Branes, TQFTs and integrable lattice models

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Based on arXiv:1504.04055 and 1606.01041 (with K. Maruyoshi)

Where do integrable lattice models come from?

Answer [Costello]:

2d TQFTs + line operators + extra dimensions

Where can we find such structures?

Answer [JY]:

Branes in string theory + supersymmetric indices

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A prominent example [Spiridonov, Yamazaki, JY]:

4d $\mathcal{N} = 1$ SUSY field theories $\longleftrightarrow$ 2d integrable lattice models

Under this correspondence,

$$\text{SUSY index} \longleftrightarrow \text{partition function}$$
$$\text{Seiberg duality} \longleftrightarrow \text{integrability}$$

The story is even richer [Maruyoshi–JY]:

surface defects $\longleftrightarrow$ transfer matrices of L-operators

In the simplest case, Sklyanin's L-operator

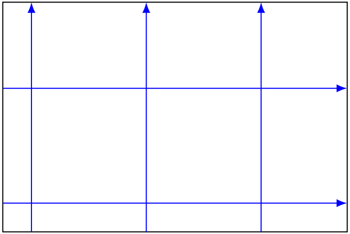
RLL relation with Baxter's R-matrix for 8-vertex model

Brane tiling configuration in Type IIB string theory:

$$\begin{array}{rcccccc} N\ \text{D5s} & S^3 & \times & S^1 & \times & T^2 & \\ \text{NS5s} & S^3 & \times & S^1 & \times & C_i & \times\ \mathbb{R} \end{array}$$

C_i: closed curves in T^2

The intersection lines C_i make a lattice on T^2:



N D5s: 6d super Yang–Mills theory with $G = \mathrm{SU}(N)$

NS5s: domain walls $\mathcal{W}_i$ on $S^3 \times S^1 \times C_i \subset S^3 \times S^1 \times T^2$

Path integral with S^1-factor + SUSY $\Longrightarrow$ SUSY index

Independent of continuous parameters, e.g. size of T^2

Leads to a 4d-2d correspondence:

6d theory on $S^3 \times S^1 \times T^2$
+ $\mathcal{W}_i$ on $S^3 \times S^1 \times C_i$

$S^3 \times S^1 \gg T^2$ (↙) $\qquad$ (↘) $S^3 \times S^1 \ll T^2$

4d theory on $S^3 \times S^1$ $\Longleftrightarrow$ 2d theory on T^2 + $\mathcal{L}_i$ on C_i

The 4d theory is an $\mathcal{N}=1$ SUSY gauge theory.

Gauge groups and matters encoded in a quiver.

Placed on $S^3 \times S^1 \implies$ SUSY index $\mathcal{I}_{\text{4d } \mathcal{N}=1 \text{ theory}}$

For the 2d theory, we get the correlator of line operators $\mathcal{L}_i(C_i)$.

Invariant under deformations of $\{C_i\}$

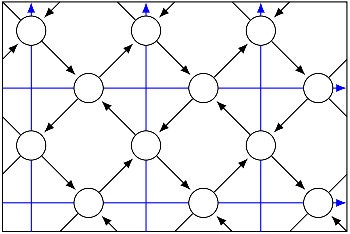
Hence, a topological theory:

$$\mathcal{I}_{\text{4d } \mathcal{N}=1 \text{ theory}} = \left\langle \prod_i \mathcal{L}_i(C_i) \right\rangle_{\text{2d TQFT}}.$$

On general grounds, $\langle \prod_i \mathcal{L}_i(C_i) \rangle_{\text{2d TQFT}} = Z_{\text{lattice model}}$.

SUSY index can be computed exactly [Römelsberger, Kinney et al.].

So we can determine the lattice model.



○ = SU(N) gauge group = N spins, $\text{diag}(z_1, \ldots, z_N) \in \text{SU}(N)$

⟶ = matter = nearest-neighbor interaction

$Z_{\text{lattice model}}$ is expressed by elliptic Γ-functions [Dolan–Osborn].

Go back to the brane configuration in Type IIB theory:

$$\begin{array}{rcccccc} N \text{ D5s} & S^3 & \times & S^1 & \times & T^2 & \\ \text{NS5s} & S^3 & \times & S^1 & \times & C_i & \times \ \mathbb{R} \end{array}$$

T-dualize along the S^1 to go to Type IIA string theory:

$$\begin{array}{rcccccc} N \text{ D4s} & S^3 & \times & \{\text{pt}\} & \times & T^2 & \\ \text{NS5s} & S^3 & \times & S^1 & \times & C_i & \times \ \mathbb{R} \end{array}$$

Lift to M-theory:

$$\begin{array}{rcccccccc} N \text{ M5s} & S^3 & \times & \{\text{pt}\} & \times & T^2 & \times & & {\color{red} S^1} \\ \text{M5s} & S^3 & \times & S^1 & \times & C_i & \times \ \mathbb{R} & \times & {\color{red}\{\text{pt}\}} \end{array}$$

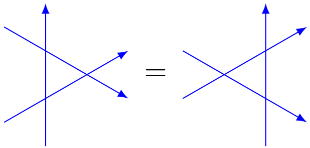
The line operators supported at points in the 11th dimension

Our TQFT has a hidden extra dimension, the M-theory circle, in which the line operators are supported at points.

