The cluster structure of geometric R-marices

Rei Inoue (Chiba)

with

Thomas Lam (Michigan)

Pavlo Pylyavskyy (Minnesota)

Plan

§1 Introduction — geometric R-matrix

§2 Cluster R-matrix

§3 Quantization

§1 Introduction

- Geometric R-matrix [Kajiwara-Noumi-Yamada 2002] [Etingov 2003]

$R \curvearrowright \mathbb{C}^n \times \mathbb{C}^n;\ (\mathbf{p}, \mathbf{q}) := (p_i, q_i)_{1 \le i \le n} \mapsto (q_i \frac{\kappa_{i+1}}{\kappa_i}, p_i \frac{\kappa_i}{\kappa_{i+1}})_{1 \le i \le n}$

where $\kappa_i := \kappa_i(\mathbf{p}, \mathbf{q}) = \sum_{j=0}^{n-1} p_{i-1} p_{i-2} \cdots p_{i-j}\, q_{i-j-2} q_{i-j-3} \quad (i \in \mathbb{Z}/n\mathbb{Z})$

§1 Introduction

• Geometric R-matrix [Kajiwara-Noumi-Yamada 2002] [Etingov 2003]

$R \curvearrowright \mathbb{C}^n \times \mathbb{C}^n;\ (\mathbf{p}, \mathbf{q}) := (p_i, q_i)_{1 \le i \le n} \mapsto (q_i \frac{\kappa_{i+1}}{\kappa_i}, p_i \frac{\kappa_i}{\kappa_{i+1}})_{1 \le i \le n}$

where $\kappa_i := \kappa_i(\mathbf{p}, \mathbf{q}) = \sum_{j=0}^{n-1} p_{i-1} p_{i-2} \cdots p_{i-j}\, q_{i-j-2} q_{i-j-3} \quad (i \in \mathbb{Z}/n\mathbb{Z})$

(Ex) $R \curvearrowright \mathbb{C}^3 \times \mathbb{C}^3; (\mathbf{p}, \mathbf{q}) \mapsto (\mathbf{p}', \mathbf{q}')$

$$p_1' = q_1 \frac{p_3 p_1 + q_2 p_1 + q_2 q_3}{p_2 p_3 + q_1 p_3 + q_1 q_2}, \quad q_1' = p_1 \frac{p_2 p_3 + q_1 p_3 + q_1 q_2}{p_3 p_1 + q_2 p_1 + q_2 q_3}$$

§1 Introduction

• Geometric R-matrix [Kajiwara-Noumi-Yamada 2002] [Etingov 2003]

$R \curvearrowright \mathbb{C}^n \times \mathbb{C}^n;\ (\mathbf{p},\mathbf{q}) := (p_i, q_i)_{1\le i\le n} \mapsto (q_i \frac{\kappa_{i+1}}{\kappa_i}, p_i \frac{\kappa_i}{\kappa_{i+1}})_{1\le i\le n}$

where $\kappa_i := \kappa_i(\mathbf{p},\mathbf{q}) = \sum_{j=0}^{n-1} p_{i-1}p_{i-2}\cdots p_{i-j}\, q_{i-j-2}q_{i-j-3} \quad (i \in \mathbb{Z}/n\mathbb{Z})$

(Ex) $R \curvearrowright \mathbb{C}^3 \times \mathbb{C}^3; (\mathbf{p},\mathbf{q}) \mapsto (\mathbf{p}',\mathbf{q}')$

$$p_1' = q_1 \frac{p_3p_1 + q_2p_1 + q_2q_3}{p_2p_3 + q_1p_3 + q_1q_2}, \quad q_1' = p_1 \frac{p_2p_3 + q_1p_3 + q_1q_2}{p_3p_1 + q_2p_1 + q_2q_3}$$

Thm [Yamada 2001]

Define $R_j : (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m) \mapsto (\mathbf{q}_1, \ldots, R(\mathbf{q}_j, \mathbf{q}_{j+1}), \ldots, \mathbf{q}_m)$.
Then R_j $(1 \le j \le m-1)$ generate the symmetric group $\mathfrak{S}_m$ action on $\mathbb{C}(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m)$:

$$R_j^2 = \mathrm{id}, \qquad R_jR_i = R_iR_j \ \ (|i-j| > 1)$$

$$R_jR_{j+1}R_j = R_{j+1}R_jR_{j+1} \ \ (j = 1, \ldots, m-2)$$

Fact Via tropicalization:

$$x + y \mapsto \min(x, y), \quad xy \mapsto x + y$$

the geometric R reduces to the combinatorial R matrix which acts on a tensor product of symmetric crystals of $\mathfrak{sl}_n$:

$$B_k = \left\{(x_1, \ldots, x_n) \in (\mathbb{Z}_{\geq 0})^n \,\Big|\, \sum_i x_i = k\right\}$$

$$= \left\{\boxed{b_1}\boxed{b_2}\boxed{\cdots}\boxed{b_k} \,\Big|\, 1 \leq b_1 \leq b_2 \leq \cdots \leq b_k \leq n\right\}$$

$$R_{\mathrm{comb}} : \; B_k \otimes B_\ell \xrightarrow{\sim} B_\ell \otimes B_k$$

Fact Via tropicalization:

$$x + y \mapsto \min(x, y), \quad xy \mapsto x + y$$

the geometric R reduces to the combinatorial R matrix which acts on a tensor product of symmetric crystals of $\mathfrak{sl}_n$:

$$B_k = \left\{(x_1, \ldots, x_n) \in (\mathbb{Z}_{\geq 0})^n \mid \sum_i x_i = k\right\}$$

$$= \left\{\boxed{b_1}\boxed{b_2}\boxed{\cdots}\boxed{b_k} \;\middle|\; 1 \leq b_1 \leq b_2 \leq \cdots \leq b_k \leq n\right\}$$

$$R_{\mathrm{comb}} : \; B_k \otimes B_\ell \overset{\sim}{\to} B_\ell \otimes B_k$$

(Ex) $\mathfrak{sl}_3$ crystals

$$R_{\mathrm{comb}} : \boxed{1}\boxed{2}\boxed{3}\boxed{3} \otimes \boxed{2}\boxed{2} \mapsto \boxed{1}\boxed{3} \otimes \boxed{2}\boxed{2}\boxed{2}\boxed{3}$$

