Super Quantum Curves as Super Virasoro Singular Vectors

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Joint works with P. Ciosmak, L. Hadasz, P. Sułkowski
based on arXiv:1512.05785, and to appear

1. Introduction

Main interest in this talk

- Quantization of Riemann surface (in topological strings)

$$A(x,y)=0,\quad x,y\in\mathbb{C}\quad\rightsquigarrow\quad\widehat{A}(\widehat{x},\widehat{y})\Psi(x)=0,\quad[\widehat{y},\widehat{x}]=\hbar$$

Brief sketch of a background / motivation

Riemann surface

$\Sigma : A(x, y) = 0$

$x, y \in \boldsymbol{C}$

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Top. B-model on local CY3		Riemann surface
$A(x, y) = uv$	embed	$\Sigma : A(x, y) = 0$
$x, y, u, v \in \mathbf{C}$	$\longleftarrow$	$x, y \in \mathbf{C}$

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$A(x, y) = uv$

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Quantization
of the complex structure

BCOV Theory [94]
(Quantum Kodaira-Spencer)

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↑ Quantization of the complex structure

BCOV Theory [94]
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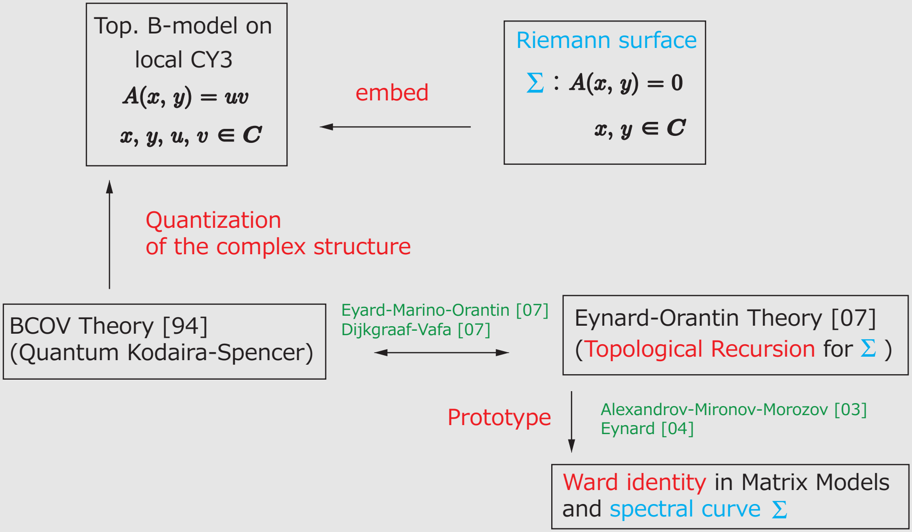
Eyard-Marino-Orantin [07]
Dijkgraaf-Vafa [07]
↔

Eynard-Orantin Theory [07]
(Topological Recursion for Σ)

Brief sketch of a background / motivation



As a prototype of topological strings, it would be instructive to discuss the quantization in the context of matrix models.

The most simplest one is hermitian 1-matrix model:

$$Z = \int dM_{N\times N} e^{-\frac{1}{\hbar}\mathrm{Tr}V(M)}$$

One can translate the matrix model language into 2d $c = 1$ CFT language via free field representation

$$\phi(x) = \frac{1}{2\hbar} V(x) - \mathrm{Tr}\log(x - M)$$

The Ward identity yields the Virasoro constraints $L_{n\geq -1} Z = 0$.

In the B-model language, $\phi(x)$ describes the Kodaira-Spencer field on target Riemann surface Σ; Aganagic-Dijkgraaf-Klemm-Marino-Vafa [03]

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A brief look at classical limit of quantum curve

