On volumes of compact and non-compact right-angled hyperbolic polyhedra

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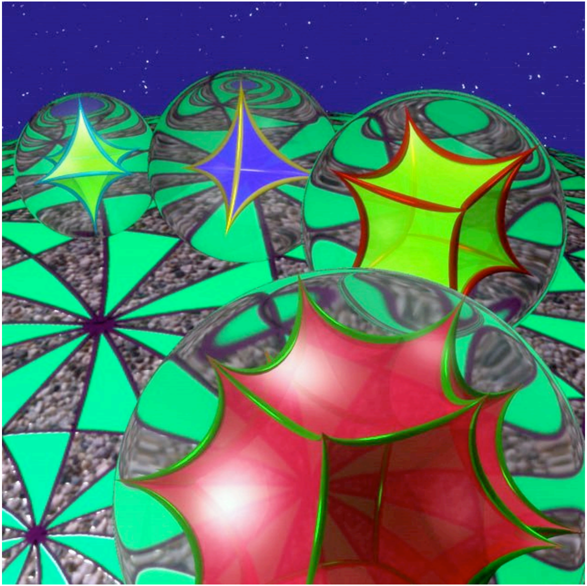
Conference "Quantum Topology", June 21, 2016

1. Uniqueness.
2. Existence.
3. Census of bounded.
4. Census of ideal.
5. Construction of 3-manifolds.



Konrad Polthier, Cover image of "Scientific Computing", 1999.

1. Uniqueness.

Let $\mathbb{H}^n$ denote an n-dimensional hyperbolic space.

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We will discuss two classes of acute-angled polyhedra:

- Coxeter polyhedra, with dihedral angles π/k;
- right-angled polyhedra, with all dihedral angled $\pi/2$.

2. Existence.

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Prokhorov, 1986: There is no finite-volume Coxeter polyhedra in $\mathbb{H}^n$ for $n > 995$.

Examples are known up to $n = 21$ only:

- Vinberg, 1972 for $n \leqslant 17$;
- Vinberg, Kaplinskaja, 1978 for $n = 18, 19$;
- Borcherds, 1987 fo $n = 21$.

For a combinatorial n-dimensional polyhedron P let $a_k(P)$ be the number of its k-dimensional faces and

$$a_k^\ell = \frac{1}{a_k} \sum_{\dim F = k} a_\ell(F)$$

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Nikulin, 1981:

$$a_k^\ell < \binom{n-\ell}{n-k} \frac{\binom{\left[\frac{n}{2}\right]}{\ell} + \binom{\left[\frac{n+1}{2}\right]}{\ell}}{\binom{\left[\frac{n}{2}\right]}{k} + \binom{\left[\frac{n+1}{2}\right]}{k}}$$

for $\ell < k \leqslant \left[\frac{n}{2}\right]$.

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for $\ell < k \leqslant \left[\frac{n}{2}\right]$. We will use this result to estimate a_2^1.

In particular, for a_2^1, the average number of sides in a 2-dimensional face, we get:

$$a_2^1 < \begin{cases} 4 + \dfrac{4}{n-2}, & \text{if} \quad n \quad \text{is even,} \\ 4 + \dfrac{4}{n-1}, & \text{if} \quad n \quad \text{is odd.} \end{cases}$$

But we already observed that $5 \leqslant a_2^1$.

Corollary from the Nikulin inequality:
There is no compact right-angled polyhedra in $\mathbb{H}^n$ for $n > 4$.

Examples for $n = 3, 4$ are well-known: dodecahedron and 120-cell.

Dufour, 2010: There is no finite volume right-angled polyhedra in $\mathbb{H}^n$ for $n > 12$.

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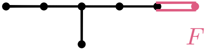
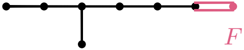
For $n = 6, 7, 8$ consider a Coxeter tetrahedron T^n with Coxeter diagram as below. In each case "black" sub-diagram corresponds to the finite Coxeter group E_n.

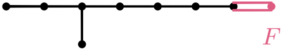
T^6 (with E_6)



T^7 (with E_7)



T^8 (with E_8)



Then the union

$$P^n = \bigcup_{g \in E_n} g(T)$$

is a right-angled polyhedron in $\mathbb{H}^n$ with faces $\partial P^n = \cup_{g \in E_n} g(F)$.

