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \left((2k+1)\tilde{t}_{2k+1} x^{k-\frac{1}{2}} + \frac{1}{x^{k+\frac{3}{2}}} \frac{\partial}{\partial t_{2k+1}} \right)$$

where the time variables are subject to the dilaton shift

$$\tilde{t}_k = t_k - \frac{\delta_{k,3}}{3\hbar}$$

Then, we can construct

$$\widehat{\mathcal{L}}(z) = \sum_{k=-\infty}^{\infty} \frac{\widehat{\mathcal{L}}_k}{z^{k+2}} = \frac{1}{2} {}^*_* \widehat{J}_o^2(z) {}^*_* + \frac{1}{16z^2}$$

where we use usual bosonic normal order so that

$$\left[\widehat{\mathcal{L}}_k, \widehat{\mathcal{L}}_m \right] = (k-m)\widehat{\mathcal{L}}_{k+m} + \frac{1}{12} k(k^2-1)\delta_{k,-m}$$

with central charge $c = 1$.

From the Virasoro constraints

$$\hat{\mathcal{L}}_k \, \tau_{KW}(\mathbf{t};\hbar) = 0, \qquad k \geq -1$$

it follows that the Kontsevich–Witten tau-function can be described by a cut-and-join operator [A. A. '10]

$$\tau_{KW}(\mathbf{t};\hbar) = e^{\hbar \widehat{W}_{KW}} \cdot 1$$

where

$$\widehat{W}_{KW} = \frac{1}{3} \sum_{k,m \geq 0} (2k+1)\,(2m+1)\, t_{2k+1} t_{2m+1} \frac{\partial}{\partial t_{2k+2m-1}} + \frac{1}{3!} \sum_{k,m \geq 0} (2k+2m+5)\, t_{2k+2m+5} \frac{\partial^2}{\partial t_{2k+1} \partial t_{2m+1}} + \frac{t_1^3}{3!} + \frac{t_3}{8}$$

Operator $\widehat{W}_{KW}$ describes a topological recursion on the level of tau-function

The moduli spaces for the open Riemann surfaces (Riemann surfaces with boundaries) were described for the disc case in [R. Pandharipande, J. Solomon and R. Tessler '14] and for the higher genera case in [R. Tessler '15].

$$\dim_{\mathbb{R}} \mathcal{M}_{h,b,k,l} = 6h - 6 + 3b + k + 2l.$$

We can consider the intersection numbers

$$\text{“}\int_{\overline{\mathcal{M}}_{h,b,k,l}} \psi_1^{\alpha_1} \ldots \psi_l^{\alpha_l} \phi_{l+1}^{\beta_1} \ldots \phi_{l+k}^{\beta_k}\text{”} = \left\langle \tau_{\alpha_1} \ldots \tau_{\alpha_l} \sigma_{\beta_1} \ldots \sigma_{\beta_k} \right\rangle_{h,b}$$

where ψ_j are the the first Chern classes of the bundles $\mathcal{L}_i$ corresponding to the interior points and ϕ_j are their analogs for the boundary points. In in [R. Pandharipande, J. Solomon and R. Tessler '14] all intersection numbers of the form

$$\text{“}\int_{\overline{\mathcal{M}}_{0,1,k,l}} \psi_1^{\alpha_1} \ldots \psi_l^{\alpha_l} \phi_{l+1}^{0} \ldots \phi_{l+k}^{0}\text{”}$$

were constructed.

The generating function of all these intersection numbers

$$\mathcal{F}_Q(\mathbf{T};\mathbf{S},\hbar) = \sum_{h=0}^{\infty}\sum_{b=0}^{\infty}\hbar^{2h-2+b}Q^b\left\langle \exp\left(\hbar\sum_{k\geq 0}(T_k\tau_k + S_k\sigma_k)\right)\right\rangle_{h,b}$$

and

$$\tau_Q(\mathbf{T};\mathbf{S},\hbar) = e^{\mathcal{F}_Q(\mathbf{T};\mathbf{S},\hbar)}$$

In [R. Tessler '15] all coefficients of the generating function for $Q = 1$ (that is the function, to which the components of the moduli spaces with different number of boundaries contributes with the same weight) and $\mathbf{S_0} = \{S_0, 0, 0, \dots\}$ (that is without descendants on the boundary),

$$\tau_1(\mathbf{T};\mathbf{S_0},\hbar).$$

were calculated. Obtained all-genera generating function is uniquely specified by the so called **open KdV** equations and the Virasoro constraints.