• Related topics

Berenstein-Fomin-Zelevinsky 1996

canonical bases and totally positive matrices

Berenstein-Zelevinsky 1996

canonical bases for quantum group

Noumi-Yamada 2001–

rational version of RSK correspondence

• Related topics

Berenstein-Fomin-Zelevinsky 1996
canonical bases and totally positive matrices

Berenstein-Zelevinsky 1996
canonical bases for quantum group

Noumi-Yamada 2001–
rational version of RSK correspondence

Fomin-Zelevinsky 2000– cluster algebras

Fock-Goncharov 2003– quantum cluster ensembles

• Related topics

Berenstein-Fomin-Zelevinsky 1996
canonical bases and totally positive matrices

Berenstein-Zelevinsky 1996
canonical bases for quantum group

Noumi-Yamada 2001–
rational version of RSK correspondence

Fomin-Zelevinsky 2000– cluster algebras

Fock-Goncharov 2003– quantum cluster ensembles

Lam-Pylyavskyy 2012– geometric R and cylindric networks

• Related topics

Berenstein-Fomin-Zelevinsky 1996
canonical bases and totally positive matrices

Berenstein-Zelevinsky 1996
canonical bases for quantum group

Noumi-Yamada 2001–
rational version of RSK correspondence

Fomin-Zelevinsky 2000– cluster algebras

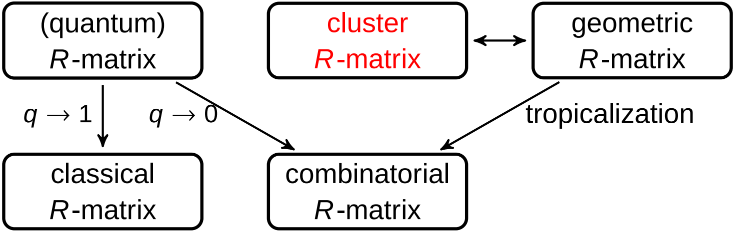
Fock-Goncharov 2003– quantum cluster ensembles

Lam-Pylyavskyy 2012– geometric R and cylindric networks

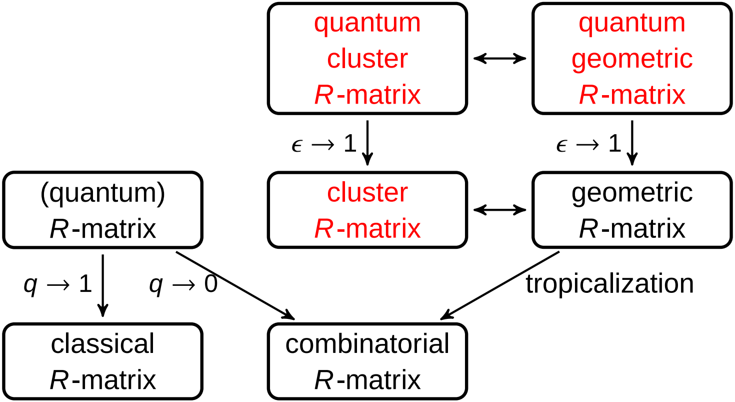
- R-matrices

$$
\begin{tikzcd}
\boxed{\text{(quantum) } R\text{-matrix}} \arrow[d, "q \to 1"'] \arrow[dr, "q \to 0"] & & \boxed{\text{geometric } R\text{-matrix}} \arrow[dl, "\text{tropicalization}"] \\
\boxed{\text{classical } R\text{-matrix}} & \boxed{\text{combinatorial } R\text{-matrix}} &
\end{tikzcd}
$$

• R-matrices


(quantum)
R-matrix
cluster
R-matrix
geometric
R-matrix
$q \to 1$
$q \to 0$
tropicalization
classical
R-matrix
combinatorial
R-matrix

• R-matrices


quantum
cluster
R-matrix
quantum
geometric
R-matrix
$\epsilon \to 1$
$\epsilon \to 1$
(quantum)
R-matrix
cluster
R-matrix
geometric
R-matrix
$q \to 1$
$q \to 0$
tropicalization
classical
R-matrix
combinatorial
R-matrix

§2 Cluster R-matrix

- Cluster mutation [Fomin-Zelevinsky 2000]

Def A y-seed $(\mathbf{B}, \boldsymbol{y})$:
$\mathbf{B} = (b_{ij})_{i,j} \in \mathrm{Mat}_N(\mathbb{Z})$: a skew-symmetric matrix
$\boldsymbol{y} = (y_1, \ldots, y_N)$, y_i: a y-variable ($\in$ the universal semifield)

§2 Cluster R-matrix

- Cluster mutation [Fomin-Zelevinsky 2000]

<u>Def</u> A y-seed $(\mathbf{B}, \boldsymbol{y})$:
$\mathbf{B} = (b_{ij})_{i,j} \in \mathrm{Mat}_N(\mathbb{Z})$: a skew-symmetric matrix
$\boldsymbol{y} = (y_1, \ldots, y_N)$, y_i: a y-variable ($\in$ the universal semifield)

The mutation $\mu_k(\mathbf{B}, \boldsymbol{y}) = (\widetilde{\mathbf{B}}, \widetilde{\boldsymbol{y}})$ at k $(k = 1, \cdots, N)$:

$$\widetilde{y}_i = \begin{cases} {y_k}^{-1} & \text{for } i = k \\ y_i\,(1 + y_k^{-1})^{-b_{ki}} & \text{for } k \neq i,\ b_{ki} \geq 0 \\ y_i\,(1 + y_k)^{b_{ik}} & \text{for } k \neq i,\ b_{ik} \geq 0. \end{cases} \qquad \widetilde{b}_{ij} = \begin{cases} -b_{ij} & \text{for } i = k \text{ or } j = k \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{o.w.} \end{cases}$$

§2 Cluster R-matrix

- Cluster mutation [Fomin-Zelevinsky 2000]

Def A y-seed $(\mathbf{B}, \boldsymbol{y})$:
$\mathbf{B} = (b_{ij})_{i,j} \in \mathrm{Mat}_N(\mathbb{Z})$: a skew-symmetric matrix
$\boldsymbol{y} = (y_1, \ldots, y_N)$, y_i: a y-variable ($\in$ the universal semifield)

The mutation $\mu_k(\mathbf{B}, \boldsymbol{y}) = (\widetilde{\mathbf{B}}, \widetilde{\boldsymbol{y}})$ at k $(k = 1, \cdots, N)$:

$$\widetilde{y}_i = \begin{cases} {y_k}^{-1} & \text{for } i = k \\ y_i\,(1 + y_k^{-1})^{-b_{ki}} & \text{for } k \neq i,\ b_{ki} \geq 0 \\ y_i\,(1 + y_k)^{b_{ik}} & \text{for } k \neq i,\ b_{ik} \geq 0. \end{cases} \qquad \widetilde{b}_{ij} = \begin{cases} -b_{ij} & \text{for } i = k \text{ or } j = k \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{o.w.} \end{cases}$$

Remark
(i) $\mu_k \circ \mu_k = \mathrm{id}$. If $b_{jk} = 0$, $\mu_k \circ \mu_j = \mu_j \circ \mu_k$.