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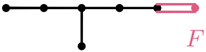

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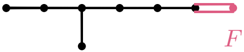

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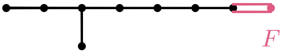

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is a right-angled polyhedron in $\mathbb{H}^n$ with faces $\partial P^n = \cup_{g \in E_n} g(F)$.

Recall order of the above groups: $|E_6| = 72 \cdot 6!$, $|E_7| = 72 \cdot 8!$, $|E_8| = 192 \cdot 10!$.

Pogorelov, 1967: A polyhedron can be realized in $\mathbb{H}^3$ as a bounded right-angled polyhedron if and only if

1. any vertex is incident to 3 edges;
2. any face has at least 5 sides;
3. any simple closed circuit on the surface of the polyhedron which separate some two faces of it (prismatic circuit), intersects at least 5 edges.

Right-angled polyhedra in $\mathbb{H}^3$

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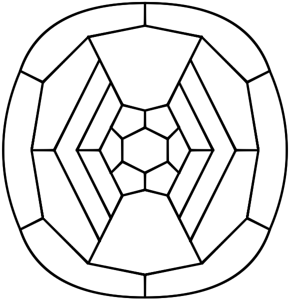
The simplest bounded $\pi/2$–polyhedron in $\mathbb{H}^3$ is the $\pi/2$–dodecahedron.

The following polyhedron satisfies (1) and (2), but not (3):

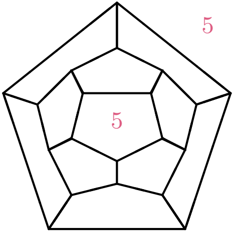


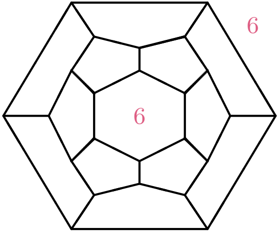
There is a closed circuit which separates two 6-gonal faces (top and bottom), but intersects only 4 edges.

3. A census of bounded right-angled polyhedra in $\mathbb{H}^3$.

By Pogorelov theorem: For any integer $n \geqslant 5$ there exists a right-angled $(2n+2)$-hedron Rn.

Polyhedra $R5$ and $R6$ look as the following:





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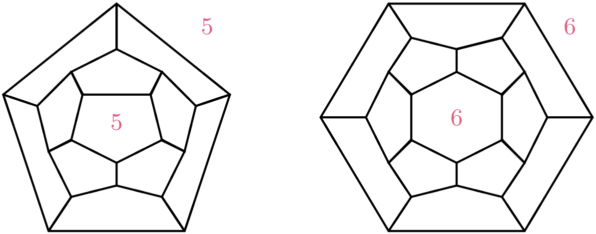

We will say Rn to be Löbelll polyhedra. Frank Richard Löbell (1893 – 1964).

Why this infinite family is important?

Two moves for bounded right-angled polyhedra

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Inoue, 2008: Two moves on $\mathcal{R}$.

- Composition / Decomposition: Let $R_1, R_2 \in \mathcal{R}$; $F_1 \subset R_1$ and $F_2 \subset R_2$ be a pair of k-gonal faces. Then a composition is $R = R_1 \cup_{F_1 = F_2} R_2$.

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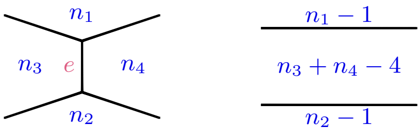
- Edge surgery: combinatorial transformation from R to $R - e$:



polyhedron R polyhedron $R - e$

If $R \in \mathcal{R}$ and e is such that faces F_1 and F_2 have at least 6 sides each and e is not a part of prismatic 5-circuit, then $R - e \in \mathcal{R}$.

Inoue, 2008: For any $P_0 \in \mathcal{R}$ there exists a sequence of unions of right-angled hyperbolic polyhedra $P_1, \ldots, P_k$ such that:

- each set P_i is obtained from P_{i-1} by decomposition or edge surgery,
- any union P_k consists of Löbell polyhedra.

