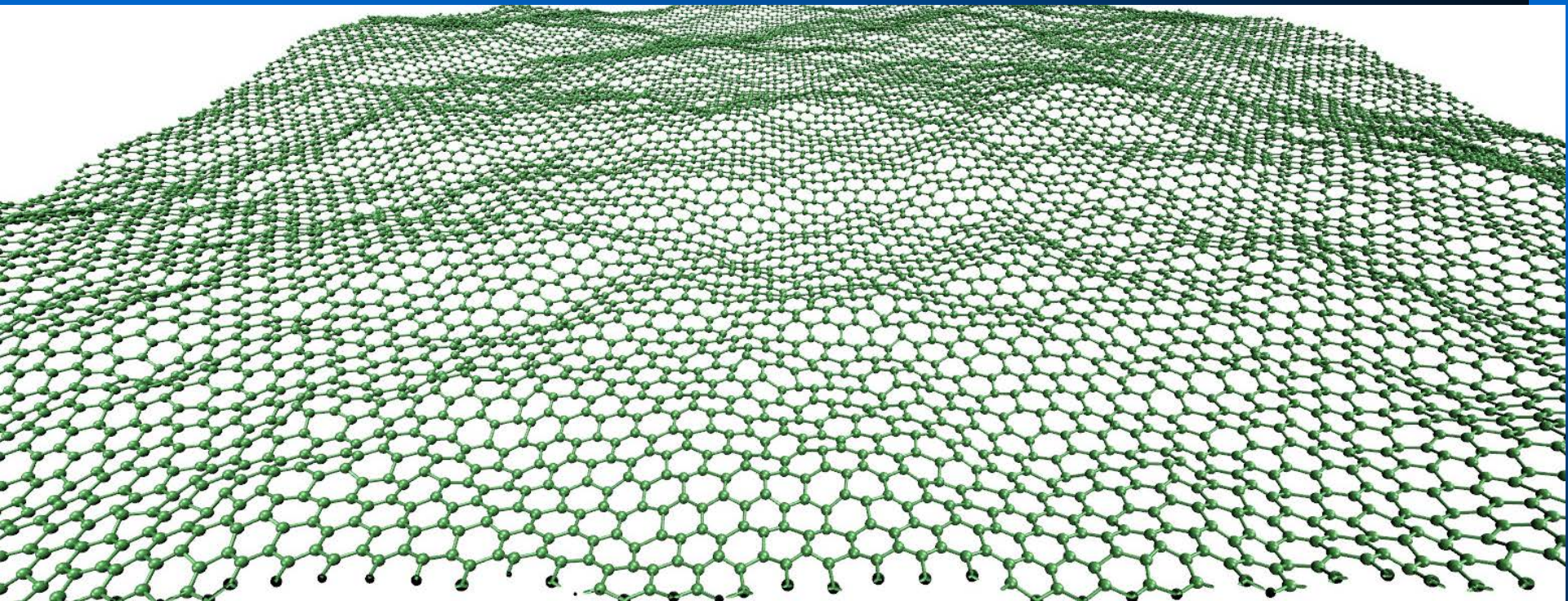
Radboud Universiteit Nijmegen

IN·DEI·NOMINE·FELICITER

IMM
Institute for Molecules and Materials

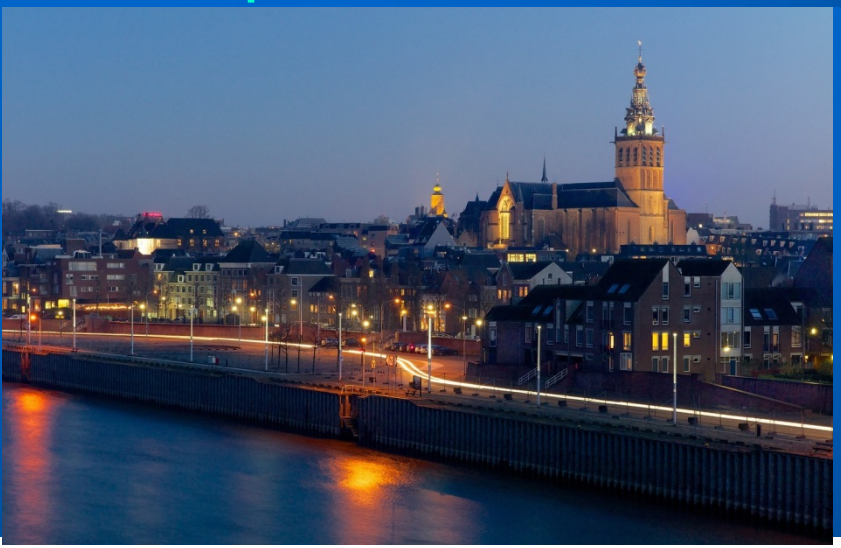
# *Theory of graphene: CERN on the desk*

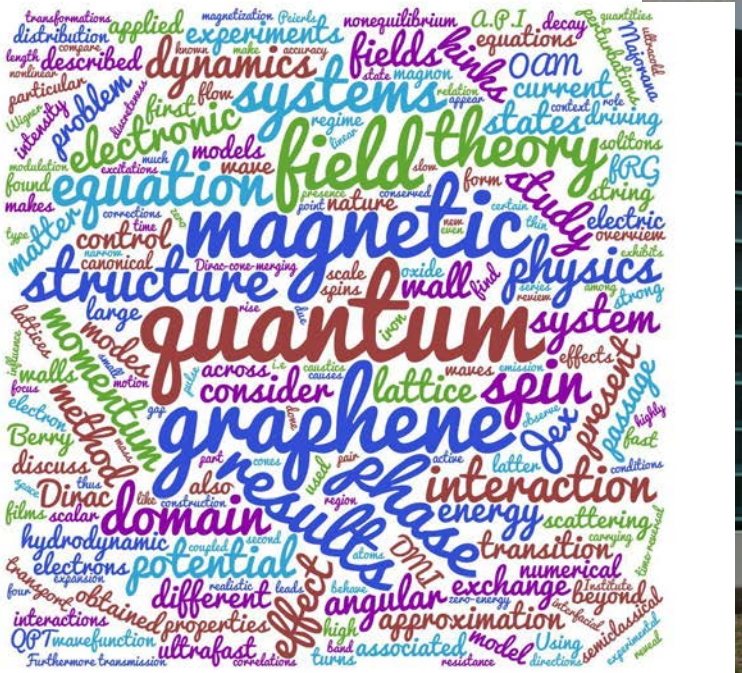
Mikhail Katsnelson



# *Theory of condensed matter group*

http://www.ru.nl/tcm/






# *Instead of epigraph*

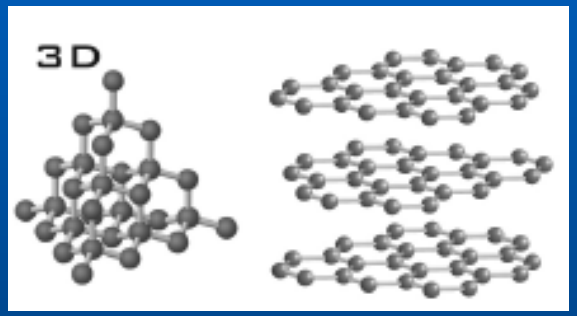
You can get much further with a *kind word* and a *gun* than you can with a *kind word* alone
(Al Capone)

You can get much further with an insight from experiment and mathematics than you can with mathematics alone