In [A. Buryak, '14] the generating function was generalized to describe the descendants on the boundary, and the Virasoro constrains for this conjectural generalized (or extended) generating function were established.

$$\tau_1(\mathbf{T};\mathbf{S},\hbar).$$

From the definition it follows that for $Q=0$ only the components without boundaries contribute, so that the generating function does not depend on S_k's and coincides with the Kontsevich-Witten tau-function

$$\tau_0(\mathbf{T};\mathbf{S},\hbar)=\tau_{KW}(\mathbf{T},\hbar).$$

$$\tau_Q(\mathbf{T};\mathbf{S},\hbar)=\tau_Q(\mathbf{T};\mathbf{S},1)\Big|_{T_k\mapsto\hbar^{\frac{2k+1}{3}}T_k,\,S_k\mapsto\hbar^{\frac{2k+2}{3}}S_k}$$

Thus, we can omit $\hbar$ and then restore it if necessary.

We **unify** two infinite sets of variables T_k and S_k, corresponding to the descendants in the interior and on the boundary:

$$T_k = (2k+1)!!\, t_{2k+1}, \quad S_k = 2^{k+1}(k+1)!\, t_{2k+2}$$

Proposition: the extended generating function of open intersection numbers $\tau_1(\mathbf{t})$ is a KP tau-function, given by the matrix integral

$$\tau_1(\mathbf{t}) = \mathcal{C}^{-1} \det(\Lambda) \int [d\Phi] \exp\left(-\mathrm{Tr}\left(\frac{\Phi^3}{3!} - \frac{\Lambda^2\Phi}{2} + \log\Phi\right)\right)$$

where

$$t_k = \frac{1}{k}\mathrm{Tr}\,\Lambda^{-k}$$

This matrix integral belongs to the family of the **generalized Kontsevich models**.

KP tau-function!

The $W_{1+\infty}$ algebra of infinitesimal symmetries of the KP hierarchy can be described in terms of the bosonic current $\widehat{J}(z) = \sum \widehat{J}_k z^{-k-1}$, where

$$\widehat{J}_k = \begin{cases} \dfrac{\partial}{\partial t_k} & \text{for} \quad k > 0, \\ 0 & \text{for} \quad k = 0, \\ -k t_{-k} & \text{for} \quad k < 0 \end{cases}$$

$\widehat{J}(z)$ generates the **Heisenberg algebra**. ${}_*^*\widehat{J}(z)^2{}_*^*$ generates the **Virasoro algebra**:

$$\widehat{L}_m = \frac{1}{2} \sum_{k+l=-m} k\, l\, t_k t_l + \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{k+l=m} \frac{\partial^2}{\partial t_k \partial t_l}$$

${}_*^*\widehat{J}(z)^3{}_*^*$ generates the $W^{(3)}$ **algebra**:

$$\widehat{M}_k = \frac{1}{3} \sum_{a+b+c=-k} a\, b\, c\, t_a\, t_b\, t_c + \sum_{c-a-b=k} a\, b\, t_a\, t_b \frac{\partial}{\partial t_c} + \sum_{b+c-a=k} a\, t_a \frac{\partial^2}{\partial t_b \partial t_c} + \frac{1}{3} \sum_{a+b+c=k} \frac{\partial^3}{\partial t_a \partial t_b \partial t_c}$$

Using the Kac–Schwarz operators we can show that the tau-function τ_1 is an eigenfunction of the Virasoro operators:

$$\widehat{L}_k^{(1)} = \widehat{L}_{2k} + (k+2)\widehat{J}_{2k} - \widehat{J}_{2k+3} + \left(\frac{1}{8} + \frac{3}{2}\right)\delta_{k,0}, \quad k \geq -1$$

$$\widehat{M}_k^{(1)} = \widehat{M}_{2k} + 2(k+3)\widehat{L}_{2k} - 2\widehat{L}_{2k+3} - 2(k+3)\widehat{J}_{2k+3} + \left(\frac{95}{12} + 6k + \frac{4}{3}k^2\right)\widehat{J}_{2k} + \widehat{J}_{2k+6} + \frac{23\,\delta_{k,0}}{3}, \quad k \geq -2$$