Moreover,

$$\mathrm{vol}(P_0) \geqslant \mathrm{vol}(P_1) \geqslant \mathrm{vol}(P_2) \geqslant \ldots \geqslant \mathrm{vol}(P_k).$$

The volume formula for Löbell polyhedra

Volumes of hyperbolic 3-polyhedra can be done in terms of the Lobachevsky function

$$\Lambda(\theta) = -\int\limits_0^\theta \log|2\sin(t)|\,\mathrm{d}t.$$

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V., 1998: For any $n \geqslant 5$ the following formula holds for volumes of Löbell polyhedra

$$\mathrm{vol}(Rn) = \frac{n}{2}\left(2\Lambda(\theta_n) + \Lambda\left(\theta_n + \frac{\pi}{n}\right) + \Lambda\left(\theta_n - \frac{\pi}{n}\right) + \Lambda\left(\frac{\pi}{2} - 2\theta_n\right)\right),$$

where

$$\theta_n = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\pi/n)}\right).$$

The census of bounded right-angled polyhedra

Inoue, 2008: The dodecahedron $R5$ and the Löbell polyhedron $R6$ are the first and the second smallest volume bounded right-angled hyperbolic polyhedra.

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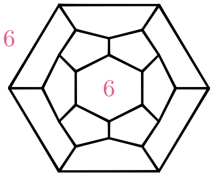
Shmel'kov – V., 2011: The first eleven smallest volume bounded right-angled hyperbolic polyhedra:

1	$4.3062\ldots$	$R5$	7	$8.6124\ldots$	$R5 \cup R5$
2	$6.0230\ldots$	$R6$	8	$8.6765\ldots$	$R6_3^3$
3	$6.9670\ldots$	$R6^1$	9	$8.8608\ldots$	$R6_1^3$
4	$7.5632\ldots$	$R7$	10	$8.9456\ldots$	$R6_2^3$
5	$7.8699\ldots$	$R6_1^2$	11	$9.0190\ldots$	$R8$
6	$8.0002\ldots$	$R6_2^2$			

The polyhedron $R6$ and possible faces to apply surgeries:



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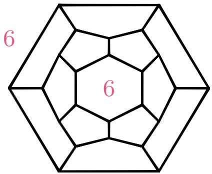
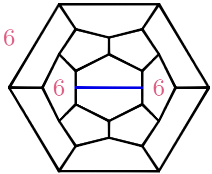
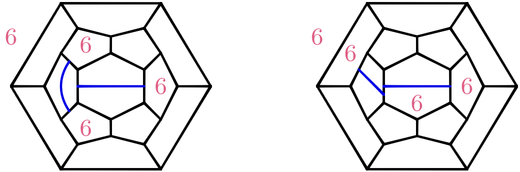


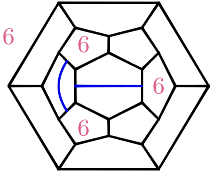
The polyhedron $R6^1$ (obtained from $R6$) and possible faces to apply surgeries:

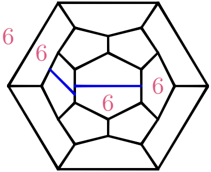


Polyhedra $R6_1^2$ and $R6_2^2$ obtained from $R6^1$ by edge surgeries.

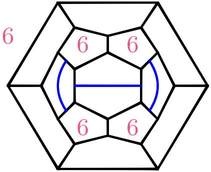


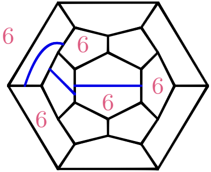
Polyhedra $R6^2_1$ and $R6^2_2$ obtained from $R6^1$ by edge surgeries.

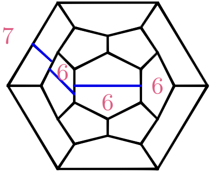




There are few possibilities to apply edge surgeries to them.
Polyhedra $R6^3_1$, $R6^3_2$ and $R6^3_3$ $(= R7^1)$:

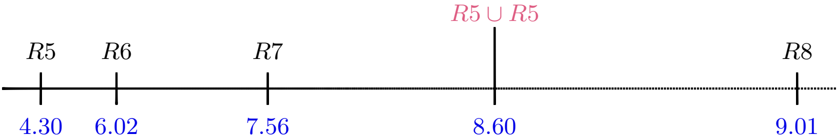




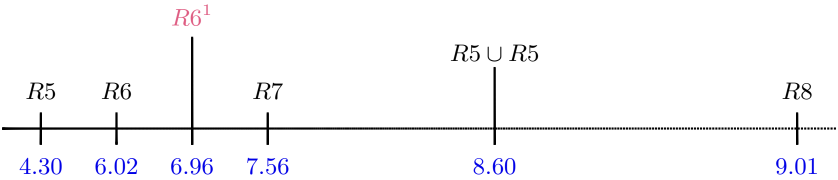


Löbell polyhedra:

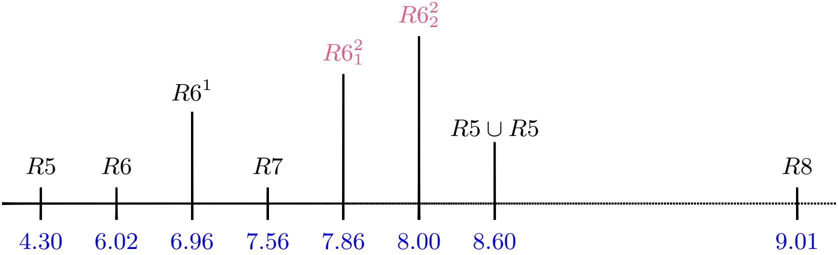
Composition of $R5$ with $R5$:



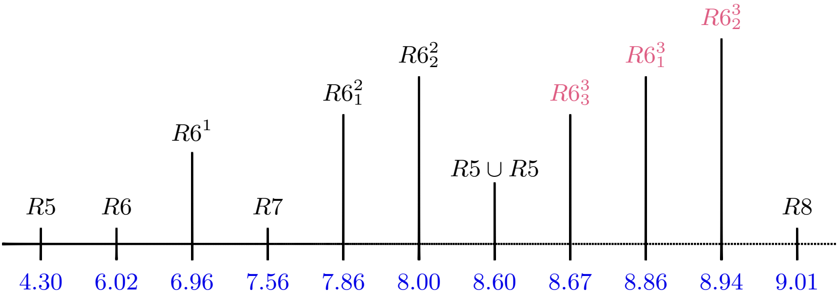
Edge surgery on $R6$:



Edge surgeries on $R6^1$:



Edge surgery on $R6_1^2$ and $R6_2^2$:



Atkinson, 2009: Let P be a bounded right-angled hyperbolic polyhedron with V vertices. Then

$$(V-2)\cdot\frac{v_8}{32}\leqslant \mathrm{vol}(P)<(V-10)\cdot\frac{5v_3}{8},$$

where $v_8=3.66\ldots$ is the maximal octahedron volume
and $v_3=1.014\ldots$ is the maximal tetrahedron volume.

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From the above census: If P is differ from the eleven smallest volume polyhedra then

$$\max\{(V-2)\cdot\frac{v_8}{32}, 9.019053\ldots\}\leqslant \mathrm{vol}(P)$$

and

$$\max\{(F-3)\cdot\frac{v_8}{16}, 9.019053\ldots\}\leqslant \mathrm{vol}(P).$$

This improves Atkinson's low bound for $V \leqslant 80$ and $F \leqslant 42$.

Inoue, arxiv:1512.0176:

The first 825 bounded right-angled polyhedra are constructed by compositions and edge surgeries. The 825th smallest right-angled polyhedron has volume $13.4203\ldots$.

The recent census of bounded right-angled polyhedra

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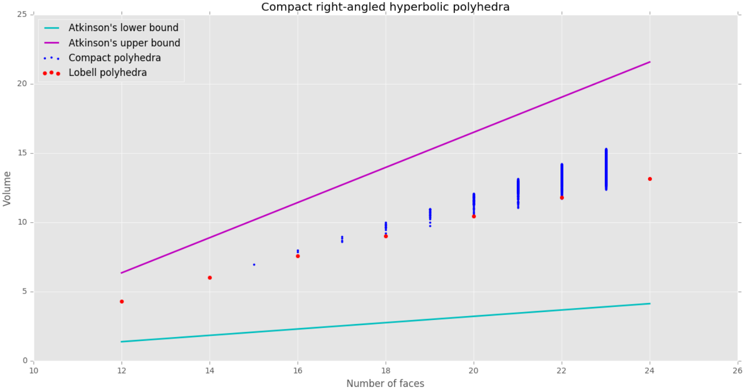
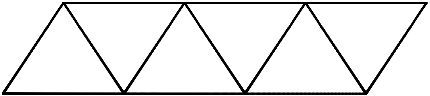
Shmel'kov – V.: about 3.000 smallest compact right-angled polyhedra.



4. A census of ideal right-angled polyhedra.

Let $\mathcal{A}_n$, $n \geqslant 3$, be an ideal (with all vertices at infinity) n-antiprism in $\mathbb{H}^3$ with dihedral angles $\pi/2$. See the figure, where left and right sides assumed to be identified.