§2 Cluster R-matrix

• Cluster mutation [Fomin-Zelevinsky 2000]

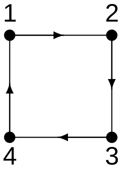
Def A y-seed $(\mathbf{B}, \boldsymbol{y})$:
$\mathbf{B} = (b_{ij})_{i,j} \in \mathrm{Mat}_N(\mathbb{Z})$: a skew-symmetric matrix
$\boldsymbol{y} = (y_1, \ldots, y_N)$, y_i: a y-variable ($\in$ the universal semifield)

The mutation $\mu_k(\mathbf{B}, \boldsymbol{y}) = (\widetilde{\mathbf{B}}, \widetilde{\boldsymbol{y}})$ at k $(k = 1, \cdots, N)$:

$$\widetilde{y}_i = \begin{cases} {y_k}^{-1} & \text{for } i = k \\ y_i\,(1 + y_k^{-1})^{-b_{ki}} & \text{for } k \neq i,\ b_{ki} \geq 0 \\ y_i\,(1 + y_k)^{b_{ik}} & \text{for } k \neq i,\ b_{ik} \geq 0. \end{cases} \qquad \widetilde{b}_{ij} = \begin{cases} -b_{ij} & \text{for } i = k \text{ or } j = k \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{o.w.} \end{cases}$$

Remark
(i) $\mu_k \circ \mu_k = \mathrm{id}$. If $b_{jk} = 0$, $\mu_k \circ \mu_j = \mu_j \circ \mu_k$.
(ii) $\mathbf{B} \longleftrightarrow$ a quiver with N vertices
$b_{ij} = \#\{\text{arrows from } i \text{ to } j\} - \#\{\text{arrows from } j \text{ to } i\}$

(Ex)



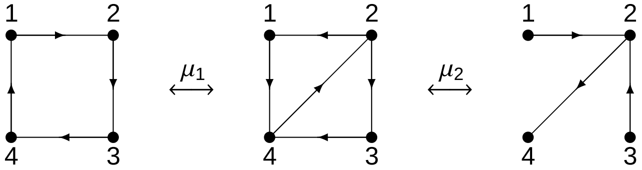
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

(Ex)

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3 \arrow[l] \end{tikzcd} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{tikzcd} 1 \arrow[d] & 2 \arrow[l] \arrow[d] \\ 4 \arrow[ur] & 3 \arrow[l] \end{tikzcd}$$

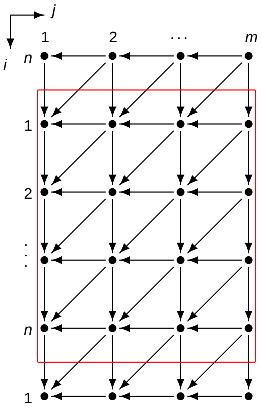
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1/y_1 \\ y_2/(1+1/y_1) := y_2' \\ y_3 \\ y_4(1+y_1) := y_4' \end{pmatrix}$$

(Ex)

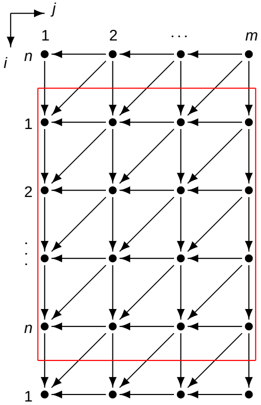


$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \qquad \begin{pmatrix} 1/y_1 \\ y_2/(1+1/y_1) := y_2' \\ y_3 \\ y_4(1+y_1) := y_4' \end{pmatrix} \qquad \begin{pmatrix} 1/y_1(1+1/y_2') \\ 1/y_2' \\ y_3/(1+1/y_2') \\ y_4'(1+y_2') \end{pmatrix}$$

- Triangular grid quiver $Q_{n,m}$ on a cylinder



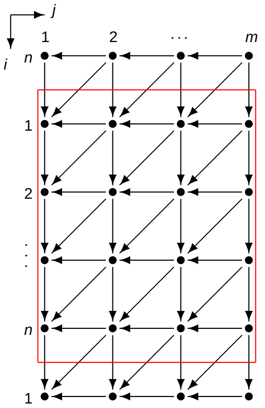
- Triangular grid quiver $Q_{n,m}$ on a cylinder



$$\boldsymbol{y} = (y_{ji})_{1 \le j \le m,\ i \in \mathbb{Z}/n\mathbb{Z}}$$

M_j: a circle $(j,1) \to (j,2) \to \cdots \to (j,n) \to (j,1)$

- Triangular grid quiver $Q_{n,m}$ on a cylinder



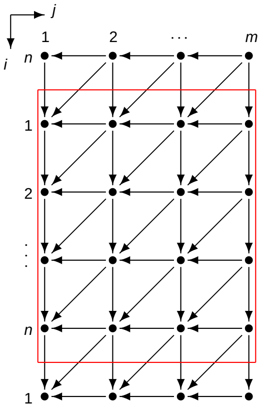
$\boldsymbol{y} = (y_{ji})_{1 \le j \le m,\ i \in \mathbb{Z}/n\mathbb{Z}}$

M_j: a circle $(j,1) \to (j,2) \to \cdots \to (j,n) \to (j,1)$

$$\mathcal{R}_{M_j,k} := s_{(j,k-2),(j,k-1)} \circ \mu_k \circ \mu_{k+1} \circ \cdots \circ \mu_{k-3} \circ \mu_{k-1} \circ \mu_{k-2} \circ \cdots \circ \mu_{k+1} \circ \mu_k$$

where $\mu_i := \mu_{j,i}$

- Triangular grid quiver $Q_{n,m}$ on a cylinder



$$\boldsymbol{y} = (y_{ji})_{1 \le j \le m,\ i \in \mathbb{Z}/n\mathbb{Z}}$$

M_j: a circle $(j,1) \to (j,2) \to \cdots \to (j,n) \to (j,1)$

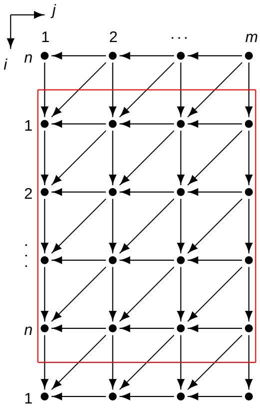
$$\mathcal{R}_{M_j,k} := s_{(j,k-2),(j,k-1)} \circ \mu_k \circ \mu_{k+1} \circ \cdots \circ \mu_{k-3} \circ \mu_{k-1} \circ \mu_{k-2} \circ \cdots \circ \mu_{k+1} \circ \mu_k$$

where $\mu_i := \mu_{j,i}$

(Ex) $n = 4$:

$$\mathcal{R}_{M_2,1} := s_{(2,3),(2,4)} \circ \mu_1 \circ \mu_2 \circ \mu_4 \circ \mu_3 \circ \mu_2 \circ \mu_1$$