These operators belong to $W_{1+\infty}$ algebra of symmetries of KP and annihilate the tau-function

$$\widehat{L}_k^{(1)}\,\tau_1 = 0, \qquad k \geq -1$$
$$\widehat{M}_k^{(1)}\,\tau_1 = 0, \qquad k \geq -2$$

The Kontsevich-Penner model

$$\tau_n = \frac{\int [d\Phi] \det\left(1+\frac{\Phi}{\Lambda}\right)^{-n} \exp\left(-\mathrm{Tr}\left(\frac{\Phi^3}{3!}+\frac{\Lambda\Phi^2}{2}\right)\right)}{\int [d\Phi] \exp\left(-\mathrm{Tr}\,\frac{\Lambda\Phi^2}{2}\right)}$$

$$= \det(\Lambda)^n \mathcal{C}^{-1} \int [d\Phi] \exp\left(-\mathrm{Tr}\left(\frac{\Phi^3}{3!}-\frac{\Lambda^2\Phi}{2}+n\log\Phi\right)\right)$$

The bilinear identity satisfied by a tau-function $\tau_n(\mathbf{t})$ of the **modified Kadomtsev–Petviashvili (MKP)** integrable hierarchy for $m \geq n$

$$\oint_\infty z^{m-n} e^{\sum_{k>0}(t_k-t'_k)z^k}\, \tau_m(\mathbf{t}-[z^{-1}])\, \tau_n(\mathbf{t}'+[z^{-1}])dz = 0$$

encodes all nonlinear equations of the hierarchy.

Using the Kac–Schwarz description of the corresponding point of the Sato Grassmannian it is easy to show that the operators from the $W_{1+\infty}$ algebra

$$\widehat{\mathrm{L}}_{-1}^{(n)} = \widehat{L}_{-2} - \frac{\partial}{\partial t_1} + 2n\, t_2,$$
$$\widehat{\mathrm{L}}_{0}^{(n)} = \widehat{L}_{0} - \frac{\partial}{\partial t_3} + \frac{1}{8} + \frac{3n^2}{2},$$
$$\widehat{\mathrm{L}}_{1}^{(n)} = \widehat{L}_{2} - \frac{\partial}{\partial t_5} + 3n\frac{\partial}{\partial t_2}$$

satisfy the commutation relation of the subalgebra of the Virasoro algebra

$$\left[\widehat{\mathrm{L}}_{i}^{(n)}, \widehat{\mathrm{L}}_{j}^{(n)}\right] = 2(i-j)\, \widehat{\mathrm{L}}_{i+j}^{(n)}, \quad i,j = -1,0,1$$
$$\widehat{\mathrm{L}}_{k}^{(n)} \tau_n = 0, \quad k = -1,0,1$$

$k = -1$ is the **string equation** ; $k = 0$ is the **dilaton equation**

The string and dilaton (?) equations were derived by [E. Brezin and S. Hikami '12].

The Virasoro operators

$$\widehat{\mathrm{L}}_k^{(n)} = \widehat{L}_{2k} - \frac{\partial}{\partial t_{2k+3}} + 3n\frac{\partial}{\partial t_{2k}} + \sum_{j=1}^{k-1} \frac{\partial^2}{\partial t_{2j}\partial t_{2k-2j}} + \left(\frac{1}{8} + \frac{3n^2}{2}\right)\delta_{k,0} + 2nt_2\delta_{k,-1}, \quad k \geq -1$$

$$\left[\widehat{\mathrm{L}}_k^{(n)}, \widehat{\mathrm{L}}_m^{(n)}\right] = 2(k-m)\widehat{\mathrm{L}}_{k+m}^{(n)}$$

annihilate the tau-function

$$\widehat{\mathrm{L}}_k^{(n)}\tau_n = 0, \quad k \geq -1$$

Remark: the situation is similar to the case of the Gaussian Hermitian matrix model. For this model we also have an infinite algebra of the Virasoro constraints, but only an $sl(2)$ subalgebra of it belongs to the $W_{1+\infty}$ algebra of KP symmetries. [M. Mulase, '94]
Proposition: the string and dilaton equations uniquely specify the solution of the KP hierarchy in the same way as the string equation specifies the KW tau-function of the KdV hierarchy.