In particular, the 3-antiprism is an octahedron.

It is known that

$$\mathrm{vol}(\mathcal{A}_n) = 2n \left[\Lambda \left(\frac{\pi}{4} + \frac{\pi}{2n} \right) + \Lambda \left(\frac{\pi}{4} - \frac{\pi}{2n} \right) \right].$$

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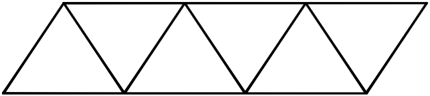

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Why this family of polyhedra is important for us?

Let $\mathcal{A}$ be the set of all ideal right-angled polyhedra in $\mathbb{H}^3$.

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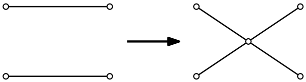
Define a move on $\mathcal{A}$. Let $A \in \mathcal{A}$.

- Edge twisting: combinatorial transformation from A to A^*:



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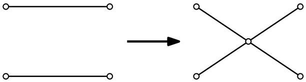

Shmel'kov – V.:

- If $A \in \mathcal{A}$ then $A^* \in \mathcal{A}$.
- The volume increases under an edge twisting move.
- Every ideal right-angled polyhedron $A \in \mathcal{A}$ can be constructed by a finitely many edge twisting moves from an n-antiprism $\mathcal{A}_n$ for some n.

Corollary: The octahedron $\mathcal{A}_3$ and polyhedron $\mathcal{A}_4$ are the first and the second smallest volume ideal right-angled polyhedra.

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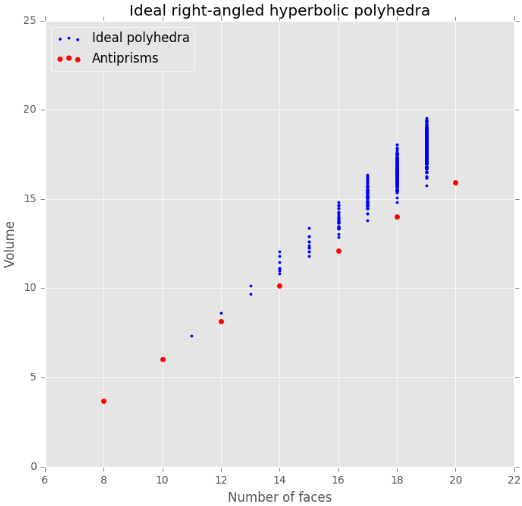
Let A be an $\mathcal{A}$-polyhedron which volume is no more than $\mathrm{vol}(\mathcal{A}_7) = 12.106298\ldots$. Then A is contained in the following list of pairwise non-isometric polyhedra:

1	$3.6638\ldots$	$\mathcal{A}_3$	11	$10.9915\ldots$	$\mathcal{A}_5^{**}(6)$
2	$6.0230\ldots$	$\mathcal{A}_4$	12	$11.1362\ldots$	$\mathcal{A}_5^{**}(5)$
3	$7.3277\ldots$	$\mathcal{A}_4^{*}$	13	$11.1362\ldots$	$\mathcal{A}_5^{**}(2)$
4	$8.1378\ldots$	$\mathcal{A}_5$	14	$11.4472\ldots$	$\mathcal{A}_5^{**}(3)$
5	$8.6124\ldots$	$\mathcal{A}_4^{**}$	15	$11.8017\ldots$	$\mathcal{A}_4^{****}(1)$
6	$9.6869\ldots$	$\mathcal{A}_5^{*}$	16	$11.8017\ldots$	$\mathcal{A}_6^{*}(1)$
7	$10.1494\ldots$	$\mathcal{A}_4^{***}$	17	$12.0460\ldots$	$\mathcal{A}_4^{****}(2)$
8	$10.1494\ldots$	$\mathcal{A}_6$	18	$12.0460\ldots$	$\mathcal{A}_6^{*}(2)$
9	$10.8060\ldots$	$\mathcal{A}_5^{**}(1)$	19	$12.1062\ldots$	$\mathcal{A}_7$
10	$10.9915\ldots$	$\mathcal{A}_5^{**}(4)$			

Census of ideal right-angled polyhedra

Shmel'kov – V.: about 2.000 smallest ideal right-angled polyhedra.