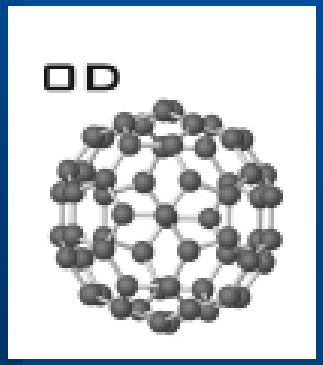
# *Carbon, an elemental solid*

Diamond

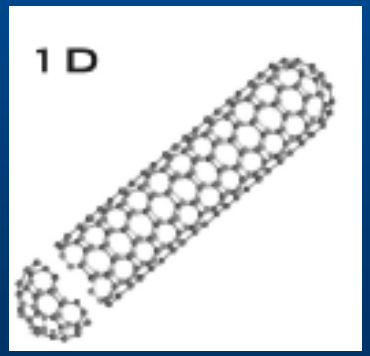
Graphite



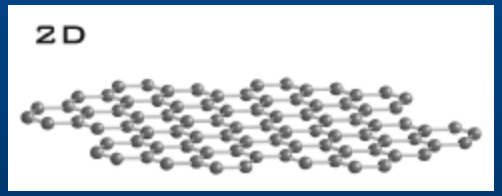

Crystal lattices




Fullerenes

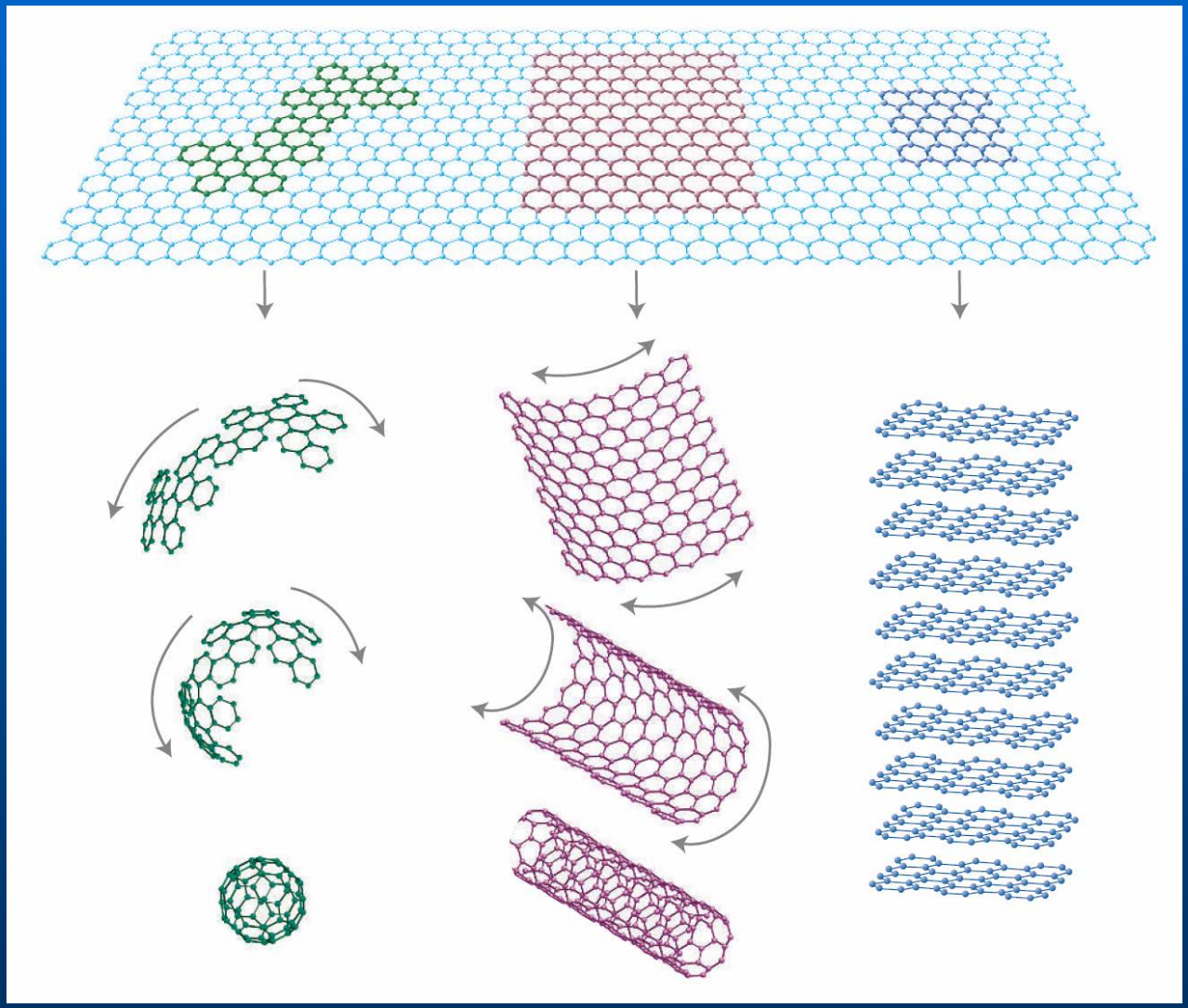



Nanotubes




Graphene

# *Mother of all graphitic forms*



Fullerenes Nanotubes Graphite

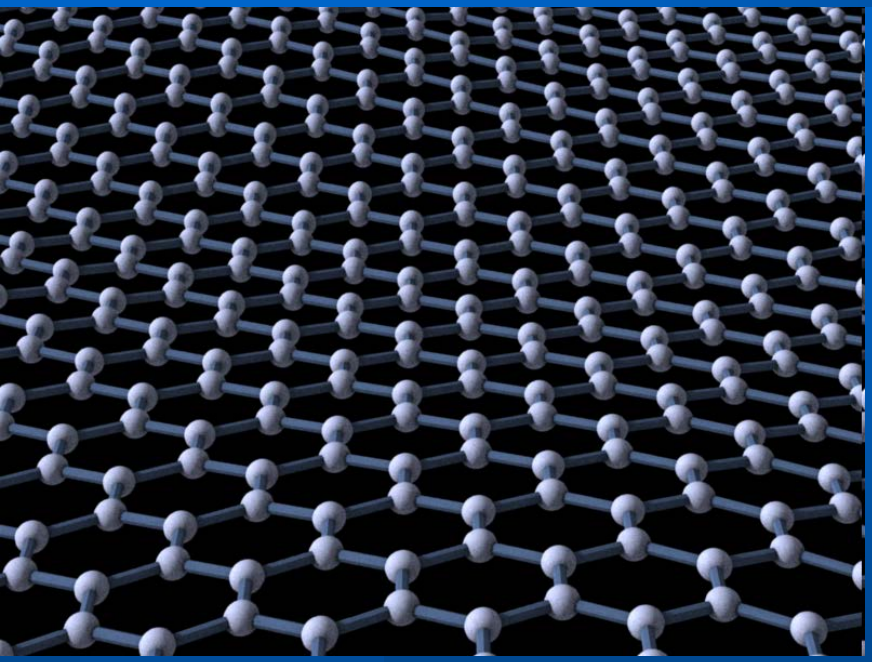
# *Why graphene is interesting?*

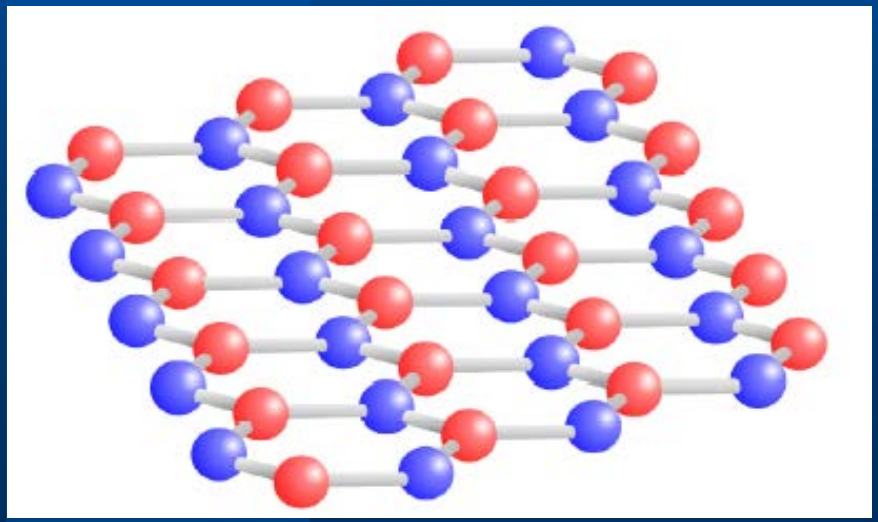
Till 2004: a way to understand graphite, nanotubes, fullerenes + theoretical interest
(Dirac point Wallace 1947, McClure 1956…)

Do we theoreticians need experimentalists?! – Yes!!!
(Klein tunneling, supercritical charge, ripples, new wave equation – bilayer, new type of transport…)

1. *Applications (modern electronics is 2D, bulk is ballast)*
2. *Prototype membrane (new drosophila for 2D statistical mechanics)*
3. *CERN on the desk (mimic high energy physics)*

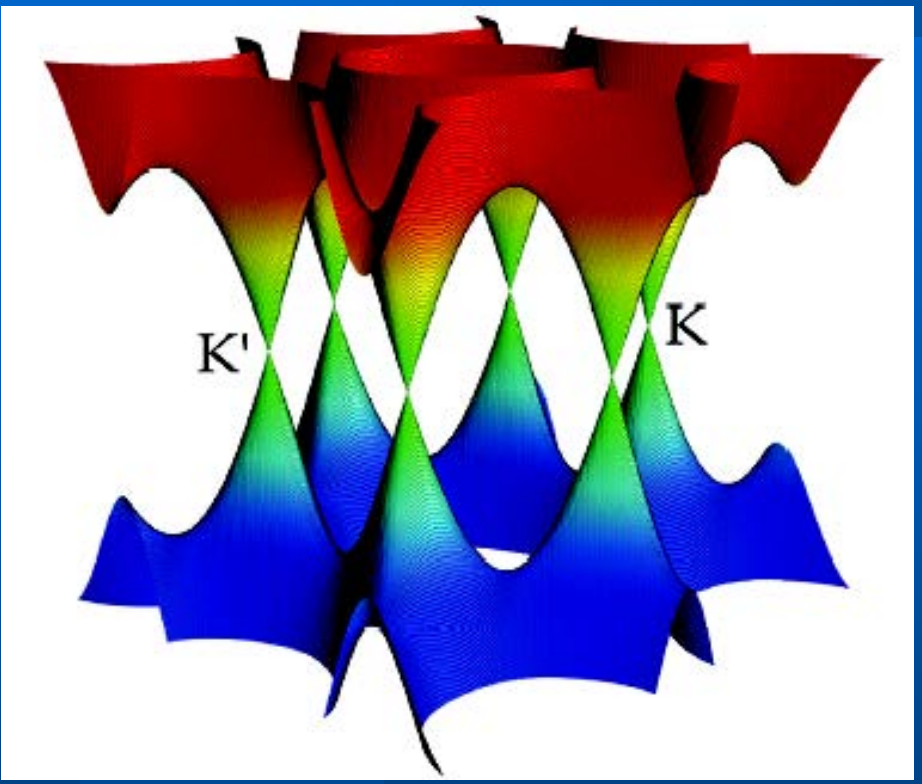
# *Honeycomb lattice*





Two equivalent sublattices, A and B (pseudospin)

# Massless Dirac fermions




*$sp^2$ hybridization, π bands crossing the neutrality point*

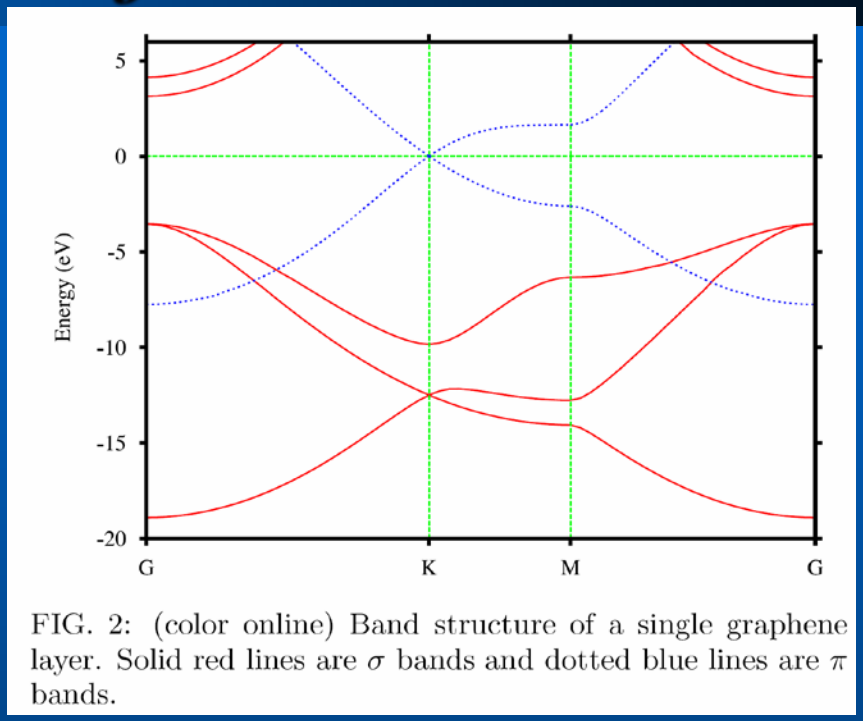
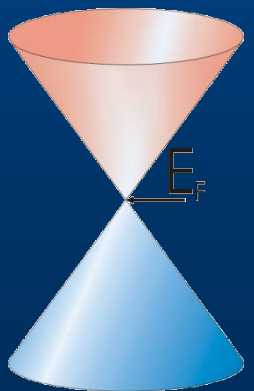



FIG. 2: (color online) Band structure of a single graphene layer. Solid red lines are $\sigma$ bands and dotted blue lines are $\pi$ bands.

Neglecting intervalley scattering: massless Dirac fermions

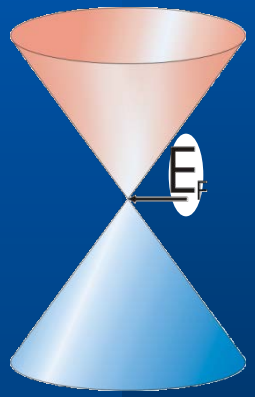
Massless relativistic particles (light cones)



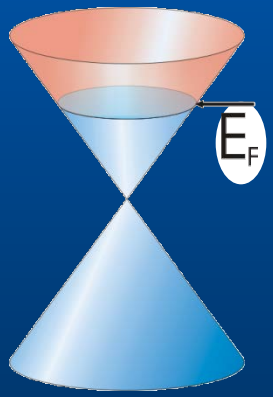

Symmetry protected (*T* and *I*)

# *Massless Dirac fermions II*

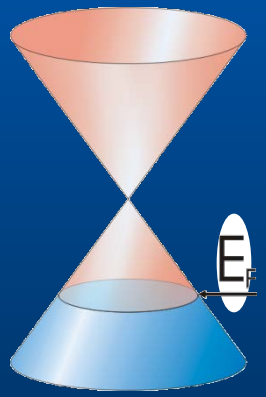
**Spectrum near *K* (*K'*) points is linear.**
**Conical cross-points: provided by symmetry and thus robust property**