Consider a "wave-function" in the matrix model

$$\Psi(x) = \left\langle e^{\phi(x)} \right\rangle$$

This function satisfies a quantum curve equation (time-dependent Schrodinger equation; Aganagic-Cheng-Dijkgraaf-Krefl-Vafa [11])

$$\widehat{A}\Psi(x) = 0, \quad \widehat{A} = \hbar^2 \partial_x^2 - \widehat{\mathcal{L}}_{-2}(x)$$

where $\widehat{\mathcal{L}}_{-2}(x)$ is a time-dependent ("∂_{t_i}") operator.
Define spectral function (which is finitely defined)

$$y(x) = \lim_{``\hbar \to 0"} \langle \hbar \partial_x \phi(x) \rangle$$

Then $\hbar \partial_x \Psi(x) \overset{``\hbar \to 0"}{\to} y(x)\Psi(x)$, and we find classical spectral curve:

$$\widehat{A}\Psi(x) = 0 \quad \overset{``\hbar \to 0"}{\longrightarrow} \quad A(x) = y(x)^2 - L(x) = 0$$

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One can generalize this matrix model/CFT with $c = 1$ to

Beta-deformed ensemble $\longleftrightarrow$ CFT with $c = 1 - 6Q^2$

where $Q \equiv \beta^{-1/2} - \beta^{1/2}$, for beta-deformed ensemble

$$Z = \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \Delta(z)^{2\beta} e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^{N} V(z_a)}$$

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Aganagic-Cheng-Dijkgraaf-Krefl-Vafa [11] $\rightsquigarrow$ "stacked version";
M.M.-Sułkowski [15]

Using Ward identity one can construct (Piotr's talk)

$$\text{Quantum Curves} \quad \stackrel{1:1}{\longleftrightarrow} \quad \text{Virasoro Singular vectors } \Phi_{r,s}$$

As a straightforward generalization we can introduce SUSY;

$$\text{Beta-deformed super-ensemble} \quad \longleftrightarrow \quad \text{SCFT with } c = \frac{3}{2} - 3Q^2$$

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$$Z = \int \prod_{a=1}^{N} dz_a d\vartheta_a \Delta(z, \vartheta)^\beta e^{-\frac{\sqrt{\beta}}{\hbar}\sum_{a=1}^{N}\left(V_B(z_a) + V_F(z_a)\vartheta_a\right)}$$

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$V_B(x)$ and $V_F(x)$ are bosonic and fermionic single-valued functions of x, and then we can describe the NS sector on the SCFT side (R-sector?).

The Ward identity yields the NS super-Virasoro constraints $G_{n\geq -1/2}Z = L_{n\geq -1}Z = 0$.

The $\beta = 1$ case was studied in the first half of the 90s; Alvarez-Gaume-Itoyama-Manes-Zadra [91], Plefka [96] etc

Even for the $\beta = 1$ case, "matrix model description" is NOT known. (Feynman diagram description by 't Hooft?)

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- Classical spectral curve in the (beta-deformed) super-ensemble
- Construction (by Ward identity) of

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where $\Delta(z,\vartheta) \equiv \prod_{1\le a<b\le N}(z_a - z_b - \vartheta_a\vartheta_b)$, and put

$$V_B(x) = \sum_{n=0}^{\infty} t_n x^n, \qquad V_F(x) = \sum_{n=0}^{\infty} \xi_{n+1/2} x^n$$

Relation with 2d SCFT w/ NS sector

$$\phi(x) = \frac{1}{\hbar} V_B(x) - \sqrt{\beta}\sum_{a=1}^{N}\log(x - z_a), \quad \psi(x) = \frac{1}{\hbar} V_F(x) - \sqrt{\beta}\sum_{a=1}^{N}\frac{\vartheta_a}{x - z_a}$$

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Actually in the expectation value, one finds annihilation operators as

$$\sum_{a=1}^{N} z_a^n \longleftrightarrow -\frac{\hbar}{\sqrt{\beta}}\partial_{t_n}, \quad \sum_{a=1}^{N} z_a^n \vartheta_a \longleftrightarrow -\frac{\hbar}{\sqrt{\beta}}\partial_{\xi_{n+1/2}}$$

and then obtains the OPEs

$$\phi(x_1)\phi(x_2) = \log(x_1 - x_2) + \cdots, \quad \psi(x_1)\psi(x_2) = \frac{1}{x_1 - x_2} + \cdots$$