- Triangular grid quiver $Q_{n,m}$ on a cylinder



$\boldsymbol{y} = (y_{ji})_{1\le j\le m,\ i\in\mathbb{Z}/n\mathbb{Z}}$

M_j: a circle $(j,1) \to (j,2) \to \cdots \to (j,n) \to (j,1)$

$$\mathcal{R}_{M_j,k} := s_{(j,k-2),(j,k-1)} \circ \mu_k \circ \mu_{k+1} \circ \cdots \circ \mu_{k-3} \circ \mu_{k-1} \circ \mu_{k-2} \circ \cdots \circ \mu_{k+1} \circ \mu_k$$

where $\mu_i := \mu_{j,i}$

(Ex) $n = 4$:

$$\mathcal{R}_{M_2,1} := s_{(2,3),(2,4)} \circ \mu_1 \circ \mu_2 \circ \mu_4 \circ \mu_3 \circ \mu_2 \circ \mu_1$$

(demo of mutation)

Thm 1

(i) We have $\mathcal{R}_{M_j,k}(Q_{n,m},\boldsymbol{y}) = (Q_{n,m},\boldsymbol{y}')$;

$$y'_{j',i} = \begin{cases} y_{j',i} & j' \notin \{j-1, j, j+1\} \\ (\alpha_{i+2})^{-1} \cdot y_{j,i+1}^{-1} \cdot \alpha_i & j' = j \\ (\alpha_i)^{-1} \cdot y_{j,i}\, y_{j-1,i} \cdot \alpha_{i+1} & j' = j-1 \\ (\alpha_{i+1})^{-1} \cdot y_{j,i+1}\, y_{j+1,i} \cdot \alpha_{i+2} & j' = j+1 \end{cases}$$

independently of k. Here $\alpha_i = \alpha_i(M_j) := 1 + \sum_{\ell=1}^{n-1} y_{j,i}\, y_{j,i+1} \cdots y_{j,i+\ell-1}$. We write $R_{M_j}(\boldsymbol{y}) = \boldsymbol{y}'$.

Thm 1

(i) We have $\mathcal{R}_{M_j,k}(Q_{n,m},\boldsymbol{y}) = (Q_{n,m},\boldsymbol{y}')$;

$$y'_{j',i} = \begin{cases} y_{j',i} & j' \notin \{j-1, j, j+1\} \\ (\alpha_{i+2})^{-1} \cdot y_{j,i+1}^{-1} \cdot \alpha_i & j' = j \\ (\alpha_i)^{-1} \cdot y_{j,i}\, y_{j-1,i} \cdot \alpha_{i+1} & j' = j-1 \\ (\alpha_{i+1})^{-1} \cdot y_{j,i+1}\, y_{j+1,i} \cdot \alpha_{i+2} & j' = j+1 \end{cases}$$

independently of k. Here $\alpha_i = \alpha_i(M_j) := 1 + \sum_{\ell=1}^{n-1} y_{j,i}\, y_{j,i+1} \cdots y_{j,i+\ell-1}$. We write $R_{M_j}(\boldsymbol{y}) = \boldsymbol{y}'$.

(ii) R_{M_j} $(1 \le j \le m)$ generate $\mathfrak{S}_{m+1}$ action on $\mathbb{C}(\boldsymbol{y})$:

$$R_{M_j}^2 = \mathrm{id}, \qquad R_{M_j} R_{M_i} = R_{M_i} R_{M_j} \ \ (|i-j| > 1)$$

$$R_{M_j} R_{M_{j+1}} R_{M_j} = R_{M_{j+1}} R_{M_j} R_{M_{j+1}} \ \ (j = 1, \ldots, m-1)$$

Thm 1

(i) We have $\mathcal{R}_{M_j,k}(Q_{n,m},\boldsymbol{y}) = (Q_{n,m},\boldsymbol{y}')$;

$$y'_{j',i} = \begin{cases} y_{j',i} & j' \notin \{j-1, j, j+1\} \\ (\alpha_{i+2})^{-1} \cdot y_{j,i+1}^{-1} \cdot \alpha_i & j' = j \\ (\alpha_i)^{-1} \cdot y_{j,i}\, y_{j-1,i} \cdot \alpha_{i+1} & j' = j-1 \\ (\alpha_{i+1})^{-1} \cdot y_{j,i+1}\, y_{j+1,i} \cdot \alpha_{i+2} & j' = j+1 \end{cases}$$

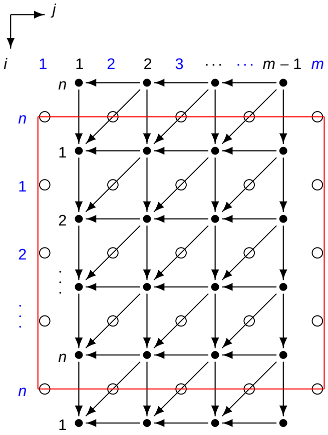
independently of k. Here $\alpha_i = \alpha_i(M_j) := 1 + \sum_{\ell=1}^{n-1} y_{j,i}\, y_{j,i+1} \cdots y_{j,i+\ell-1}$.
We write $R_{M_j}(\boldsymbol{y}) = \boldsymbol{y}'$.

(ii) R_{M_j} $(1 \le j \le m)$ generate $\mathfrak{S}_{m+1}$ action on $\mathbb{C}(\boldsymbol{y})$:

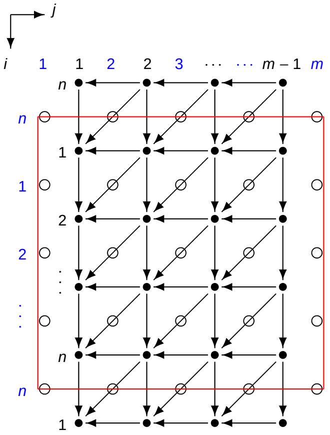
$$R_{M_j}^2 = \mathrm{id}, \qquad R_{M_j} R_{M_i} = R_{M_i} R_{M_j} \ \ (|i-j| > 1)$$
$$R_{M_j} R_{M_{j+1}} R_{M_j} = R_{M_{j+1}} R_{M_j} R_{M_{j+1}} \ \ (j = 1, \ldots, m-1)$$

(Ex) $n = 3$: $(y'_{j-1,1}, y'_{j,1}, y'_{j+1,1})$
$$= \left(\frac{y_{j-1,1} y_{j,1} (1+y_2+y_2y_3)}{1+y_1+y_1y_2}, \frac{1+y_1+y_1y_2}{y_{j,2}(1+y_3+y_3y_1)}, \frac{y_{j,2} y_{j+1,1}(1+y_3+y_3y_1)}{1+y_2+y_2y_3} \right);\ y_i := y_{j,i}$$

- Cluster R and geometric R



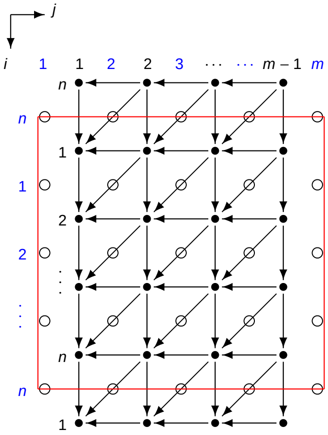
- Cluster R and geometric R



Consider the y-seed $(Q_{n,m-1}, \boldsymbol{y})$.