Undoped




Electron




Hole

# *Massless Dirac fermions III*

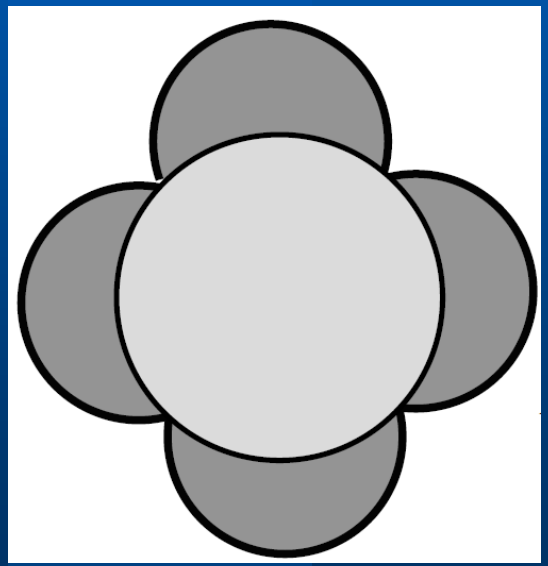
If Umklapp-processes K-K’ are neglected:
2D Dirac massless fermions with the Hamiltonian

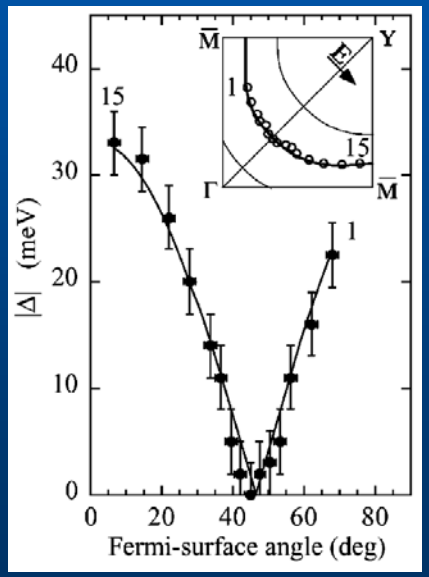
$$H = -i\hbar c^* \begin{pmatrix} 0 & \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} & 0 \end{pmatrix} \qquad \hbar c^* = \frac{\sqrt{3}}{2}\gamma_0 a$$

“Spin indices’’ label sublattices *A* and *B*
rather than real spin

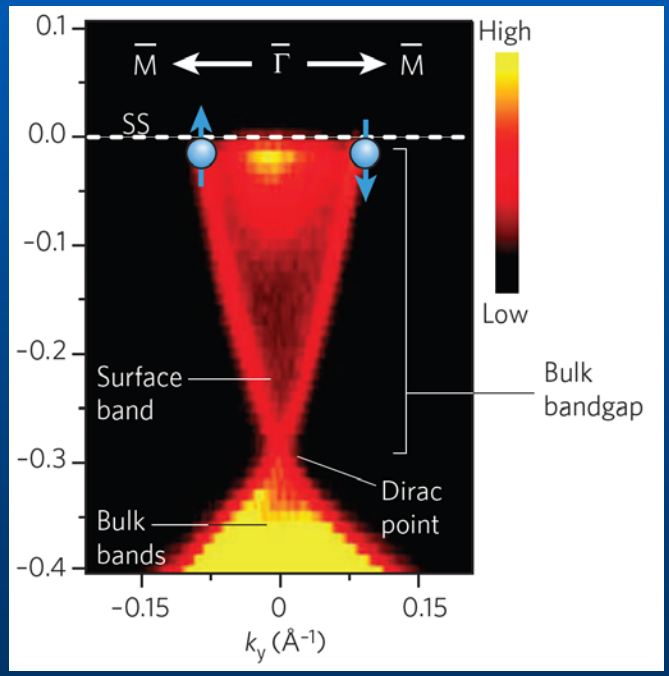
# *Massless Dirac fermions in condensed matter physics*

1. d-wave superconductors
2. Vortices in superconductors and in superfluid helium-3
3. Topological insulators
4. Graphene



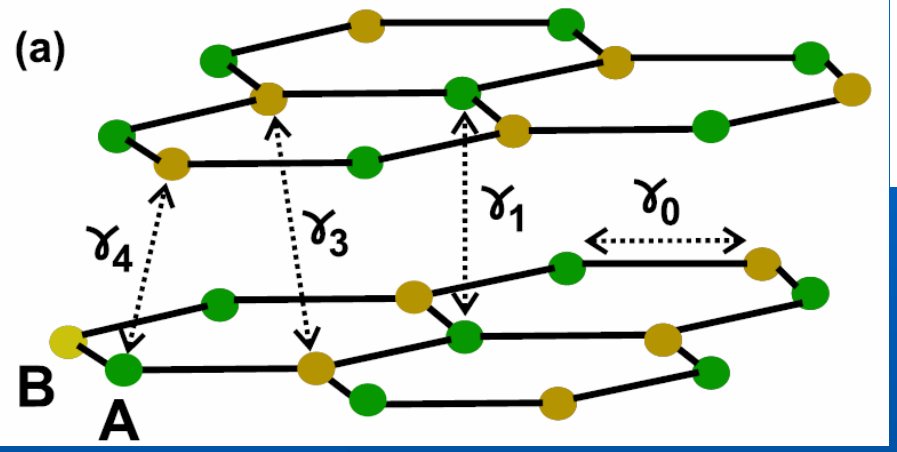



Gap in high-Tc cuprates




Electronic structure on surface of $Bi_2Se_3$

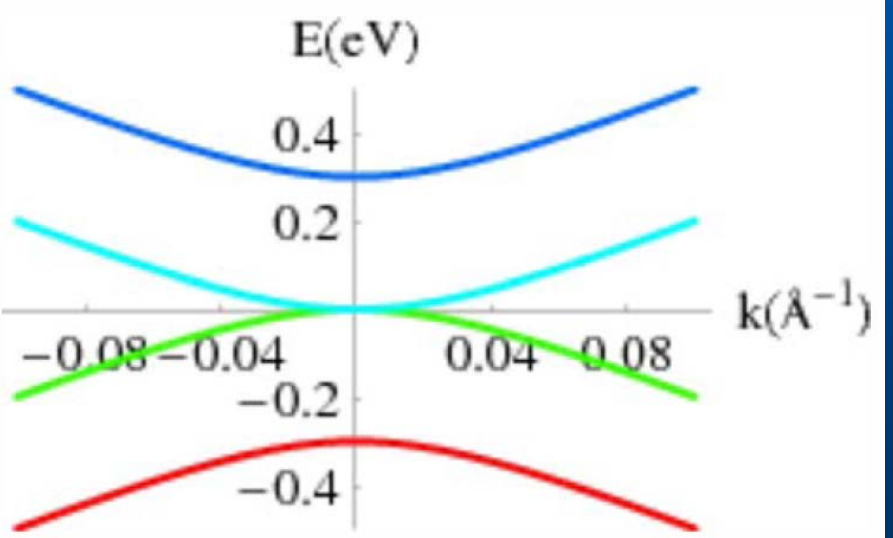
# Bilayer graphene – TB description

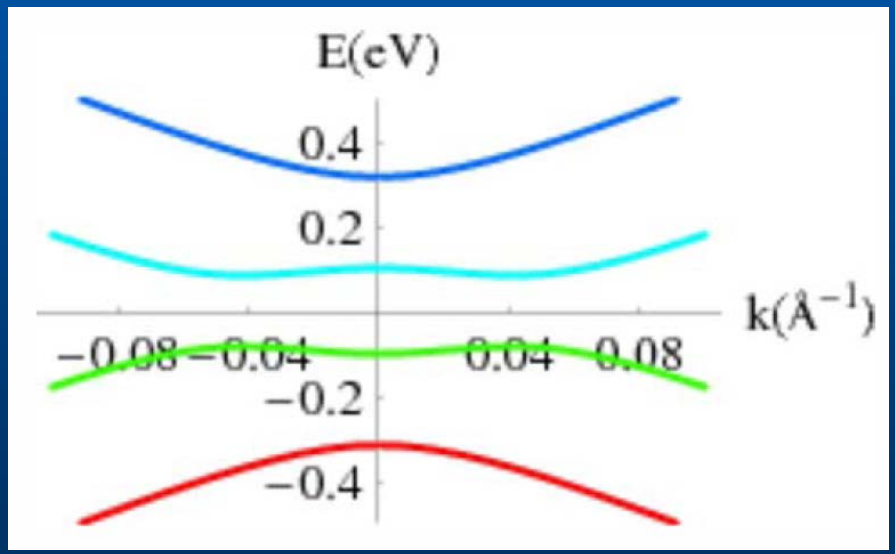



$$H = \begin{pmatrix} 0 & -(p_x - ip_y)^2/2m \\ -(p_x + ip_y)^2/2m & 0 \end{pmatrix}$$

$$m^* \approx 0.028 m_e$$

(neglecting $\gamma_3$)




Gapless, parabolic

Electric field perp. layers

# Bilayer graphene II

Trigonal warping, many-body effects and spectrum reconstruction at small energies

Single-particle Hamiltonian:

$$\hat{H}_{\mathrm{K}} = \begin{pmatrix} 0 & \dfrac{\left(\hat{p}_x - i\hat{p}_y\right)^2}{2m^*} + \dfrac{3\gamma_3 a}{\hbar}\left(\hat{p}_x + i\hat{p}_y\right) \\ \dfrac{\left(\hat{p}_x + i\hat{p}_y\right)^2}{2m^*} + \dfrac{3\gamma_3 a}{\hbar}\left(\hat{p}_x - i\hat{p}_y\right) & 0 \end{pmatrix}$$

**Interaction-Driven Spectrum Reconstruction in Bilayer Graphene**

A. S. Mayorov,[1] D. C. Elias,[1] M. Mucha-Kruczynski,[2] R. V. Gorbachev,[3] T. Tudorovskiy,[4] A. Zhukov,[3] S. V. Morozov,[5] M. I. Katsnelson,[4] V. I. Fal'ko,[2] A. K. Geim,[3] K. S. Novoselov[1*]

12 AUGUST 2011 VOL 333 SCIENCE

# *Outline*

*Graphene in magnetic field: a physical realization of index theorem*

*Minimal conductivity problem and transport via evanescent waves*

*Klein tunneling and inhomogeneities*

*Chiral tunneling - bilayer*

*Relativistic collapse for supercritical charges*

*Psedomagnetic gauge field and “strain engineering”*

# Effective Hamiltonian in magnetic field: Peierls

$$H = \frac{\hat{\vec{\pi}}^2}{2m} + V(\vec{r})$$

$$\hat{\vec{\pi}} = \hat{\vec{p}} - \frac{e}{c}\vec{A},$$

$$\vec{p} = -i\hbar\vec{\nabla}$$

$$\vec{B} = \vec{\nabla} \times \vec{A},$$

$V(\vec{r})$ is a periodic crystal potential

$$\left[\hat{\pi}_x, \hat{\pi}_y\right] = -\left[\hat{\pi}_y, \hat{\pi}_x\right] = \frac{ie}{\hbar c} B$$

Exact formulation

Simplifications:

$$l_B >> a$$

up to 10,000 T – OK!

$$l_B = \sqrt{\frac{\hbar c}{|e|B}}$$

magnetic length

# Effective Hamiltonian II

Single band model (neglecting magnetic breakdown) – fine for $\pi$-band in graphene, no restrictions, actually

$$\hat{H}_{ef} = t\left(\hat{\vec{\pi}}\right)$$

$$t\left(\hat{\vec{\pi}}\right)\psi = E\psi$$

Bosonic operators

$$\hat{\pi}_{-} = \hat{\pi}_x - i\hat{\pi}_y = \sqrt{\frac{2|e|\hbar B}{c}}\hat{b},$$

$$\hat{\pi}_{+} = \hat{\pi}_x + i\hat{\pi}_y = \sqrt{\frac{2|e|\hbar B}{c}}\hat{b}^{+}$$

$$\left[\hat{b}, \hat{b}^{+}\right] = 1$$

Dirac electrons

$$\hat{H} = v\begin{pmatrix} 0 & \hat{\pi}_{-} \\ \hat{\pi}_{+} & 0 \end{pmatrix} = \sqrt{\frac{2|e|\hbar B v^2}{c}}\begin{pmatrix} 0 & \hat{b} \\ \hat{b}^{+} & 0 \end{pmatrix}$$