The superconformal current and the EM tensor are realized as

$$S(x) =: \psi(x)\partial_x\phi(x) : + Q\partial_x\psi(x)$$
$$T(x) = \frac{1}{2} : \partial_x\phi(x)\partial_x\phi(x) : + \frac{1}{2} : \partial_x\psi(x)\psi(x) : + \frac{1}{2}Q\partial_x^2\phi(x)$$

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Then by the mode expansions

$$S(x) = \sum_{r\in\mathbb{Z}+1/2} G_r x^{-r-3/2}, \qquad T(x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2}$$

one obtains the NS super-Virasoro algebra

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

with the central charge

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Ward identities (positive frequency modes)

$$\langle S_+(x) \rangle = \langle T_+(x) \rangle = 0$$

Actually $S_+(x)$ and $T_+(x)$ generates infinitesimal shifts

$$S_+(x): \quad z_a \to z_a + \frac{\vartheta_a \delta}{x - z_a}, \quad \vartheta_a \to \vartheta_a + \frac{\delta}{x - z_a}$$

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Ward identities (positive frequency modes)

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$S_+(x)$ is obtained as

$$S_+(x) = \beta \sum_{a,b=1}^{N} \frac{\vartheta_a}{(x-z_a)(x-z_b)} + (1-\beta) \sum_{a=1}^{N} \frac{\vartheta_a}{(x-z_a)^2} - \frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^{N} \frac{1}{x-z_a} \left(V_B'(z_a)\vartheta_a + V_F(z_a) \right)$$

Super-Virasoro constraints (by the mode expansion)

$$\langle S_+(x) \rangle = 0 \quad \Longrightarrow \quad G_{n+1/2} Z = 0, \qquad n \geq -1$$

where

$$G_{n+1/2} = \sum_{k=1}^{\infty} k t_k \partial_{\xi_{k+n+1/2}} + \sum_{k=0}^{\infty} \xi_{k+1/2} \partial_{t_{k+n+1}} + \hbar^2 \sum_{k=0}^{n} \partial_{\xi_{k+1/2}} \partial_{t_{n-k}} + (\sqrt{\beta} - \sqrt{\beta^{-1}})\hbar(n+1)\partial_{\xi_{n+1/2}}$$

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$$T_+(x) = \frac{\beta}{2}\sum_{a,b=1}^{N}\frac{1}{(x-z_a)(x-z_b)} + \frac{\beta}{2}\sum_{a,b=1}^{N}\frac{\vartheta_a\vartheta_b}{(x-z_a)(x-z_b)^2}$$

$$+\frac{1}{2}(1-\beta)\sum_{a=1}^{N}\frac{1}{(x-z_a)^2}$$

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where

$$L_n = \sum_{k=1}^{\infty} k t_k \partial_{t_{k+n}} + \frac{\hbar^2}{2}\sum_{k=1}^{n}\partial_{t_k}\partial_{t_{n-k}} + \sum_{k=0}^{\infty}\left(k+\frac{n+1}{2}\right)\xi_{k+1/2}\partial_{\xi_{k+n+1/2}}$$

$$+\frac{\hbar^2}{2}\sum_{k=1}^{n} k\,\partial_{\xi_{n-k+1/2}}\partial_{\xi_{k-1/2}} + \frac{1}{2}\left(\sqrt{\beta}-\sqrt{\beta^{-1}}\right)\hbar(n+1)\partial_{t_n}$$

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3. Classical spectral curve via Ward identity

't Hooft limit (denoted by "$\hbar \to 0$")

$$N \to \infty, \quad \hbar \to 0, \quad \beta \to 1, \quad \text{with} \quad \hbar N = \text{fixed}$$

of the Ward identities

$$\langle S_+(x) \rangle = \langle T_+(x) \rangle = 0$$

Define spectral functions

$$y_B(x) = \lim_{\text{“}\hbar\to 0\text{”}} \langle \hbar \partial_x \phi(x) \rangle, \quad y_F(x) = \lim_{\text{“}\hbar\to 0\text{”}} \langle \hbar \psi(x) \rangle$$