$$\phi : \mathbb{C}(\boldsymbol{y}) \hookrightarrow \mathbb{C}(\mathbf{q}_1, \dots, \mathbf{q}_m);\ y_{j,i} \mapsto \frac{q_{j+1,i-1}}{q_{j,i}}$$

• Cluster R and geometric R



Consider the y-seed $(Q_{n,m-1}, \boldsymbol{y})$.

$$\phi : \mathbb{C}(\boldsymbol{y}) \hookrightarrow \mathbb{C}(\mathbf{q}_1, \dots, \mathbf{q}_m); \; y_{j,i} \mapsto \frac{q_{j+1,i-1}}{q_{j,i}}$$

<u>Thm 2</u>

For $j = 1, \dots, m-1$, the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{C}(\boldsymbol{y}) \arrow[r, "\phi"] \arrow[d, "R_{M_j}"'] & \mathbb{C}(\mathbf{q}_1, \dots, \mathbf{q}_m) \arrow[d, "R_j"] \\
\mathbb{C}(\boldsymbol{y}) \arrow[r, "\phi"'] & \mathbb{C}(\mathbf{q}_1, \dots, \mathbf{q}_m)
\end{tikzcd}$$

Remark

$\widetilde{Q}_{n,m+1}$: a quiver $Q_{n,m+1}+$ (some ‘frozen vertices’)
$(\widetilde{Q}_{n,m+1}, \boldsymbol{x}, \boldsymbol{X})$: a seed with ‘frozen variables’ $\boldsymbol{X}$

We can define the cluster R-matrix

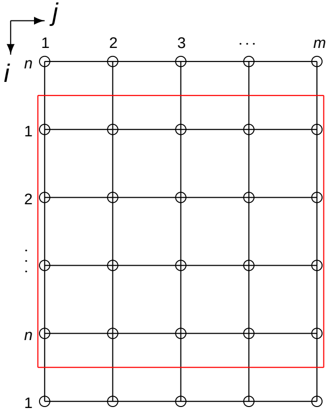
$$\widetilde{R}_{M_j} = (\text{permutations on frozen vertices}) \circ R_{M_j}$$

which makes the following diagram commutative:

$$\begin{tikzcd}
\mathbb{C}(\boldsymbol{y}) \arrow[r, "\phi"] \arrow[d, "R_{M_j}"'] & \mathbb{C}(\mathbf{q}_1, \ldots, \mathbf{q}_m) \arrow[r, "\psi"] \arrow[d, "R_j"'] & \mathbb{C}(\boldsymbol{x}, \boldsymbol{X}) \arrow[d, "\widetilde{R}_{M_j}"] \\
\mathbb{C}(\boldsymbol{y}) \arrow[r, "\phi"] & \mathbb{C}(\mathbf{q}_1, \ldots, \mathbf{q}_m) \arrow[r, "\psi"'] & \mathbb{C}(\boldsymbol{x}, \boldsymbol{X})
\end{tikzcd}$$

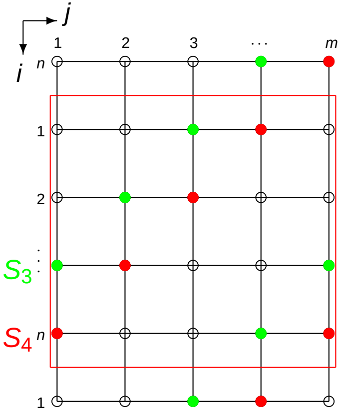
for $j = 1, \ldots, m-1$. Here ψ is given by a form $q_{j,i} = X\dfrac{x\,x}{x\,x}$.

- Quantum geometric R



$$\boldsymbol{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_m), \ \mathbf{q}_j = (q_{ji})_{i \in \mathbb{Z}/n\mathbb{Z}}$$

- Quantum geometric R

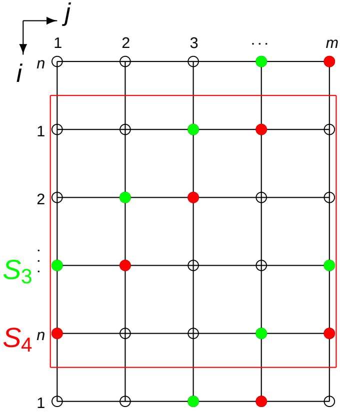


$$\boldsymbol{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_m), \ \mathbf{q}_j = (q_{ji})_{i \in \mathbb{Z}/n\mathbb{Z}}$$

$$S_k := \{q_{1,k}, q_{2,k-1}, \ldots, q_{m,k-m+1}\}$$

§3 Quantization

- Quantum geometric R



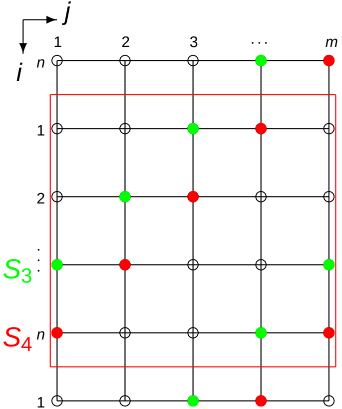
$$\boldsymbol{q} = (\mathbf{q}_1, \dots, \mathbf{q}_m),\ \mathbf{q}_j = (q_{ji})_{i \in \mathbb{Z}/n\mathbb{Z}}$$

$$S_k := \{q_{1,k}, q_{2,k-1}, \dots, q_{m,k-m+1}\}$$

$\mathbb{C}_\epsilon\langle \boldsymbol{q} \rangle$: a skew field with relations:

$$q_{ji}q_{j'i'} = \begin{cases} \epsilon\, q_{j'i'}q_{ji} & q_{ji} \in S_{k+1}, q_{j'i'} \in S_k, j \le j' \\ \epsilon^{-1} q_{j'i'}q_{ji} & q_{ji} \in S_{k+1}, q_{j'i'} \in S_k, j > j' \\ \epsilon^{-2} q_{j'i'}q_{ji} & q_{ji}, q_{j'i'} \in S_k, j < j' \\ \epsilon^{2} q_{j'i'}q_{ji} & q_{ji}, q_{j'i'} \in S_k, j > j' \\ q_{j'i'}q_{ji} & \text{otherwise} \end{cases}$$