Visualize it as the direction perpendicular to the screen.

Suppose the lines sit at different points there. Then,

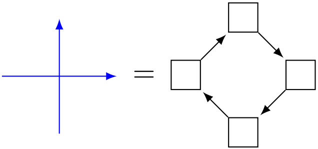
1. Lines carry a spectral parameter, their X^{11} coordinate.
2. The Yang–Baxter equation



holds due to the topological invariance on the screen. No phase transition.

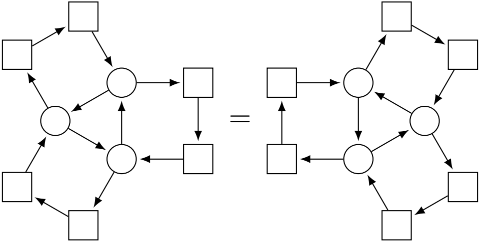
The lattice model is integrable [Costello].

Crossing (R-operator) = diamond of arrows:



□ = SU(N) flavor group = spin site but no summation

The most general known solution of YBE [Bazhanov–Sergeev]

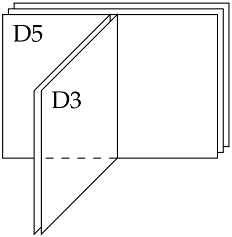


Follows from an integral identity for elliptic Γ [Spiridonov, Rains]

Physically, a consequence of Seiberg duality (quiver mutation)

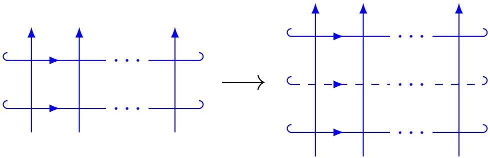
Add a stack of D3-branes [Maruyoshi–JY]:

N D5s	S^3	$\times$	S^1	$\times$	T^2		
NS5s	S^3	$\times$	S^1	$\times$	C_i	$\times$	$\mathbb{R}$
D3s	S^1	$\times$	S^1	$\times$	C	$\times$	$\mathbb{R}$

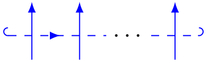


In the 4d theory, creates a surface defect

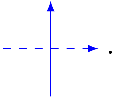
In the lattice model, inserts a new line C:



Surface defect is represented by transfer matrix



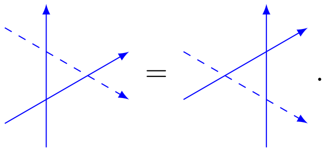
constructed from the L-operator



A dashed line is labeled by a pair (R_1, R_2) of reps of $\mathrm{SU}(N)$.

So is the L-operator.

Using a dashed line, we can write down the YBE



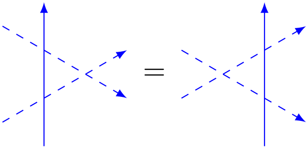
It's called an RLL relation.

For $(R_1, R_2) = (\emptyset, \square)$ of SU(2), the relation coincides with the one studied by Derkachov and Spiridonov:

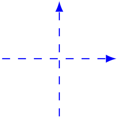
$$\check{L} = \text{Sklyanin's L-operator}\,,$$

a 2×2 matrix whose entries are difference operators.

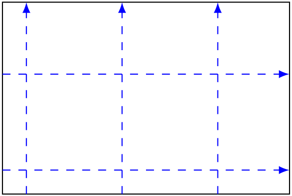
Another RLL relation



For Sklyanin's L-operator,

 = Baxter's R-matrix for 8-vertex model .

So we can make the 8-vertex model using branes:



Further directions:

- What is the L-operator for $\mathrm{SU}(N)$? [Maruyoshi–JY, in progress]
- Other kinds of defects. Their meanings in lattice models?
- Replace S^3 with another M_3. For $M_3 = \Sigma_2 \times S^1$, we get Gromov–Witten invariants.
- 3d lattice model and tetrahedron equation
- Relations to AGT, Nekrasov–Shatashvili, cluster integrable systems, knots, AdS/CFT, Little String Theory, ...
- Apply string theory/gauge theory techniques (dualities, large-N expansion, ...) to integrable models.
- Apply integrable model techniques to gauge theories.