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"Spectral curves"

$$\langle S_+(x)\rangle = 0 \quad \xrightarrow{\text{"}\hbar\to 0\text{"}} \quad A_F(x, y_B|y_F) \equiv y_B(x)y_F(x) - G(x) = 0$$

$$\langle T_+(x)\rangle = 0 \quad \xrightarrow{\text{"}\hbar\to 0\text{"}} \quad A_B(x, y_B|y_F) \equiv y_B(x)^2 + y_F'(x)y_F(x) - 2L(x) = 0$$

where

$$G(x) = V_B'(x)V_F(x) + h_{cl}(x), \quad L(x) = \frac{1}{2}V_B'(x)^2 + \frac{1}{2}V_F'(x)V_F(x) + f_{cl}(x)$$

and

$$h_{cl}(x) = -\lim_{\text{"}\hbar\to 0\text{"}} \left\langle \hbar \sum_{a=1}^{N} \left(\frac{V_F(x) - V_F(z_a)}{x - z_a} + \frac{(V_B'(x) - V_B'(z_a))\vartheta_a}{x - z_a} \right) \right\rangle$$

$$f_{cl}(x) = -\lim_{\text{"}\hbar\to 0\text{"}} \left\langle \hbar \sum_{a=1}^{N} \left(\frac{V_B'(x) - V_B'(z_a)}{x - z_a} + \frac{(V_F'(x) - V_F'(z_a))\vartheta_a}{2(x - z_a)} + \frac{V_F^{(2)}(x, z_a)\vartheta_a}{2(x - z_a)^2} \right) \right\rangle$$

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4. Ward identity for wave-function

Wave-function with a momenta α

$$\Psi_\alpha(x,\theta) = \left\langle e^{\frac{\alpha}{\hbar}\phi(x)+\frac{\alpha}{\hbar}\psi(x)\theta} \right\rangle$$

This operator insertion induces a back reaction to the potentials

$$\begin{aligned} V_B(z_a) &\longrightarrow \widetilde{V}_B(z_a;x) = V_B(z_a) + \alpha\log(x-z_a) \\ V_F(z_a) &\longrightarrow \widetilde{V}_F(z_a;,x,\theta) = V_F(z_a) - \frac{\alpha\theta}{x-z_a} \end{aligned}$$

As the result the original superconformal current and the EM tensor are deformed

$$\begin{aligned} S(y) &\longrightarrow S(y;x,\theta) \\ T(y) &\longrightarrow T(y;x,\theta) \end{aligned}$$

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On the other hand, the negative frequency modes

$$\mathcal{G}_{-n+1/2}\Psi_\alpha(x,\theta) = \oint_{y=x}\frac{dy}{2\pi i}(y-x)^{1-n}\left\langle S_-(y;x,\theta)e^{\frac{\alpha}{\hbar}\phi(x)+\frac{\alpha}{\hbar}\psi(x)\theta}\right\rangle$$

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Explicitly we obtain

$$\mathcal{G}_{-1/2} = \theta\partial_x - \partial_\theta,$$
$$\mathcal{G}_{-n+1/2} = \frac{1}{\hbar^2(n-2)!}\Big(\partial_x^{n-2}\big(V_B'(x)V_F(x)\big) + Q\hbar\partial_x^{n-1}V_F(x) + \partial_x^{n-2}\widehat{h}(x) + \big[\partial_x^{n-2}\widehat{h}(x), \log Z\big]\Big), \quad \text{for } n \geq 2$$

$$\widehat{h}(x) \equiv \hbar^2 \sum_{n=0}^{\infty} x^n \sum_{k=n+2}^{\infty} \left(\xi_{k-1/2}\partial_{t_{k-n-2}} + kt_k\partial_{\xi_{k-n-3/2}}\right)$$

$$\mathcal{L}_{-1} = \partial_x,$$
$$\mathcal{L}_{-n} = \frac{1}{\hbar^2(n-2)!}\Big(\frac{1}{2}\partial_x^{n-2}\big(V_B'(x)^2\big) + \frac{1}{2}\partial_x^{n-2}\big(V_F'(x)V_F(x)\big) + \frac{1}{2}Q\hbar\partial_x^{n}V_B(x) + \partial_x^{n-2}\widehat{f}(x) + \big[\partial_x^{n-2}\widehat{f}(x), \log Z\big]\Big), \quad \text{for } n \geq 2,$$