- Quantum geometric R



$$\boldsymbol{q} = (\mathbf{q}_1, \dots, \mathbf{q}_m), \ \mathbf{q}_j = (q_{ji})_{i \in \mathbb{Z}/n\mathbb{Z}}$$

$$S_k := \{q_{1,k}, q_{2,k-1}, \dots, q_{m,k-m+1}\}$$

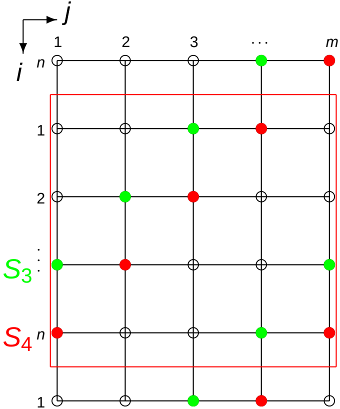
$\mathbb{C}_\epsilon\langle \boldsymbol{q} \rangle$: a skew field with relations:

$$q_{ji}q_{j'i'} = \begin{cases} \epsilon\, q_{j'i'}q_{ji} & q_{ji} \in S_{k+1}, q_{j'i'} \in S_k, j \le j' \\ \epsilon^{-1} q_{j'i'}q_{ji} & q_{ji} \in S_{k+1}, q_{j'i'} \in S_k, j > j' \\ \epsilon^{-2} q_{j'i'}q_{ji} & q_{ji}, q_{j'i'} \in S_k, j < j' \\ \epsilon^{2} q_{j'i'}q_{ji} & q_{ji}, q_{j'i'} \in S_k, j > j' \\ q_{j'i'}q_{ji} & \text{otherwise} \end{cases}$$

Def $R^\epsilon;\ (\mathbf{p}, \mathbf{q}) := (p_i, q_i)_{1 \le i \le n} \mapsto ((\kappa_i^\epsilon)^{-1} \cdot q_i \cdot \kappa_{i+1}^\epsilon,\ (\kappa_{i+1}^\epsilon)^{-1} \cdot p_i \cdot \kappa_i^\epsilon)_{1 \le i \le n}$

where $\kappa_i^\epsilon(\mathbf{p}, \mathbf{q}) := \sum_{j=0}^{n-1} p_{i-1}p_{i-2} \cdots p_{i-j}\, q_{i-j-2}q_{i-j-3} \cdots q_{i-n}\ (i \in \mathbb{Z}/n\mathbb{Z})$

- Quantum geometric R



$$\boldsymbol{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_m),\ \mathbf{q}_j = (q_{ji})_{i\in\mathbb{Z}/n\mathbb{Z}}$$

$$S_k := \{q_{1,k}, q_{2,k-1}, \ldots, q_{m,k-m+1}\}$$

$\mathbb{C}_\epsilon\langle\boldsymbol{q}\rangle$: a skew field with relations:

$$q_{ji}q_{j'i'} = \begin{cases} \epsilon\, q_{j'i'}q_{ji} & q_{ji}\in S_{k+1}, q_{j'i'}\in S_k, j\le j' \\ \epsilon^{-1}q_{j'i'}q_{ji} & q_{ji}\in S_{k+1}, q_{j'i'}\in S_k, j> j' \\ \epsilon^{-2}q_{j'i'}q_{ji} & q_{ji}, q_{j'i'}\in S_k, j<j' \\ \epsilon^{2}q_{j'i'}q_{ji} & q_{ji}, q_{j'i'}\in S_k, j>j' \\ q_{j'i'}q_{ji} & \text{otherwise} \end{cases}$$

Def $\quad R^\epsilon;\ (\mathbf{p},\mathbf{q}) := (p_i, q_i)_{1\le i\le n} \mapsto ((\kappa_i^\epsilon)^{-1}\cdot q_i\cdot \kappa_{i+1}^\epsilon,\ (\kappa_{i+1}^\epsilon)^{-1}\cdot p_i\cdot\kappa_i^\epsilon)_{1\le i\le n}$

where $\kappa_i^\epsilon(\mathbf{p},\mathbf{q}) := \sum_{j=0}^{n-1} p_{i-1}p_{i-2}\cdots p_{i-j}\, q_{i-j-2}q_{i-j-3}\cdots q_{i-n}\ (i\in\mathbb{Z}/n\mathbb{Z})$

$R_j^\epsilon(\mathbf{q}_1,\ldots,\mathbf{q}_m) := (\mathbf{q}_1,\ldots,R^\epsilon(\mathbf{q}_j,\mathbf{q}_{j+1}),\ldots,\mathbf{q}_m)\ (j=1,\ldots,m-1)$

• Quantization of y-seed [Fock-Goncharov]

<u>Def</u> A quantum y-seed $(\mathbf{B}, \boldsymbol{y})$:
$\mathbf{B} = (b_{ij})_{i,j} \in \mathrm{Mat}_N(\mathbb{Z})$: a skew-symmetric matrix
$\boldsymbol{y} = (y_1, \ldots, y_N)$; $y_i y_j = \epsilon^{2b_{ji}} y_j y_i$: quantum y-variables

• Quantization of y-seed [Fock-Goncharov]

Def A quantum y-seed $(\mathbf{B}, \boldsymbol{y})$:
$\mathbf{B} = (b_{ij})_{i,j} \in \mathrm{Mat}_N(\mathbb{Z})$: a skew-symmetric matrix
$\boldsymbol{y} = (y_1, \ldots, y_N)$; $y_i y_j = \epsilon^{2b_{ji}} y_j y_i$: quantum y-variables

The quantum mutation $\mu_k^\epsilon(\mathbf{B}, \boldsymbol{y}) = (\widetilde{\mathbf{B}}, \widetilde{\boldsymbol{y}})$ at k $(k = 1, \cdots, N)$:

$$\widetilde{y}_i = \begin{cases} y_k^{-1} & i = k \\ y_i \prod_{m=1}^{b_{ki}} \left(1 + \epsilon^{2m-1} y_k^{-1}\right)^{-1} & i \neq k,\ b_{ki} \geq 0 \\ y_i \prod_{m=1}^{-b_{ki}} (1 + \epsilon^{2m-1} y_k) & i \neq k,\ b_{ki} \leq 0. \end{cases}$$

$\widetilde{b}_{ij}$: same as classical case

• Quantization of y-seed [Fock-Goncharov]

Def A quantum y-seed $(\mathbf{B}, \boldsymbol{y})$:
$\mathbf{B} = (b_{ij})_{i,j} \in \mathrm{Mat}_N(\mathbb{Z})$: a skew-symmetric matrix
$\boldsymbol{y} = (y_1, \ldots, y_N)$; $y_i y_j = \epsilon^{2b_{ji}} y_j y_i$: quantum y-variables

The quantum mutation $\mu_k^\epsilon(\mathbf{B}, \boldsymbol{y}) = (\widetilde{\mathbf{B}}, \widetilde{\boldsymbol{y}})$ at k $(k = 1, \cdots, N)$:

$$\widetilde{y}_i = \begin{cases} y_k^{-1} & i = k \\ y_i \prod_{m=1}^{b_{ki}} \left(1 + \epsilon^{2m-1} y_k^{-1}\right)^{-1} & i \neq k,\ b_{ki} \geq 0 \\ y_i \prod_{m=1}^{-b_{ki}} (1 + \epsilon^{2m-1} y_k) & i \neq k,\ b_{ki} \leq 0. \end{cases}$$