# Solution

$$\hat{b}\psi_2 = \varepsilon\psi_1$$
$$\hat{b}^+\psi_1 = \varepsilon\psi_2$$

$$E = \sqrt{\frac{2|e|\hbar B v^2}{c}}\varepsilon \equiv \frac{\sqrt{2\hbar}v}{l_B}\varepsilon$$

Zero-energy solution: $\psi_1 = 0$ $\psi_2 \equiv |0\rangle$ $b|0\rangle = 0$

Complete spectrum:

$$\hat{b}^+\hat{b}\psi_2 = \varepsilon^2\psi_2$$

$$\varepsilon_\nu{}^2 = \nu = 0,1,2,\ldots$$

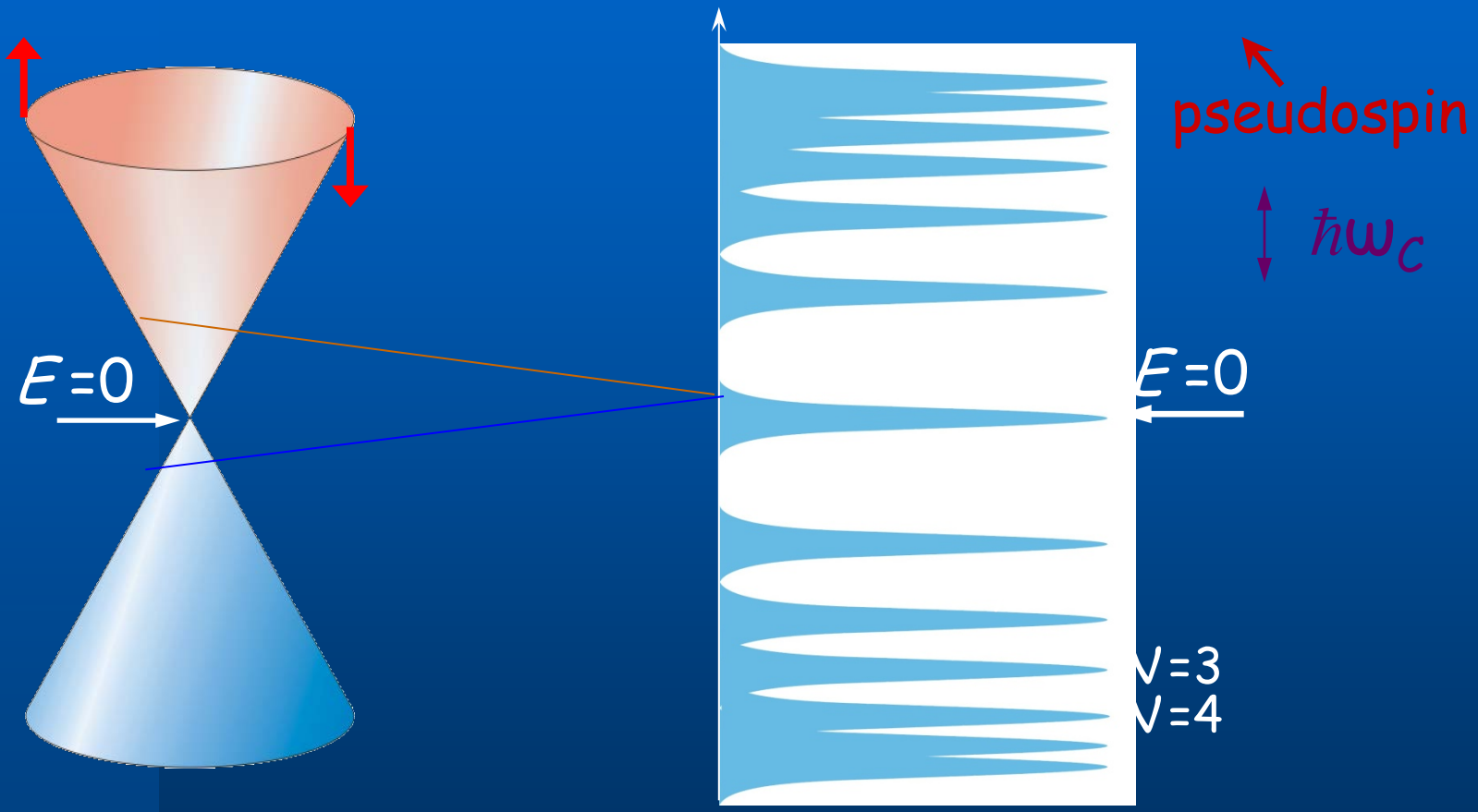
$$E_\nu{}^{(\pm)} = \pm\hbar\omega_c\sqrt{\nu}$$

$$\hbar\omega_c = \frac{\sqrt{2\hbar}v}{l_B} = \sqrt{\frac{2\hbar|e|Bv^2}{c}}$$

# Anomalous Quantum Hall Effect

$E = \hbar v k$

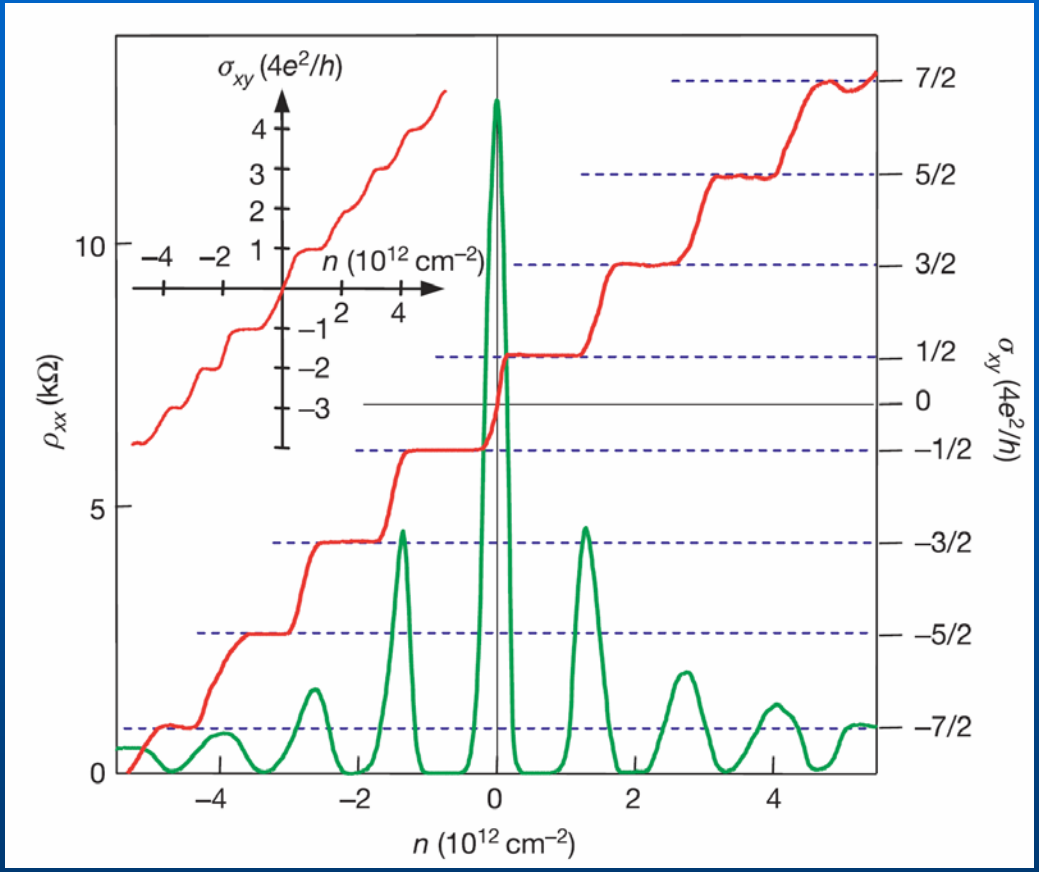
$E_N = [2e\hbar v^2 B(N + \frac{1}{2} \pm \frac{1}{2})]^{1/2}$




The lowest Landau level is *at ZERO energy* and shared equally by electrons and holes

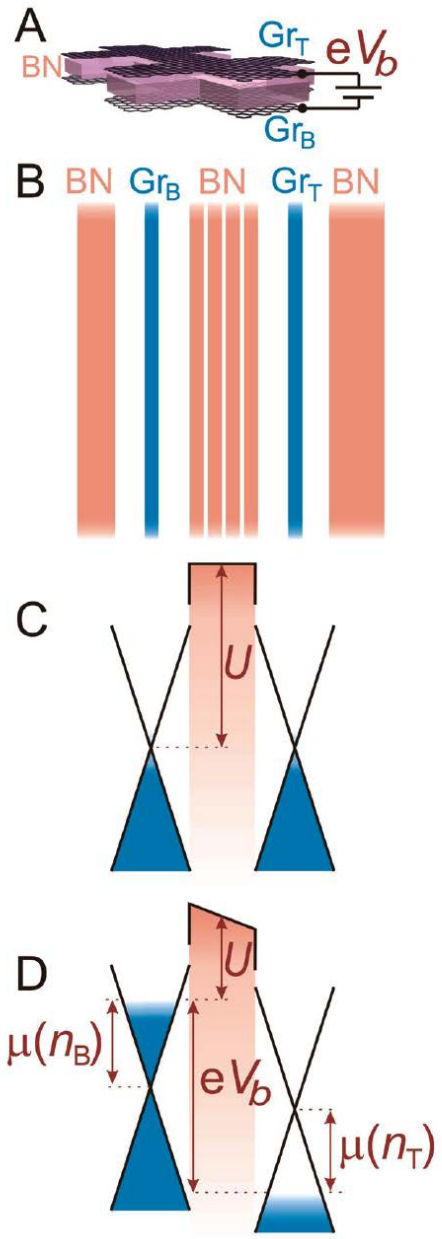
(McClure 1956)

# Anomalous QHE in single-layer graphene




**Single-layer: half-integer quantization since zero-energy Landau level is equally shared by electrons and holes (Novoselov et al 2005, Zhang et al 2005)**

# Quantum capacitance




$$eV_b = eF_b d - \mu(n_T) + \mu(n_B)$$

Assuming for simplicity $\mu(-n) = -\mu(n)$

$$\frac{e^2 d}{\varepsilon_0 \varepsilon_r} n_T + \mu(n_T) + \mu\left(n_T + \frac{\varepsilon_0 \varepsilon_r F_g}{e}\right) + eV_b$$

For small bias:

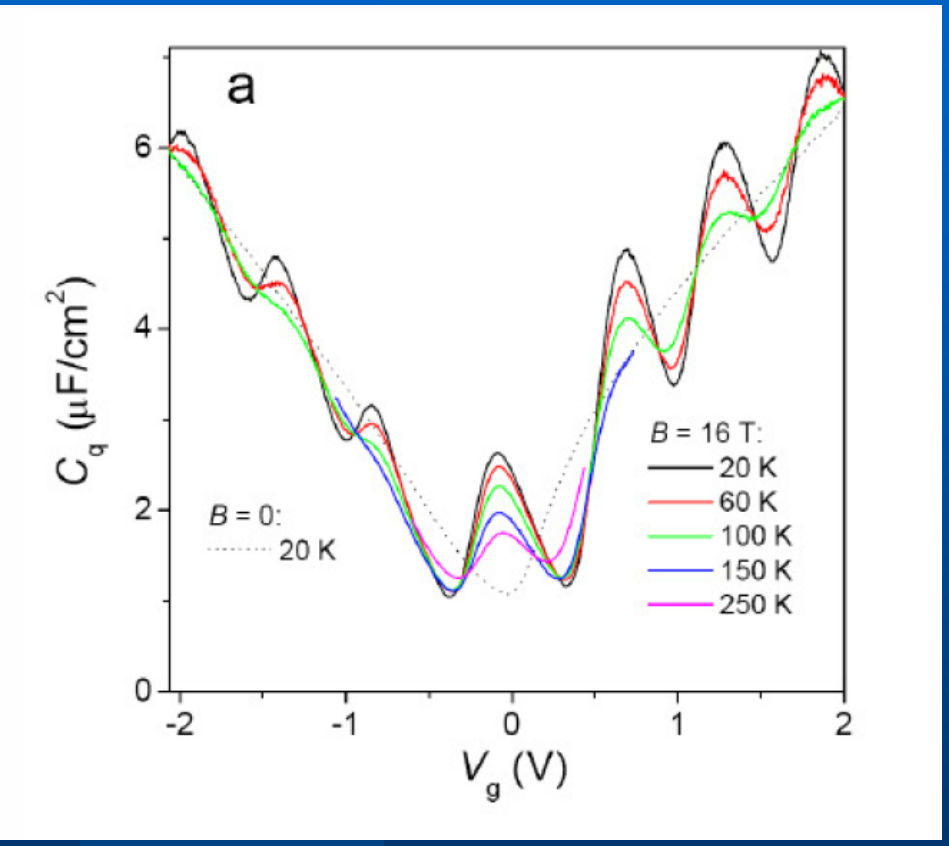
$$1/C = 1/C_G + 1/C_Q,$$

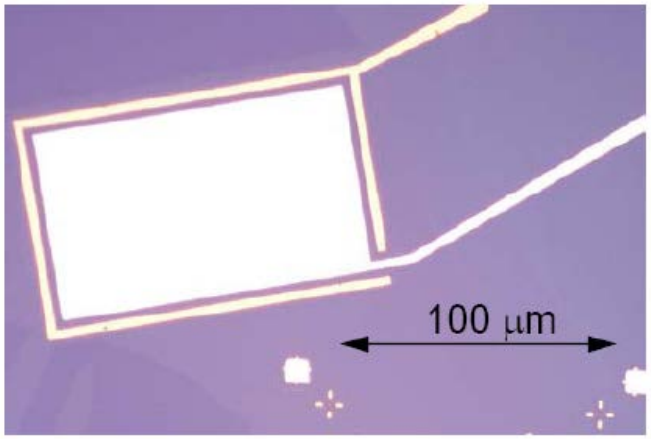
G means “geometrical”

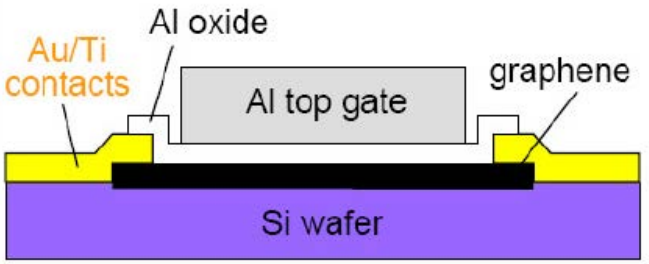
$$C_Q = Se^2\, dn/d\mu.$$

# Quantum capacitance II

Ponomarenko et al, PRL 2010



# Quantum capacitance III

Yu et al, PNAS 2013

Graphene on hBN: one can see many-body effects etc.

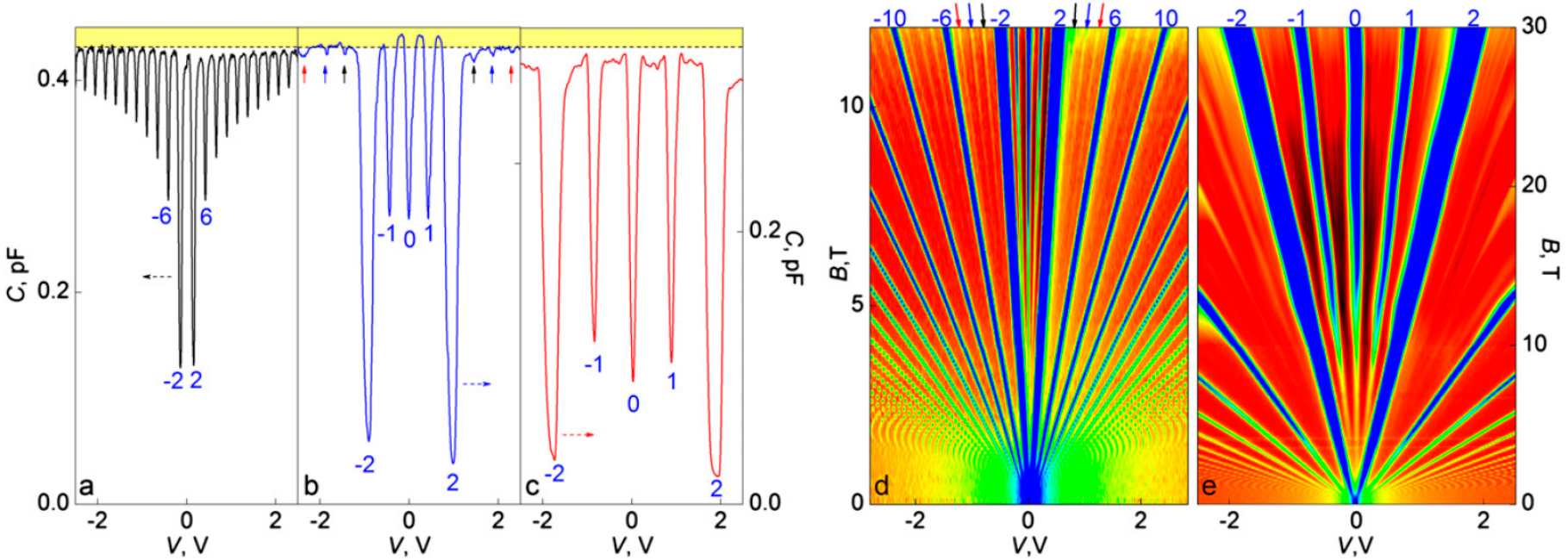



**Fig. 3.** Graphene capacitors in quantizing fields. (*A*–*C*) Examples of graphene capacitance for $B = 3$, 15, and 30 T, respectively. Blue numbers are $\nu$ for the corresponding minima. The arrows in *B* mark $\nu = \pm 3$ (black arrows), $\pm 4$ (blue) and $\pm 5$ (red). The dashed line indicates $C_G = 0.433 \pm 0.002$ pF (for a device shown in *A*) and $0.335 \pm 0.002$ pF (for a device shown in *B* and *C*). The 15-T curve reveals that the total capacitance becomes higher than the geometrical one at $\nu$ around $\pm 1/2$ and $\pm 3/2$, indicating a negative contribution from $C_Q$, that is, negative compressibility. (*D*) Two-dimensional map of differential capacitance as a function of *B* and *V* for the same device as in *A*. Color scale is 0.37 pF to $C_G$ (blue to green to red) and $C_G$ to 0.454 pF (red to black). Numbers and arrows are as in *B*. (*E*) Two-dimensional map in *B* up to 30 T for the device in *B* and *C*. Scale: 0.23 pF to $C_G$ (blue to green to red) and $C_G$ to 0.349 pF (red to black). The dark regions (*D* and *E*) correspond to negative $C_Q$.

# Semiclassics

$$S(E_\nu) = \frac{2\pi|e|B}{\hbar c}\left(\nu + \frac{1}{2} - \frac{\gamma}{2\pi}\right)$$

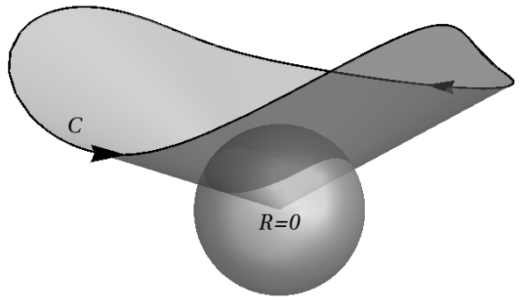
$\nu = 0,1,2\ldots$ $\gamma$ is the Berry phase

$$\gamma_n(C) = -\int d\vec{S}\vec{V}_n(\vec{k})$$

$$\vec{V}_n = \mathrm{Im}\sum_{m\neq n}\frac{\langle n|\vec{\nabla}_{\vec{k}}\hat{H}_{ef}|m\rangle\times\langle m|\vec{\nabla}_{\vec{k}}\hat{H}_{ef}|n\rangle}{(E_m - E_n)^2}$$

$$\hat{H}_{ef} = \frac{1}{2}\vec{R}(\vec{k})\vec{\sigma}$$

$$\gamma_\pm(C) = \mp\frac{1}{2}\Omega(C)$$




2D Dirac: $\gamma = \pi$

# *Half-integer quantum Hall effect and "index theorem"*

Atiyah-Singer index theorem: number of chiral modes with zero energy for massless Dirac fermions with gauge fields

Simplest case: 2D, electromagnetic field

$$N_+ - N_- = \phi / \phi_0$$

(magnetic flux in units of the flux quantum)

Magnetic field can be inhomogeneous!!!

# *Anomalous QHE for bilayer graphene*

Low-energy description:
***Massive*** chiral fermions

$$H = \begin{pmatrix} 0 & -(p_x - ip_y)^2/2m \\ -(p_x + ip_y)^2/2m & 0 \end{pmatrix}$$

Homogeneous magnetic Field ($v$ =0,1,2…)

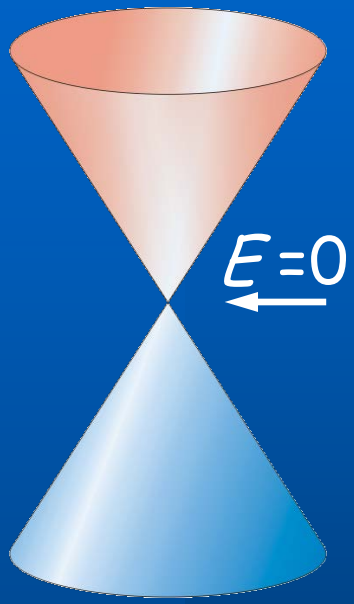
$$E_\nu \propto \sqrt{\nu(\nu - 1)}$$

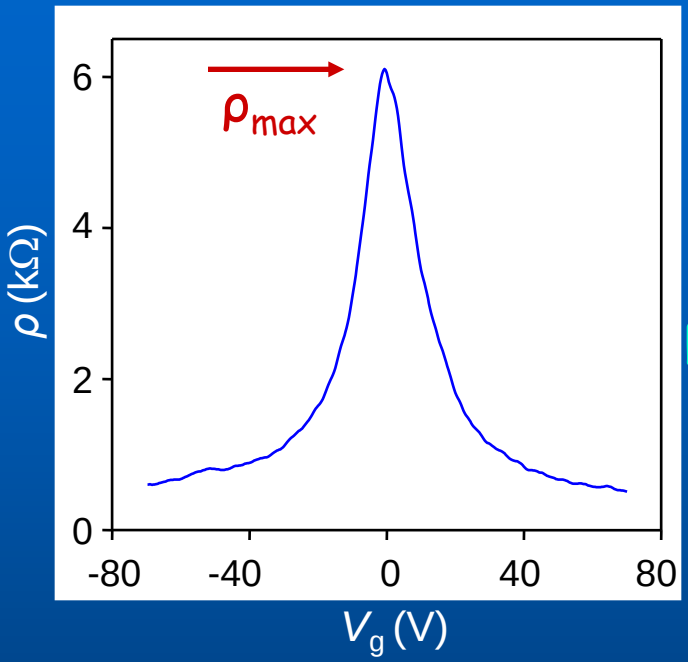
**Index theorem for bilayer graphene (MIK and M. Prokhorova, PR B 2008)**