$$\widehat{\mathrm{M}}_k^{(n)} = \widehat{M}_{2k} - 2\widehat{L}_{2k+3} + \widehat{J}_{2k+6} + \left(3(k+1)n^2 + \frac{1}{4}\right)\widehat{J}_{2k}$$
$$+(k+4)n\left(\widehat{L}_{2k} - \widehat{J}_{2k+3}\right) + 2\left(n^2 + \frac{1}{4}\right)n\delta_{k,0} + 4\,n^2 t_2\delta_{k,-1} + 16\,n^2 t_4\delta_{k,-2}$$
$$+(k-2)n\sum_{j=1}^{k-1}\frac{\partial^2}{\partial t_{2j}\partial t_{2k-2j}} - \frac{4}{3}\sum_{i+j+l=k}\frac{\partial^3}{\partial t_{2i}\partial t_{2j}\partial t_{2l}}$$

for $k \geq -2$. Commutation relations between the Virasoro and W-operators

$$\left[\widehat{\mathrm{L}}_k^{(n)}, \widehat{\mathrm{M}}_l^{(n)}\right] = 2\,(2k-l)\widehat{\mathrm{M}}_{k+l}^{(n)} - 4\left(k(k-1) - 2\delta_{k,-1}\right)\,n\widehat{\mathrm{L}}_{k+l}^{(n)} + 8\sum_{j=1}^{k-1} j\frac{\partial}{\partial t_{2k-2j}}\widehat{\mathrm{L}}_{l+j}^{(n)}$$

for $k \geq -1$ and $l \geq -2$, so that

$$\widehat{\mathrm{M}}_k^{(n)}\tau_n = 0, \quad k \geq -2.$$

$W^{(n)}$ algebra can be naturally described in terms of free bosonic fields

[A. B. Zamolodchikov '85]
[V. A. Fateev and A. B. Zamolodchikov '87]
[V. A. Fateev and S. L. Lukyanov '88]

For the case of $sl(n)$ it in can be represented in terms of the vector of $n-1$ independent bosonic currents $\vec{J} = (J_{(1)}, J_{(2)}, \ldots, J_{(n-1)})$

$$J_{(k)}(x) = \partial_x \phi_{(k)}(x) = \sum_{m=-\infty}^{\infty} J_m^{(k)} x^{-m-1}, \qquad \left[J_m^{(k)}, J_n^{(l)}\right] = m\,\delta_{k,l}\,\delta_{m,-n}$$

and is generated by

$$R_n(u) = -{}^*_* \prod_{m=1}^{n} (u - \vec{h}_m \cdot \vec{J}) {}^*_*$$

Here the $\vec{h}_m$'s are the weight vectors of the fundamental representation of $sl(n)$.

In particular, for $n = 3$, the $W^{(3)}$ algebra is generated by

$$R_3(u) = -{}^*_*\prod_{m=1}^{3}(u - \vec{h}_m\vec{J}){}^*_* = -u^3 - u\,{}^*_*\prod_{i<j}(\vec{h}_i\cdot\vec{J})(\vec{h}_j\cdot\vec{J}){}^*_* + {}^*_*\prod_i \vec{h}_i\cdot\vec{J}\,{}^*_*$$
$$= -u^3 + u\,\mathcal{L}(x) + \mathcal{M}(x)$$

Then

$$\mathcal{L}(x) = \sum_{k=\infty}^{\infty}\frac{\mathcal{L}_k}{x^{k+2}} = \frac{1}{2}\left({}^*_*J_{(1)}(x)^2 + J_{(2)}(x)^2{}^*_*\right),$$
$$\mathcal{M}(x) = \sum_{k=-\infty}^{\infty}\frac{\mathcal{M}_k}{x^{k+3}} := \frac{1}{\sqrt{6}}\left({}^*_*J_{(1)}(x)^2J_{(2)}(x) - \frac{1}{3}J_{(2)}(x)^3{}^*_*\right)$$

generate $W^{(3)}$ algebra with $c = 2$.