$\widetilde{b}_{ij}$: same as classical case

• Quantum cluster R

$(Q_{n,m}, \boldsymbol{y})$

$$\mathcal{R}^\epsilon_{M_j,k} := s_{(j,k-2),(j,k-1)} \circ \mu_k^\epsilon \circ \mu_{k+1}^\epsilon \circ \cdots \circ \mu_{k-3}^\epsilon$$
$$\circ \mu_{k-1}^\epsilon \circ \mu_{k-2}^\epsilon \circ \cdots \circ \mu_{k+1}^\epsilon \circ \mu_k^\epsilon \quad (j = 1, \ldots, m)$$

where $\mu_k^\epsilon := \mu_{j,k}^\epsilon$

Thm 3

(i) We have $\mathcal{R}^{\epsilon}_{M_j,k}(Q_{n,m},\boldsymbol{y}) = (Q_{n,m},\boldsymbol{y}')$;

$$y'_{j',i} = \begin{cases} y_{j',i} & j' \notin \{j-1, j, j+1\} \\ (\alpha^{\epsilon}_{i+2})^{-1} \cdot y^{-1}_{j,i+1} \cdot \alpha^{\epsilon}_{i} & j' = j \\ (\alpha^{\epsilon}_{i})^{-1} \cdot \epsilon\, y_{j,i}\, y_{j-1,i} \cdot \alpha^{\epsilon}_{i+1} & j' = j-1 \\ (\alpha^{\epsilon}_{i+1})^{-1} \cdot \epsilon\, y_{j,i+1}\, y_{j+1,i} \cdot \alpha^{\epsilon}_{i+2} & j' = j+1 \end{cases}$$

independently of k. Here $\alpha^{\epsilon}_{i} := \alpha^{\epsilon}_{i}(M_j) = 1 + \sum_{\ell=1}^{n-1} \epsilon^{\ell} y_{j,i}\, y_{j,i+1} \cdots y_{j,i+\ell-1}$.
We write $R^{\epsilon}_{M_j}(\boldsymbol{y}) = \boldsymbol{y}'$.

(ii) $R^{\epsilon}_{M_j}$ $(1 \le j \le m)$ generate $\mathfrak{S}_{m+1}$ action on $\mathbb{C}_{\epsilon}\langle \boldsymbol{y} \rangle$.

Thm 3

(i) We have $\mathcal{R}^{\epsilon}_{M_j,k}(Q_{n,m},\boldsymbol{y}) = (Q_{n,m},\boldsymbol{y}')$;

$$y'_{j',i} = \begin{cases} y_{j',i} & j' \notin \{j-1, j, j+1\} \\ (\alpha^{\epsilon}_{i+2})^{-1}\cdot y^{-1}_{j,i+1}\cdot\alpha^{\epsilon}_i & j' = j \\ (\alpha^{\epsilon}_i)^{-1}\cdot \epsilon\, y_{j,i}\, y_{j-1,i}\cdot\alpha^{\epsilon}_{i+1} & j' = j-1 \\ (\alpha^{\epsilon}_{i+1})^{-1}\cdot \epsilon\, y_{j,i+1}\, y_{j+1,i}\cdot\alpha^{\epsilon}_{i+2} & j' = j+1 \end{cases}$$

independently of k. Here $\alpha^{\epsilon}_i := \alpha^{\epsilon}_i(M_j) = 1 + \sum_{\ell=1}^{n-1}\epsilon^{\ell} y_{j,i}\, y_{j,i+1}\cdots y_{j,i+\ell-1}$.
We write $R^{\epsilon}_{M_j}(\boldsymbol{y}) = \boldsymbol{y}'$.

(ii) $R^{\epsilon}_{M_j}$ $(1 \le j \le m)$ generate $\mathfrak{S}_{m+1}$ action on $\mathbb{C}_{\epsilon}\langle\boldsymbol{y}\rangle$.

Key (Thm 3 (ii))

the equivalence of a ‘mutation period’ among classical, quantum and tropical y-seeds
[Fock-Goncahrov 2009] [I-Iyama-Keller-Kuniba-Nakanishi 2013]

• Quantum cluster R and quantum geometric R

Consider the y-seed $(Q_{n,m-1}, \boldsymbol{y})$ and define
$\phi^{\epsilon} : \mathbb{C}_{\epsilon}\langle \boldsymbol{y} \rangle \hookrightarrow \mathbb{C}_{\epsilon}\langle \mathbf{q}_1, \ldots, \mathbf{q}_m \rangle;\ y_{j,i} \mapsto \epsilon^{-1} q_{j,i}^{-1} q_{j+1,i-1}.$

• Quantum cluster R and quantum geometric R

Consider the y-seed $(Q_{n,m-1}, \boldsymbol{y})$ and define
$\phi^\epsilon : \mathbb{C}_\epsilon\langle \boldsymbol{y} \rangle \hookrightarrow \mathbb{C}_\epsilon\langle \mathbf{q}_1, \ldots, \mathbf{q}_m \rangle;\ y_{j,i} \mapsto \epsilon^{-1} q_{j,i}^{-1} q_{j+1,i-1}$.

Thm 4
(i) For $j = 1, \ldots, m-1$, the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{C}_\epsilon\langle \boldsymbol{y} \rangle \arrow[r, "\phi^\epsilon"] \arrow[d, "R^\epsilon_{M_j}"'] & \mathbb{C}_\epsilon\langle \mathbf{q}_1, \ldots, \mathbf{q}_m \rangle \arrow[d, "R^\epsilon_j"] \\
\mathbb{C}_\epsilon\langle \boldsymbol{y} \rangle \arrow[r, "\phi^\epsilon"'] & \mathbb{C}_\epsilon\langle \mathbf{q}_1, \ldots, \mathbf{q}_m \rangle
\end{tikzcd}$$

(ii) R^ϵ_j $(j = 1, \ldots, m-1)$ generate $\mathfrak{S}_m$ action on $\mathbb{C}_\epsilon\langle \mathbf{q}_1, \ldots, \mathbf{q}_m \rangle$.

● Invariants of quantum geomtric R

Fact [Yamada 2001] [Lam-Pylyyavskyy 2010-] [L-P-Sakamoto 2014]
The loop analogues of elementary symmetric funct, Schur funct, skew Schur funct are defined in $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]$, and shown to belong to $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]^{\mathfrak{S}_m}$.

• Invariants of quantum geomtric R

Fact [Yamada 2001] [Lam-Pylyyavskyy 2010-] [L-P-Sakamoto 2014]
The loop analogues of elementary symmetric funct, Schur funct, skew Schur funct are defined in $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]$, and shown to belong to $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]^{\mathfrak{S}_m}$.

Claim We have its quantum analogue.

• Invariants of quantum geomtric R

Fact [Yamada 2001] [Lam-Pylyyavskyy 2010-] [L-P-Sakamoto 2014]
The loop analogues of elementary symmetric funct, Schur funct, skew Schur funct are defined in $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]$, and shown to belong to $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]^{\mathfrak{S}_m}$.