5. Constructing hyperbolic 3-manifolds from right-angled polyhedra.

Suppose

- P be a bounded $\pi/2$–polyhedron in $\mathbb{H}^3$;
- G be the group generated by reflections in faces of P.

Consider P as a Coxeter orbifold.

For each vertex v of P its stabilizer in G is isomorphic to the eight-element abelian group $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}_2^3$.

This group can be regarded as the finite vector space over the field $GF(2)$ with a basis

$$\{(1,0,0),(0,1,0),(0,0,1)\}.$$

Al-Jubouri, 1980: The kernel Ker φ of an epimorphism $\varphi : G \to \mathbb{Z}_2^3$ is torsion-free if and only if for any vertex v of P images of reflections in faces incident to v are linearly independent in $\mathbb{Z}_2^3$.

Therefore, if φ satisfies this "local linear independence" property then $M = \mathbb{H}^3/\text{Ker}\,\varphi$ is a closed hyperbolic 3-manifold (orientable or non-orientable) constructed from eight copies of P.

Elements $\alpha = (1,0,0)$, $\beta = (0,1,0)$, $\gamma = (0,0,1)$ and $\delta = \alpha + \beta + \gamma = (1,1,1)$ are such that any three of them are linearly independent in $\mathbb{Z}_2^3$.

Four colors

Elements $\alpha = (1, 0, 0)$, $\beta = (0, 1, 0)$, $\gamma = (0, 0, 1)$ and $\delta = \alpha + \beta + \gamma = (1, 1, 1)$ are such that any three of them are linearly independent in $\mathbb{Z}_2^3$.

V., 1987: If $\varphi : G \to \mathbb{Z}_2^3$ is such that for any generator g of G its image $\varphi(g)$ belongs to $\{\alpha, \beta, \gamma, \delta\}$ then $\mathrm{Ker}\, \varphi$ does not contain orientation-reversing elements.

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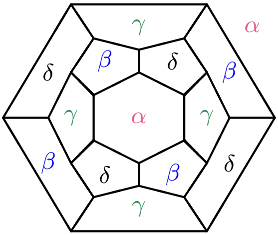
Corollary: If an epimorphism $\varphi : G \to \mathbb{Z}_2^3$ is such that

- for any generator g of G its image $\varphi(g)$ belongs to $\{\alpha, \beta, \gamma, \delta\}$;
- for any two adjacent faces their images are different;

then $M = \mathbb{H}^3 / \mathrm{Ker}\, \varphi$ is a closed orientable hyperbolic 3-manifold.

Thus any 4-coloring of faces of P defines a closed orientable hyperbolic 3-manifold.

The Löbell manifold (the first example of closed orientable hyperbolic 3-manifold) can be obtained from the following 4-coloring of $R6$:



[F. Löbell, Beispiele geschlossene dreidimensionaler Clifford–Kleinischer Räume negative Krümmung, Ber. Verh. Sächs. Akad. Lpz., Math.-Phys. Kl. **83** (1931), 168–174.]

Let $P \in \mathcal{R}$; G be generated by reflections in faces of P; Σ be the symmetry group of P.

A group G is said to be naturally maximal if $\langle G, \Sigma \rangle$ is a maximal discrete group, i.e. is not a proper subgroup of any discrete group of isometries of $\mathbb{H}^3$.

V.: Let G be a non-arithmetic naturally maximal group. Let $\varphi_1, \varphi_2 : G \to \mathbb{Z}_2^3$ be epimorphisms induced by two 4-colorings. Manifolds $\mathbb{H}^3 / \operatorname{Ker}(\varphi_1)$ and $\mathbb{H}^3 / \operatorname{Ker}(\varphi_2)$ are homeomorphic if and only if 4-colorings are symmetric under the group Σ action.

When two 4-colorings induce homeomorphic manifolds?

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Example. Let Rn be the Löbell polyhedron and $G(n)$ be generated by its reflections.

1. It $n \neq 5, 6, 7, 8, 10, 12, 18$ then group $G(n)$ is non-arithmetic (follows from Takahashi's classification of arithmetic triangle groups);
2. By Mednykh's result, if $n \geq 6$ then $G(n)$ is naturally maximal.

An n-dimensional manifold M^n is said to be hyperelliptic if it has an involution τ such that $M^n/\langle\tau\rangle$ is homeomorphic to S^n.
Such an involution τ is said to be a hyperelliptic involution.

If M^n has a geometric structure, we assume that τ is an isometry.