Twisted field

Let us introduce two bosonic currents

$$\widehat{J}_e(x) = \sum_{k=0}^{\infty} \left(\sqrt{\frac{2}{3}} k \, \tilde{t}_{2k} x^{k-1} + \sqrt{\frac{3}{2}} \frac{1}{x^{k+1}} \frac{\partial}{\partial t_{2k}} \right) + \sqrt{\frac{3}{2}} \frac{n}{x},$$

$$\widehat{J}_o(x) = \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \left((2k+1) \tilde{t}_{2k+1} x^{k-\frac{1}{2}} + \frac{1}{x^{k+\frac{3}{2}}} \frac{\partial}{\partial t_{2k+1}} \right)$$

with the dilaton shift

$$\tilde{t}_k = t_k - \frac{\delta_{k,3}}{3}$$

We see that the odd current $\widehat{J}_o(z)$ is the same as the current from the description of the Kontsevich-Witten tau-function and $\widehat{J}_e(z)$ (up to trivial rescailing of the times) is the untwisted current.

Then

$$\widehat{\mathcal{L}}^{(n)}(x) = \sum_{k=\infty}^{\infty} \frac{\widehat{\mathcal{L}}_k^{(n)}}{x^{k+2}} = \frac{1}{2}\left({}^*_* \widehat{J}_o(x)^2 + \frac{1}{8x^2} + \widehat{J}_e(x)^2 {}^*_* \right),$$

$$\widehat{\mathcal{M}}^{(n)}(x) = \sum_{k=-\infty}^{\infty} \frac{\widehat{\mathcal{M}}_k^{(n)}}{x^{k+3}} := \frac{1}{\sqrt{6}}\left({}^*_* \widehat{J}_e(x)\left(\widehat{J}_o(x)^2 + \frac{1}{8x^2}\right) - \frac{1}{3}\widehat{J}_e(x)^3 {}^*_* \right)$$

generate a representation of the $W^{(3)}$ algebra with central charge $c = 2$

$$\left[\widehat{\mathcal{L}}_k^{(n)}, \widehat{\mathcal{L}}_m^{(n)}\right] = (k-m)\widehat{\mathcal{L}}_{k+m}^{(n)} + \frac{1}{6}k(k^2-1)\delta_{k,-m},$$

$$\left[\widehat{\mathcal{L}}_k^{(n)}, \widehat{\mathcal{M}}_m^{(n)}\right] = (2k+m)\widehat{\mathcal{M}}_{k+m}^{(n)}$$

and

$$\left(\widehat{\mathcal{L}}^{(n)}(x)\right)_- \tau_n = 0$$

$$\left(\widehat{\mathcal{M}}^{(n)}(x)\right)_- \tau_n = 0$$

Topological expansion:

$$\tau_n(\mathbf{t};\hbar) = \exp\left(\sum_{\chi<0} \hbar^{-\chi} F_n^{(\chi)}(\mathbf{t})\right) = 1 + \sum_{k=1}^{\infty} \hbar^k \tau_n^{(k)}(\mathbf{t})$$

where

$$\chi = 2 - 2\#\text{handles} - \#\text{boundaries} - \#\text{points}$$

$\tau_n(\mathbf{t};\hbar)$ satisfies the cut-and-join type equation

$$\hbar \frac{\partial}{\partial \hbar} \tau_n(\mathbf{t},\hbar) = \left(\hbar \widehat{\mathrm{W}}_1 + \hbar^2 \widehat{\mathrm{W}}_2\right) \tau_n(\mathbf{t},\hbar)$$

so that $\tau_n^{(k)}$ are uniquely defined by a recursion

$$\tau_n^{(k)} = \frac{1}{k}\left(\widehat{\mathrm{W}}_1\, \tau_n^{(k-1)} + \widehat{\mathrm{W}}_2\, \tau_n^{(k-2)}\right)$$

with the initial conditions $\tau_n^{(0)} = 1$, $\tau_n^{(-1)} = 0$.