Claim We have its quantum analogue.

Def Quantum loop elementary symmetric function:

$$e_k^{(r)}(\mathbf{q}_1, \ldots, \mathbf{q}_m) := \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} q_{j_1, r+1-j_1} q_{j_2, r+2-j_2} \cdots q_{j_k, r+k-j_k} \in \mathbb{C}_\epsilon \langle \boldsymbol{q} \rangle$$

for $1 \le k \le m, r \in \mathbb{Z}/n\mathbb{Z}$, and $e_0^{(r)} = 1$ and $e_s^{(r)} = 0$ for $s < 0$ or $s > m$.

• Invariants of quantum geomtric R

Fact [Yamada 2001] [Lam-Pylyyavskyy 2010-] [L-P-Sakamoto 2014]
The loop analogues of elementary symmetric funct, Schur funct, skew Schur funct are defined in $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]$, and shown to belong to $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]^{\mathfrak{S}_m}$.

Claim We have its quantum analogue.

Def Quantum loop elementary symmetric function:

$$e_k^{(r)}(\mathbf{q}_1, \ldots, \mathbf{q}_m) := \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} q_{j_1, r+1-j_1} q_{j_2, r+2-j_2} \cdots q_{j_k, r+k-j_k} \in \mathbb{C}_\epsilon \langle \boldsymbol{q} \rangle$$

for $1 \le k \le m, r \in \mathbb{Z}/n\mathbb{Z}$, and $e_0^{(r)} = 1$ and $e_s^{(r)} = 0$ for $s < 0$ or $s > m$.
$\left(\overset{\epsilon \to 1}{\to} \text{ loop elementary sym } \overset{q_{j,i} \to q_j}{\to} \text{ elementary sym} \right)$

• Invariants of quantum geomtric R

Fact [Yamada 2001] [Lam-Pylyyavskyy 2010-] [L-P-Sakamoto 2014]
The loop analogues of elementary symmetric funct, Schur funct, skew Schur funct are defined in $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]$, and shown to belong to $\mathbb{C}[\mathbf{q}_1, \ldots, \mathbf{q}_m]^{\mathfrak{S}_m}$.

Claim We have its quantum analogue.

Def Quantum loop elementary symmetric function:

$$e_k^{(r)}(\mathbf{q}_1, \ldots, \mathbf{q}_m) := \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} q_{j_1, r+1-j_1} q_{j_2, r+2-j_2} \cdots q_{j_k, r+k-j_k} \quad \in \mathbb{C}_\epsilon \langle \boldsymbol{q} \rangle$$

for $1 \le k \le m, r \in \mathbb{Z}/n\mathbb{Z}$, and $e_0^{(r)} = 1$ and $e_s^{(r)} = 0$ for $s < 0$ or $s > m$.
$\left(\overset{\epsilon \to 1}{\to} \text{ loop elementary sym } \overset{q_{j,i} \to q_j}{\to} \text{ elementary sym} \right)$

(Ex) $n = m = 3$:

$$e_1^{(1)} = q_{1,1} + q_{2,3} + q_{3,2}$$
$$e_2^{(1)} = q_{1,1}q_{2,1} + q_{1,1}q_{3,3} + q_{2,3}q_{3,3}, \qquad e_3^{(1)} = q_{1,1}q_{2,1}q_{3,1}$$

Key (Cf [Noumi-Yamada])

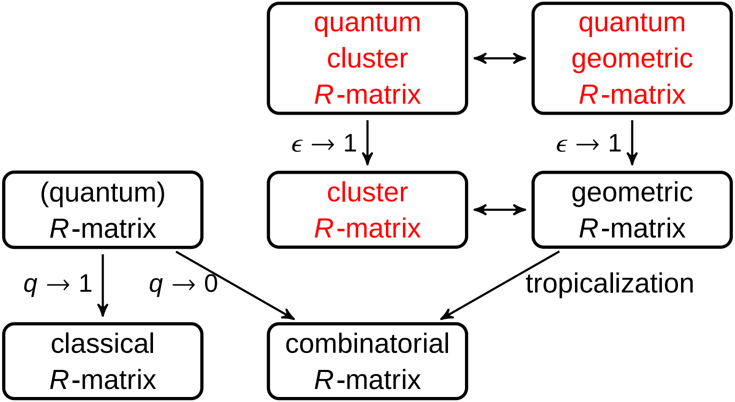
(i) Define

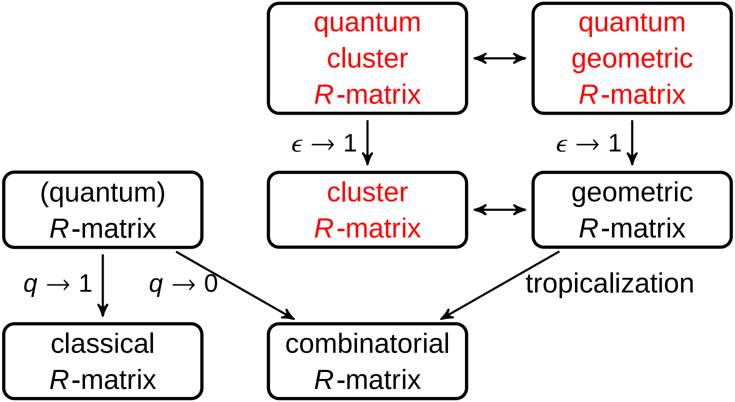
$$M(\mathbf{q};t) := \begin{pmatrix} q_1 & & & t \\ 1 & q_2 & & \\ & \ddots & \ddots & \\ & & 1 & q_n \end{pmatrix}.$$

When $R^\epsilon(\mathbf{p},\mathbf{q}) = (\mathbf{p}',\mathbf{q}')$, then it holds that $M(\mathbf{p};t)M(\mathbf{q};t) = M(\mathbf{p}';t)M(\mathbf{q}';t)$.

(ii) The $e_k^{(r)}$ are the entries of $M(\mathbf{q}_1;t)M(\mathbf{q}_2;t)\cdots M(\mathbf{q}_m;t)$.

Thus $e_k^{(r)} \in \mathbb{C}_\epsilon\langle \boldsymbol{q}\rangle^{\mathfrak{S}_m}$.


quantum
cluster
R-matrix
quantum
geometric
R-matrix
$\epsilon \to 1$
$\epsilon \to 1$
(quantum)
R-matrix
cluster
R-matrix
geometric
R-matrix
$q \to 1$
$q \to 0$
tropicalization
classical
R-matrix
combinatorial
R-matrix


quantum cluster R-matrix
quantum geometric R-matrix
ϵ → 1
ϵ → 1
(quantum) R-matrix
cluster R-matrix
geometric R-matrix
q → 1
q → 0
tropicalization
classical R-matrix
combinatorial R-matrix

Thank you!