$$N_+ - N_- = 2\phi / \phi_0$$

**Anomalous QHE in inhomogeneous field!**

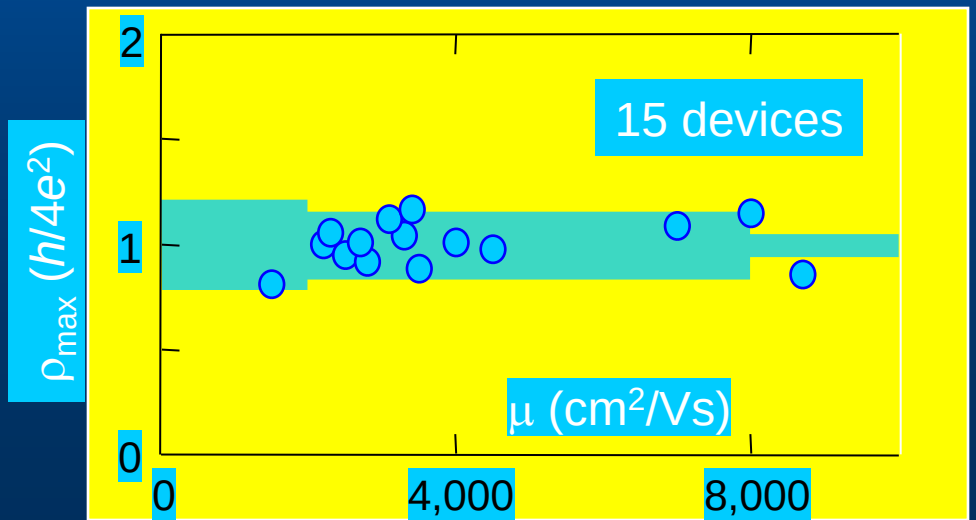
# Quantum-Limited Resistivity




no temperature dependence in the peak between 3 and 80K

zero-gap semiconductor




Novoselov et al, Nature 2005

# *Transport via evanescent waves*

MIK, EPJ B 51, 157 (2006)

Conductance = $e^2/h$ *Tr T* per valley per spin

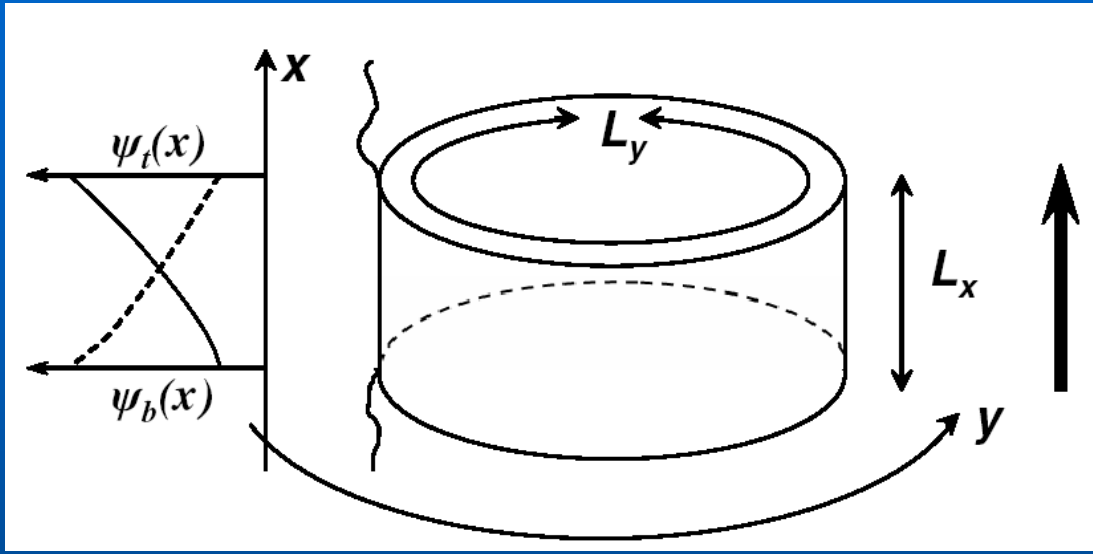
*T* is the transmission probability matrix

The wave functions of massless Dirac fermions at zero energy:

$$\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)\psi_{\pm}(x,y) = 0 \qquad \psi_{\pm}(x,y) = f(x \pm iy) \qquad \forall f$$

Boundary conditions determine the functions *f*

# *Transport via evanescent waves II*




$$\psi_{\pm}(x,y) = \exp\left[\frac{2\pi n}{L_y}(x \pm iy)\right] \qquad n = 0, \pm 1, \pm 2, \ldots$$

$f(y+L_y) = f(y)$ Edge states near the top and bottom of the sample

New type of electron transport: via evanescent waves – different from both ballistic and diffusive

# *Transport via evanescent waves III*

Leads from doped graphene

$$T_n = |t(k_y)|^2 = \frac{\cos^2\phi}{\cosh^2(k_y L_x) - \sin^2\phi}$$

$$\sin\phi = k_y / k_F$$

$$TrT = \sum_{n=-\infty}^{\infty} \frac{1}{\cosh^2(k_y L_x)} \simeq \frac{L_y}{\pi L_x}$$

Conductivity per channel: $e^2/(\pi h)$

Seems to be in agreement with the latest experiments

# Transport via evanescent waves IV

Recent experiment

In very clean samples min conductivity is close to theor. prediction

**How Close Can One Approach the Dirac Point in Graphene Experimentally?** *Nano Lett.* 2012, 12, 4629–4634

Alexander S. Mayorov,*,† Daniel C. Elias,† Ivan S. Mukhin,‡ Sergey V. Morozov,§ Leonid A. Ponomarenko,† Kostya S. Novoselov,† A. K. Geim,†,‡ and Roman V. Gorbachev†,‡

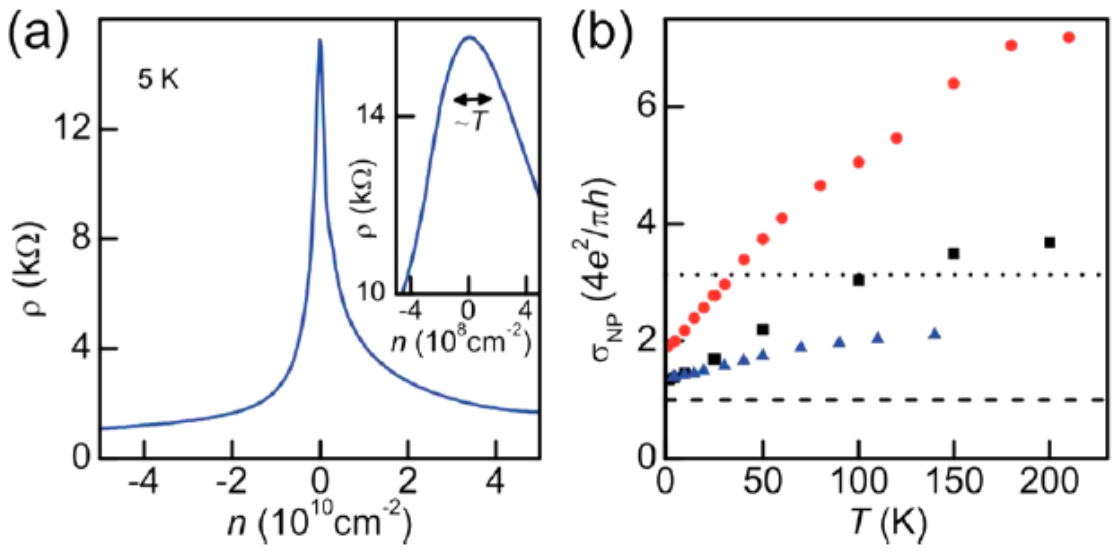


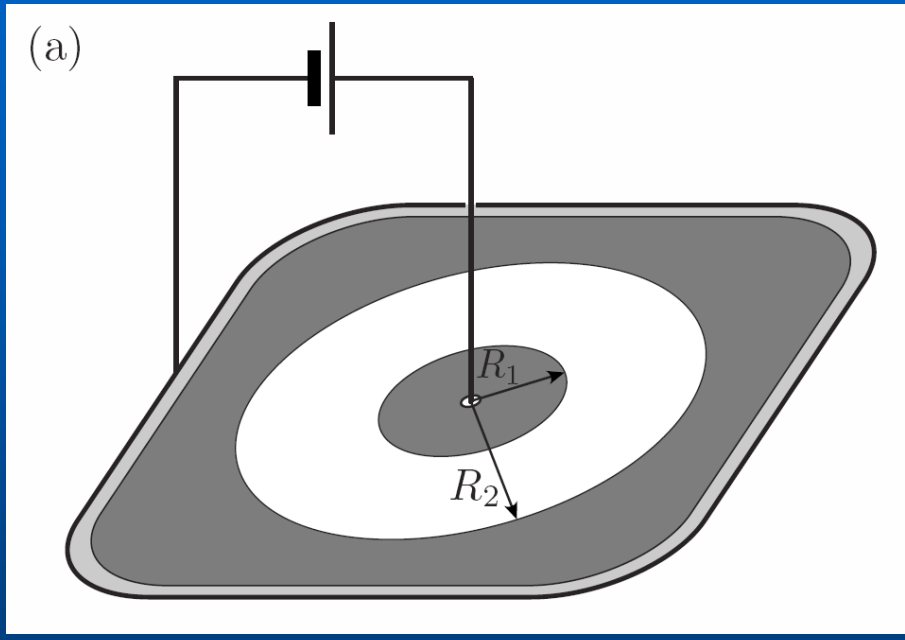

**Figure 3.** Height and broadening of the resistivity peak. (a) $\rho(n)$ for one of our devices. Inset: thermal smearing of the tip. (b) $T$ dependence of $\sigma$ at the NP for three devices. The dashed line indicates the ballistic limit; the dotted one is $4e^2/h$. For the sake of generality, no contact resistance is subtracted, which would result in slightly higher values of $\sigma_{NP}$.