Operators $\widehat{W}_1$ and $\widehat{W}_2$ are not unique.

$$v_o(z) = \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} (2k+1) x^{k-\frac{1}{2}} t_{2k+1}$$

$$v_e(z) = \sqrt{\frac{2}{3}} \sum_{k=1}^{\infty} k\, x^{k-1} t_{2k}$$

$$\widehat{W}_1 = \frac{1}{\sqrt{2}} \frac{1}{2\pi i} \oint {:}\, \frac{v_o(x)}{2} \left(\widehat{J}_o'(x)^2 + \frac{1}{8x^2} + \widehat{J}_e(x)^2 \right) + \frac{2 v_e(x)}{\sqrt{3}} \widehat{J}_o'(x) \widehat{J}_e(x) \,{:}\, \frac{dx}{\sqrt{x}},$$

$$\widehat{W}_2 = -\frac{1}{3} \frac{1}{2\pi i} \oint v_e(x) \left({:}\, \widehat{J}_e(x) \left(\widehat{J}_o'(x)^2 + \frac{1}{8x^2} \right) - \frac{1}{3} \widehat{J}_e(x)^3 \,{:} \right) \frac{dx}{x^2}$$

Not from $W_{1+\infty}$!

$$
\begin{aligned}
\mathcal{F}_n(\mathbf{t}) = {} & \left(\frac{1}{8} + \frac{3}{2}\,n^2\right) t_3 + \frac{1}{6}\,{t_1}^3 + 2n\,t_1 t_2 + 6\,nt_1 t_2 t_3 + 4n\left(1 + n^2\right) t_6 \\
& + \frac{4}{3}\,n{t_2}^3 + \left(\frac{9}{4}\,n^2 + \frac{3}{16}\right){t_3}^2 + \frac{1}{2}\,t_3{t_1}^3 + 8\,n^2 t_2 t_4 + 4\,{t_1}^2 n t_4 + \left(\frac{15}{2}\,n^2 + \frac{5}{8}\right) t_1 t_5 \\
& + 8\,n{t_2}^3 t_3 + 15\left(3\,n^2 + \frac{1}{4}\right) t_1 t_3 t_5 + 24\,n{t_1}^2 t_3 t_4 + 30\,n^2{t_2}^2 t_5 + \frac{105}{8}\left(\frac{1}{16} + \frac{7}{2}\,n^2 + n^4\right) t_9 \\
& + 35\left(n^3 + \frac{3}{4}\,n\right) t_7 t_2 + 35\left(\frac{1}{16} + \frac{3}{4}\,n^2\right) t_7{t_1}^2 + 32\,n\left(n^2 + 1\right) t_8 t_1 + 32n^2\,t_1{t_4}^2 \\
& + 48\,n^2 t_2 t_3 t_4 + 18\,nt_1 t_2{t_3}^2 + 20\,n\left(1 + 2\,n^2\right) t_5 t_4 + 24n\left(n^2 + 1\right) t_6 t_3 + 8\,n{t_1}^3 t_6 \\
& + \frac{3}{2}\,{t_1}^3{t_3}^2 + 48\,n^2 t_1 t_2 t_6 + 16\,nt_1{t_2}^2 t_4 + \frac{5}{8}\,{t_1}^4 t_5 + \left(\frac{9}{2}\,n^2 + \frac{3}{8}\right){t_3}^3 + 15\,n{t_1}^2 t_2 t_5 + \dots
\end{aligned}
$$

A complete analog of the Kontsevich–Witten description for open case.

Open and closed models are of the similar complexity: **simple!**

n	0	1	arbitrary
Intersection numbers	Closed	Open	**Refined Open**
Integrable hierarchy	KdV	KP	MKP
Algebra of constraints	Heisenberg+ Virasoro	Virasoro + $W^{(3)}$	Virasoro + $W^{(3)}$
Specified by	String	String+Dilaton	String+Dilaton
Cut-and-join operator	$e^{W_{KW}} \cdot 1$	"$e^{W_1+W_2/2}$" $\cdot$ 1	"$e^{W_1+W_2/2}$" $\cdot$ 1

Open questions

- Prove the refined conjecture [A.A., A. Buryak, R. Tessler, to appear]
- r-spin version of open intersection numbers
- Open topological string models for more complicated target spaces (CP^1)
- Open version of Topological recursion/Givental theory
- Simple formulas for $\omega_{g,n}(z_1, \ldots, z_n)$ [E. Brezin and S. Hikami, '15]