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5. Super quantum curves

NS super Virasoro singular vectors

Consider NS highest weight vector $|\Delta_\alpha\rangle$ in SCFT:

$$\mathrm{L}_0 |\Delta_\alpha\rangle = \frac{1}{2}\Delta_\alpha |\Delta_\alpha\rangle\,, \quad \mathrm{L}_{n\geq 1} |\Delta_\alpha\rangle = \mathrm{G}_{n\geq 1/2} |\Delta_\alpha\rangle = 0, \quad \Delta_\alpha \equiv \frac{\alpha}{\hbar}(\frac{\alpha}{\hbar} - Q)$$

with

$$c = \frac{3}{2} - 3Q^2, \quad Q = -\beta^{1/2} + \beta^{-1/2}$$

Then only the cases for

$$\frac{\alpha}{\hbar} = \frac{(r-1)\beta^{1/2} - (s-1)\beta^{-1/2}}{2}, \qquad \text{with } r - s \in 2\mathbb{Z},$$

the NS Verma module generated by the operations $\mathrm{L}_{n\leq -1}$ and $\mathrm{G}_{n\leq -1/2}$ on $|\Delta_\alpha\rangle$ contains a unique degenerate vector at level

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We can construct super quantum curves corresponding to these singular vectors from the Ward identities

$$\left\langle S_{+}(y;x,\theta)e^{\frac{\alpha}{\hbar}\phi(x)+\frac{\alpha}{\hbar}\psi(x)\theta}\right\rangle = \left\langle T_{+}(y;x,\theta)e^{\frac{\alpha}{\hbar}\phi(x)+\frac{\alpha}{\hbar}\psi(x)\theta}\right\rangle = 0$$

The most simplest one is the level $3/2$, and it is constructed from

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We find that only for

$$\frac{\alpha}{\hbar} = 0, \quad \sqrt{\beta}, \quad -\sqrt{\beta^{-1}}$$

the Ward identity is expressed as a differential equation

$$\theta \widehat{A}^{(0)}_{3/2} \Psi_\alpha(x,\theta) = 0, \quad \widehat{A}^{(0)}_{3/2} \equiv -\partial_x \partial_\theta - \frac{\alpha^2}{\hbar^2} \mathcal{G}_{-3/2}$$

$$\mathcal{G}_{-3/2} = \frac{1}{\hbar^2} \Big(V'_B(x) V_F(x) + Q\hbar V'_F(x) + \widehat{h}(x) + [\widehat{h}(x), \log Z] \Big)$$

Let us decompose the wave-function into components

$$\Psi_\alpha(x,\theta) = \left\langle e^{\frac{\alpha}{\hbar}\phi(x) + \frac{\alpha}{\hbar}\psi(x)\theta} \right\rangle = \Psi_{B,\alpha}(x) + \Psi_{F,\alpha}(x)\theta$$

Then the above differential equation yields

$$\partial_x \Psi_{F,\alpha}(x) - \frac{\alpha^2}{\hbar^2} \mathcal{G}_{-3/2} \Psi_{B,\alpha}(x) = 0$$

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To compensate the incompleteness, firstly, using the Virasoro generators acting on the each components let us rewrite

$$\left(L_{-1}G_{-1/2} - \frac{\alpha^2}{\hbar^2}G_{-3/2}\right)\Psi_{B,\alpha}(x) = 0$$

Here we defined $\{L_{-n}\}$ and $\{G_{-n+1/2}\}$, e.g. as

$$\begin{aligned}\mathcal{G}_{-1/2}\Psi_\alpha(x,\theta) &= \Psi_{F,\alpha}(x) + \partial_x\Psi_{B,\alpha}(x)\theta \\ &= G_{-1/2}\Psi_{B,\alpha}(x) + G_{-1/2}\Psi_{F,\alpha}(x)\theta\end{aligned}$$