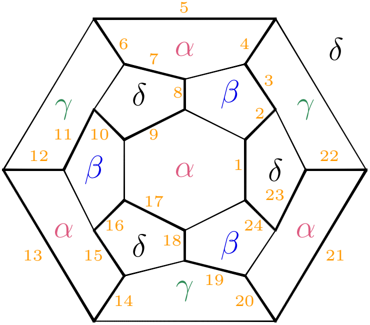
We will discuss the case of hyperbolic 3-manifolds.

Reni – Zimmermann, 2001: If 3-manifold is hyperbolic then it has at most 9 hyperelliptic involutions.

The following result gives a way to construct hyperbolic hyperelliptic 3-manifolds from bounded right-angled polyhedra.

Mednykh – V.: Let P be a bounded right-angled polyhedron in $\mathbb{H}^3$ and $\Delta(P)$ be the group generated by reflections in faces of P. Assume that P is Hamiltonian.
Then there exists a subgroup $\Gamma \triangleleft \Delta(P)$ of index $|\Delta(P) : \Gamma| = 8$ acting on $\mathbb{H}^3$ without fixed points such that the quotient space $M^3 = \mathbb{H}^3 / \Gamma$ is a hyperelliptic manifold.

Consider a Hamiltonian cycle $\mathcal{C}$ on the Löbell polyhedron $R6$ and enumerate its edges: $1, 2, \ldots, 24$. The closed curve $\mathcal{C}$ divides the surface $R6$ into two regions. Let us color faces of the first region by colours α and β, and faces of the second region by colors γ and δ.



colors of faces:
$\alpha = (1,0,0)$, $\beta = (0,1,0)$,
$\gamma = (0,0,1)$, $\delta = (1,1,1)$;
colors of edges:
$a = \alpha + \gamma = \beta + \delta = (1,0,1)$
$b = \alpha + \delta = \beta + \gamma = (0,1,1)$
$c = \alpha + \beta = \gamma + \delta = (1,1,0)$

Colorings of faces and edges induce epimorphisms

$$\varphi : \Delta(P) \quad \longrightarrow \quad \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle \alpha, \beta, \gamma, \delta \rangle$$

and

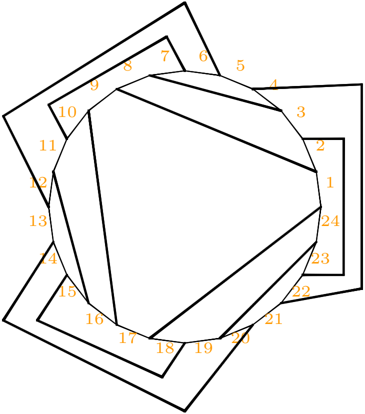
$$\psi : \Delta^+(P) \quad \longrightarrow \quad \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle a, b, c \rangle.$$

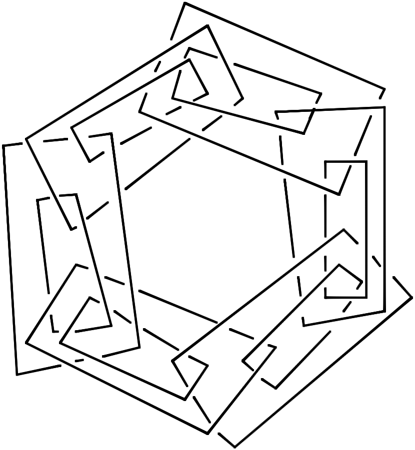
Denote $\Gamma = \mathrm{Ker}\, \varphi = \mathrm{Ker}\, \psi$ and $\Gamma_1 = \psi^{-1}(c)$, where the color "c" is associated with edges not belonging to the Hamiltonian cycle. Then

$$M = \mathbb{H}^3/\Gamma \quad \xrightarrow{2} \quad \mathbb{H}^3/\Gamma_1 \quad \xrightarrow{2} \quad \mathbb{H}^3/\Delta^+(P) \quad \xrightarrow{2} \quad \mathbb{H}^3/\Delta(P).$$

The quotient space $\mathbb{H}^3/\Gamma_1$ is homeomorphic to S^3. Thus, M is a hyperelliptic manifold.

To describe branching set let us redraw the Hamiltonian cycle as a circle.
M is the 2-fold branched covering of S^3 branched over the 12-component link formed by preimage of edges of color "c".





Thank you!