# *Other geometries by conformal mapping*

(MIK and Guinea, PR B 2008; Rycerz, Recher and Wimmer, PR B 2010)

$$T_j = \frac{1}{\cosh^2[gj \ln \Lambda\{z(w)\}]}$$

$$j = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$$




Λ transforms the region to circular ring (Corbino geometry)

$$G_{\text{diff}} = \frac{2\pi\sigma_0}{\ln(R_2/R_1)}$$

# *Aharonov-Bohm effect at zero doping*

MIK, EPL 89, 17001 (2009)

Magnetic flux tube within the ring

$$G = G_0 \left[ 1 - \frac{4\pi^2}{\ln\left(R_2/R_1\right)} \exp\left( -\frac{\pi^2}{\ln\left(R_2/R_1\right)} \right) \cos\left( \frac{e\Phi}{\hbar c} \right) \right]$$

General shape (topologically equivalent to rings)

$$G = G_0 \left[ 1 - \frac{4\pi^2}{\beta} \exp\left(-\pi^2/\beta\right) \cos\left( \frac{e\Phi}{\hbar c} \right) \right]$$

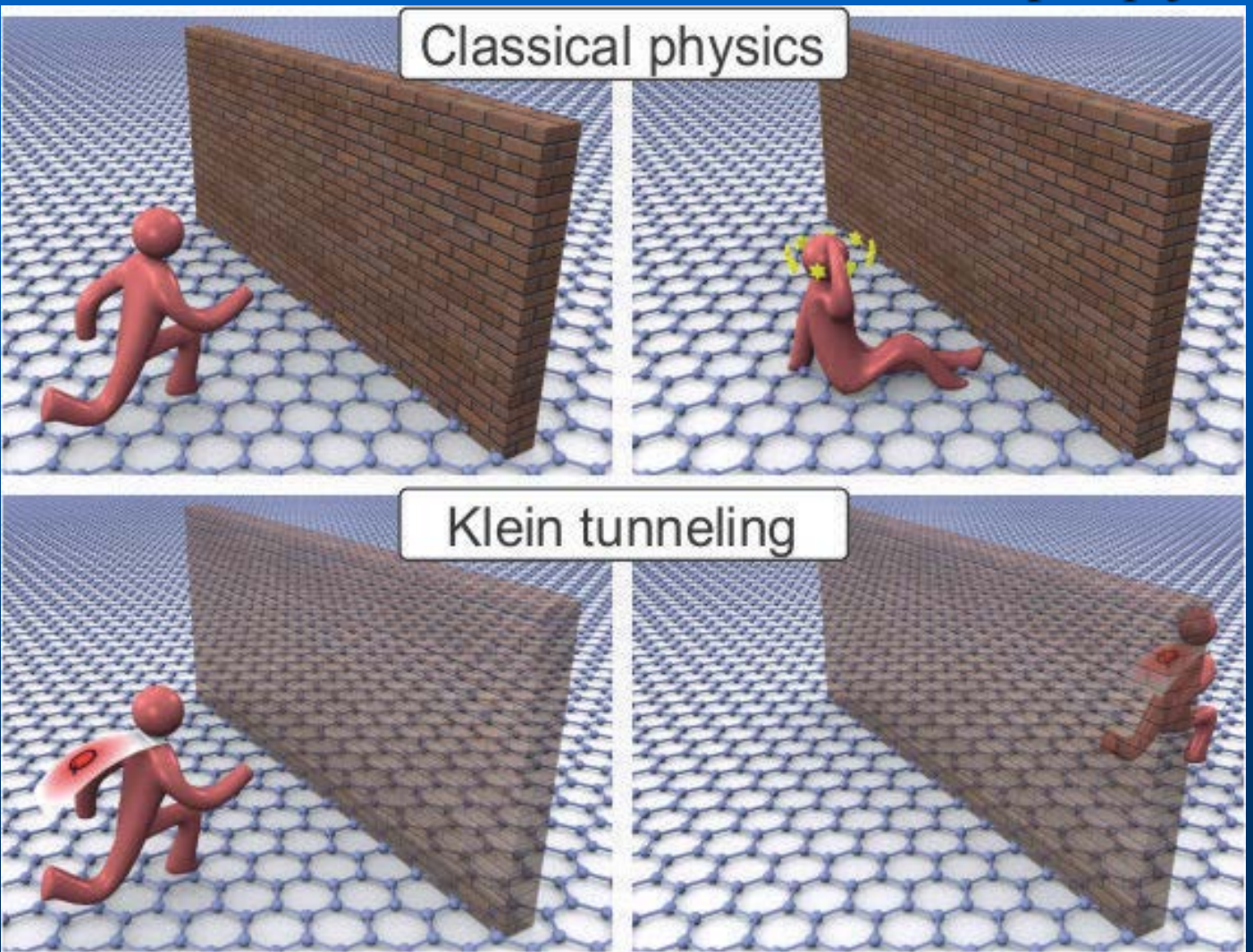
$$\beta = 2e^2/hG_0$$

$$\beta \ll \pi^2$$

# *Chiral tunneling and Klein paradox*

MIK, Novoselov, Geim, Nat. Phys. 2, 620 (2006)

Electronics: heterostructures (*p-n-p* junctions etc.)

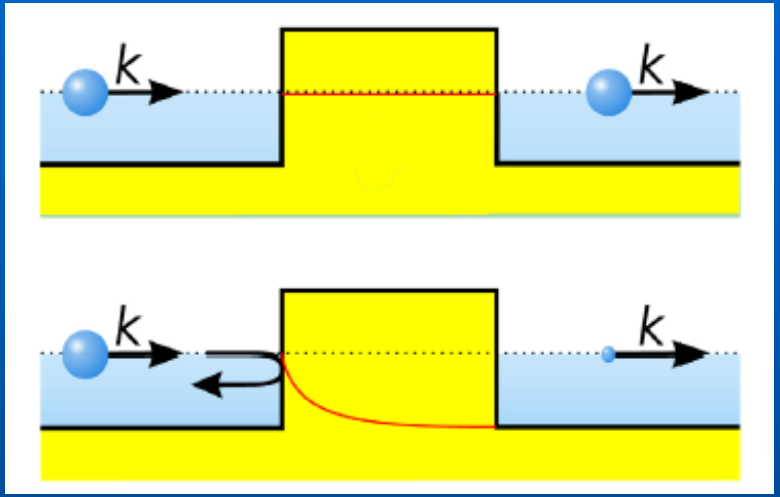



(C) Florian Sterl

# *Klein paradox II*

Ultrarelativisic

Nonrelativistic




Tunnel effect: momentum and coordinate are complementary variables, kinetic and potential energy are not measurable simultaneously

Relativistic case: even the *coordinate itself* is not measurable, particle-antiparticle pair creation

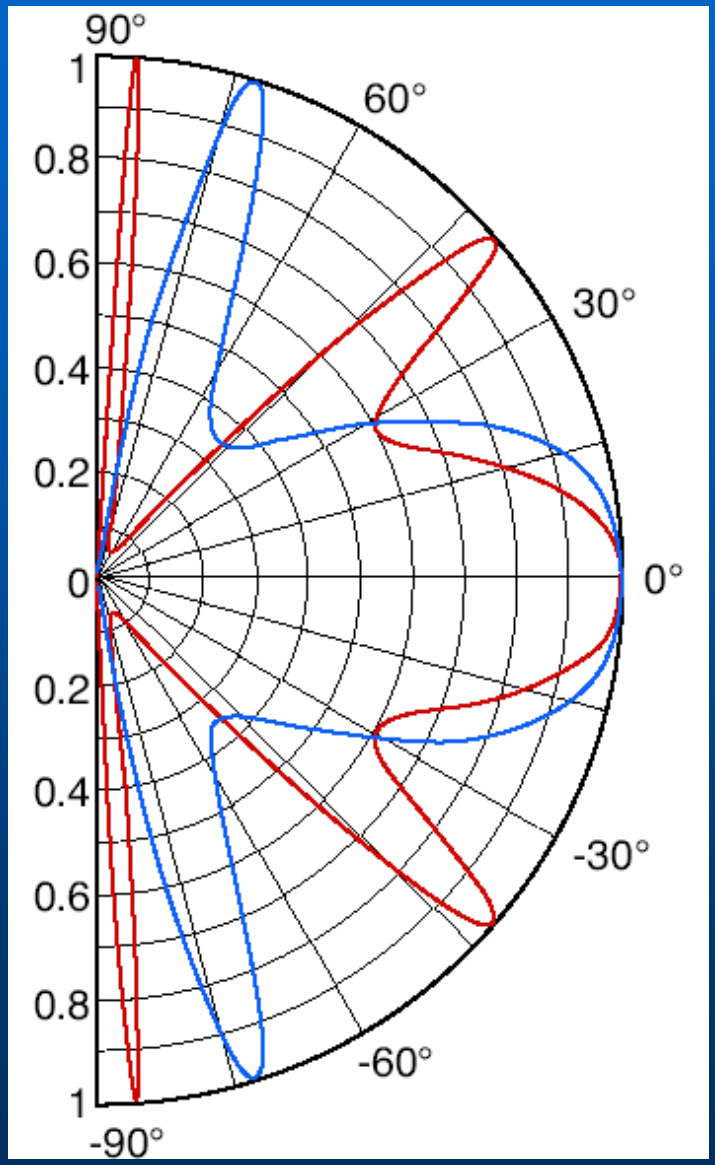
# *Klein paradox III*

Transmission probability

Barrier width 100 *nm*

Electron concentration outside barrier $0.5 \times 10^{12}$ $cm^{-2}$

Hole concentration inside barrier $1 \times 10^{12}$ $cm^{-2}$ (red) and $3 \times 10^{12}$ $cm^{-2}$ (blue)



# Klein tunneling: Experimental confirmation

PRL **102,** 026807 (2009) PHYSICAL REVIEW LETTERS week ending 16 JANUARY 2009

## Evidence for Klein Tunneling in Graphene *p-n* Junctions

N. Stander, B. Huard, and D. Goldhaber-Gordon*

*Department of Physics, Stanford University, Stanford, California 94305, USA*

(Received 13 June 2008; published 16 January 2009)

Transport through potential barriers in graphene is investigated using a set of metallic gates capacitively coupled to graphene to modulate the potential landscape. When a gate-induced potential step is steep enough, disorder becomes less important and the resistance across the step is in quantitative agreement with predictions of Klein tunneling of Dirac fermions up to a small correction. We also perform magnetoresistance measurements at low magnetic fields and compare them to recent predictions.