By the operation of $G_{-1/2}$ on this equation and using the super-Virasoro algebra we can find second differential equation

$$\theta\widehat{A}^{(1)}_{3/2}\Psi_\alpha(x,\theta) = 0, \quad \widehat{A}^{(1)}_{3/2} \equiv \partial_x^2 - \frac{2\alpha^2}{\hbar^2}\mathcal{L}_{-2} - \frac{\alpha^2}{\hbar^2}\mathcal{G}_{-3/2}\partial_\theta$$

$$\mathcal{L}_{-2} = \frac{1}{\hbar^2}\left(\frac{1}{2}V_B'(x)^2 + \frac{1}{2}V_F'(x)V_F(x) + \frac{1}{2}Q\hbar V_B''(x) + \widehat{f}(x) + [\widehat{f}(x), \log Z]\right)$$

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By packing the above two differential equation into a one place, we obtain

$$\widehat{A}_{3/2}\Psi_\alpha(x,\theta) = 0,$$

$$\widehat{A}_{3/2} \equiv \widehat{A}^{(0)}_{3/2} + \theta\left(\widehat{A}^{(0)}_{3/2}\partial_\theta - \widehat{A}^{(1)}_{3/2}\right) = -\partial_x\partial_\theta - \frac{\alpha^2}{\hbar^2}\mathcal{G}_{-3/2} - \theta\left(\partial_x^2 - \frac{2\alpha^2}{\hbar^2}\mathcal{L}_{-2}\right)$$

By the similar way we can also construct higher level super quantum curve equations, e.g. at level $5/2$:

$$\widehat{A}_{5/2}\Psi_\alpha(x,\theta) = 0,$$

$$\begin{aligned}\widehat{A}_{5/2} = &-\partial_x^2\partial_\theta + \frac{2\alpha(\alpha^2+Q\hbar\alpha-\hbar^2)}{\hbar^2(3\alpha+2Q\hbar)}\mathcal{L}_{-2}\partial_\theta - \frac{\alpha(2\alpha^2+Q\hbar\alpha+\hbar^2)}{\hbar^2(3\alpha+2Q\hbar)}\mathcal{G}_{-3/2}\partial_x \\ &+ \frac{\alpha^2(2\alpha^3+3Q\hbar\alpha^2+(Q^2-5)\hbar^2\alpha-3Q\hbar^3)}{\hbar^4(3\alpha+2Q\hbar)}\mathcal{G}_{-5/2} - \theta\Big(\partial_x^3 - \frac{2\alpha^2}{\hbar^2}\mathcal{L}_{-2}\partial_x \\ &+ \frac{\alpha(\alpha^2+Q\hbar\alpha-\hbar^2)}{\hbar^2(3\alpha+2Q\hbar)}\mathcal{G}_{-5/2}\partial_\theta + \frac{2\alpha^2(2\alpha^3+3Q\hbar\alpha^2+(Q^2-5)\hbar^2\alpha-3Q\hbar^3)}{\hbar^4(3\alpha+2Q\hbar)}\mathcal{L}_{-3}\Big)\end{aligned}$$

Valid only for $\quad \frac{\alpha}{\hbar} = 0, \quad \sqrt{\beta}, \quad -\sqrt{\beta^{-1}}, \quad -\frac{Q}{2}, \quad 2\sqrt{\beta}, \quad -2\sqrt{\beta^{-1}}$

This expression is valid not only at level $5/2$ but also at level $3/2$ and 2!

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Classical limit of level 3/2 quantum curve

Consider the level 3/2 quantum curve equation for $\beta = 1$:

$$\widehat{A}\Psi(x,\theta) = 0, \quad \widehat{A} = -\hbar^2\partial_x\partial_\theta - \mathcal{G}_{-3/2} - \theta(\hbar^2\partial_x^2 - 2\mathcal{L}_{-2})$$

Remember the spectral function

$$y_B(x) = \lim_{\text{“}\hbar\to 0\text{”}} \langle \hbar\partial_x\phi(x)\rangle, \quad y_F(x) = \lim_{\text{“}\hbar\to 0\text{”}} \langle \hbar\psi(x)\rangle$$

Then by

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