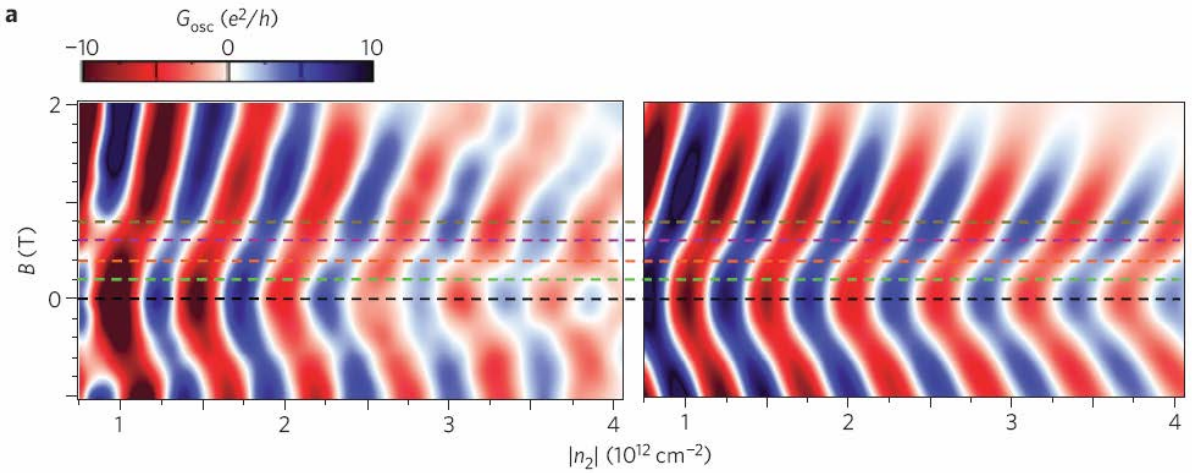
nature physics

LETTERS

PUBLISHED ONLINE: 1 FEBRUARY 2009 | DOI: 10.1038/NPHYS1198

## Quantum interference and Klein tunnelling in graphene heterojunctions

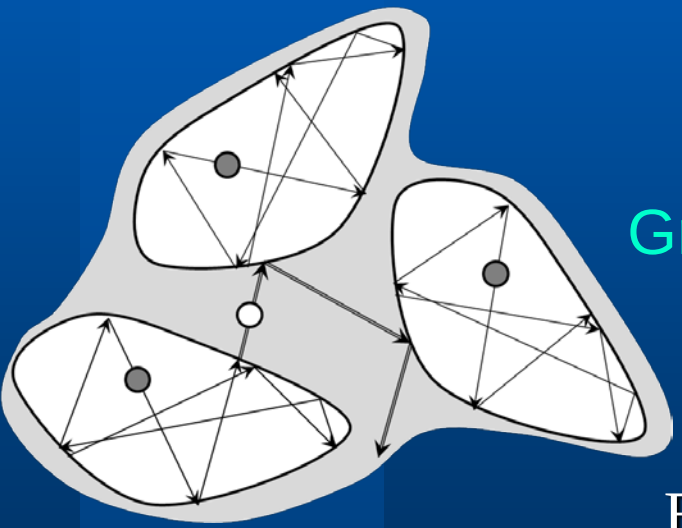
Andrea F. Young and Philip Kim*



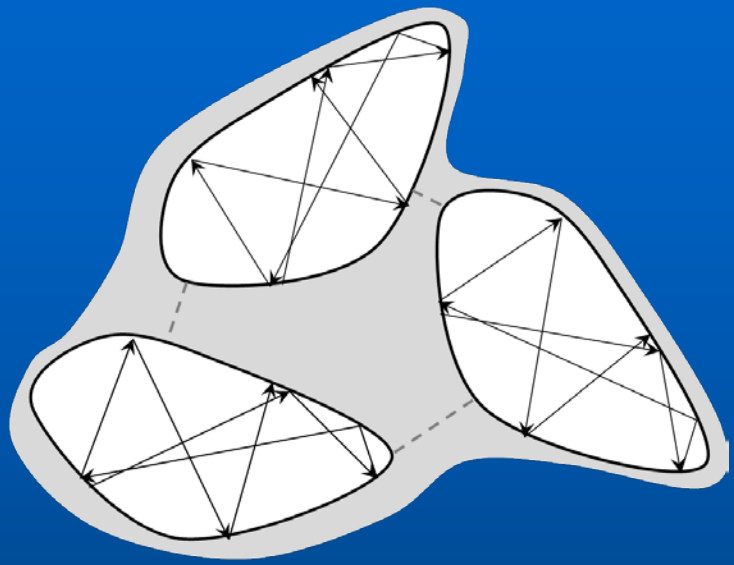
# *Klein tunneling prevents localization*

Back scattering is forbidden for chiral fermions! Magic angle = 0

Nonuniversal magic angle for bilayer exists!



Graphene



Conventional semiconductors

Electrons cannot be locked by random potential relief neither for single-layer nor for bilayer graphene – absence of localization and minimal conductivity?!

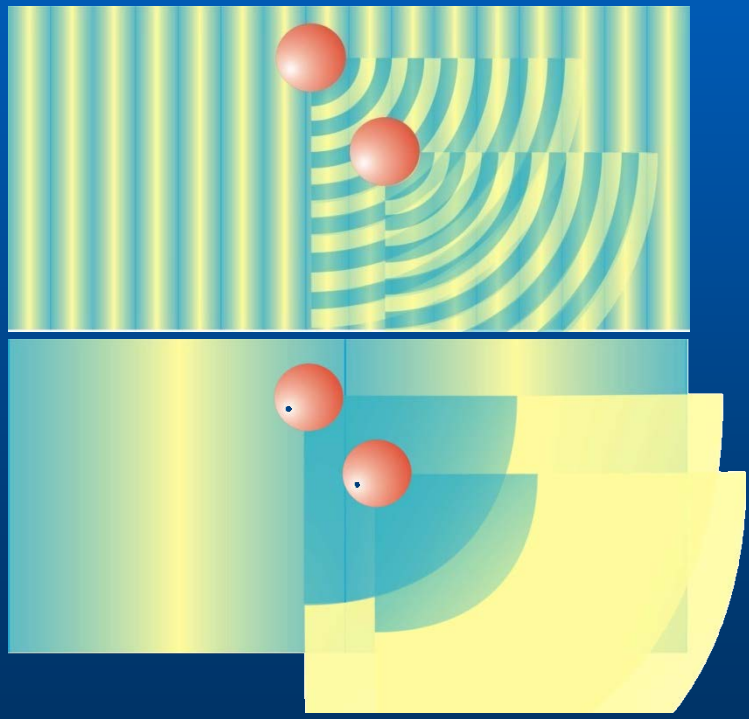
# Quantum-Limited Resistivity

$$\sigma = ne\mu = \frac{e^2}{h} k_F l$$

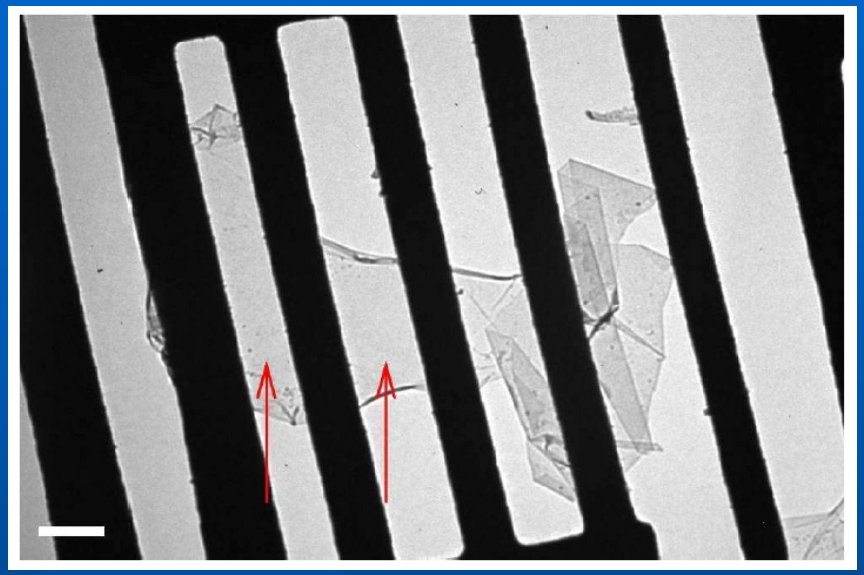
**Mott argument:** $l \geq \lambda_F$

$$\sigma \geq \frac{e^2}{h}$$

(in the absence of localization)



# *Inhomogeneities are unavoidable*



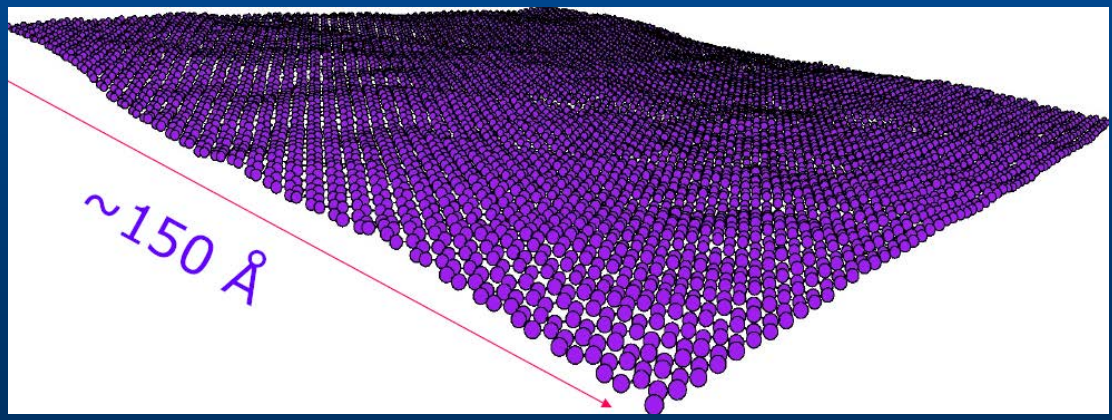
Freely suspended graphene membrane is corrugated

Meyer et al, *Nature 446, 60 (2007)*

2D crystals in 3D space cannot be flat, due to bending instability

Atomistic simulations of intrinsic ripples

Fasolino, Los & MIK, Nature Mater. 6, 858 (2007)



# Ripples and puddles

Gibertini, Tomadin, Polini, Fasolino & MIK, PR B 81, 125437 (2010)

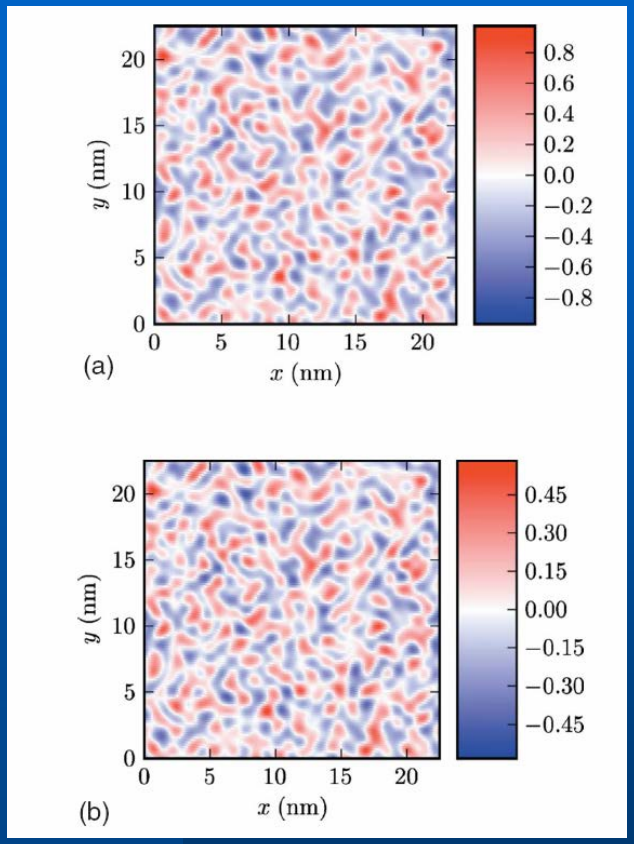



FIG. 4. (Color online) Top panel: fully self-consistent electronic density profile $\delta n(\boldsymbol{r})$ (in units of $10^{12}$ cm$^{-2}$) in a corrugated graphene sheet. The data reported in this figure have been obtained by setting $g_1=3$ eV, $\alpha_{ee}=0.9$ (this value of $\alpha_{ee}$ is the commonly used value for a graphene sheet on a $SiO_2$ substrate), and an average carrier density $\bar{n}_c \simeq 0.8\times10^{12}$ cm$^{-2}$. Bottom panel: same as in the top panel but for $\alpha_{ee}=2.2$ (this value of $\alpha_{ee}$ corresponds to suspended graphene).

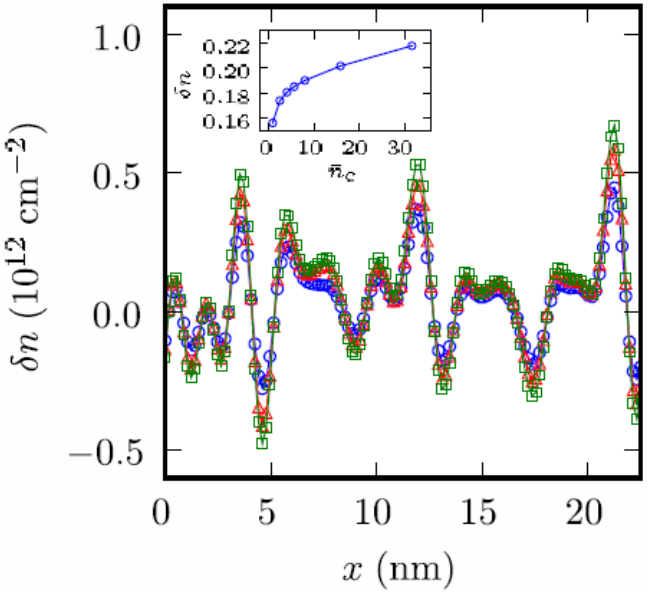


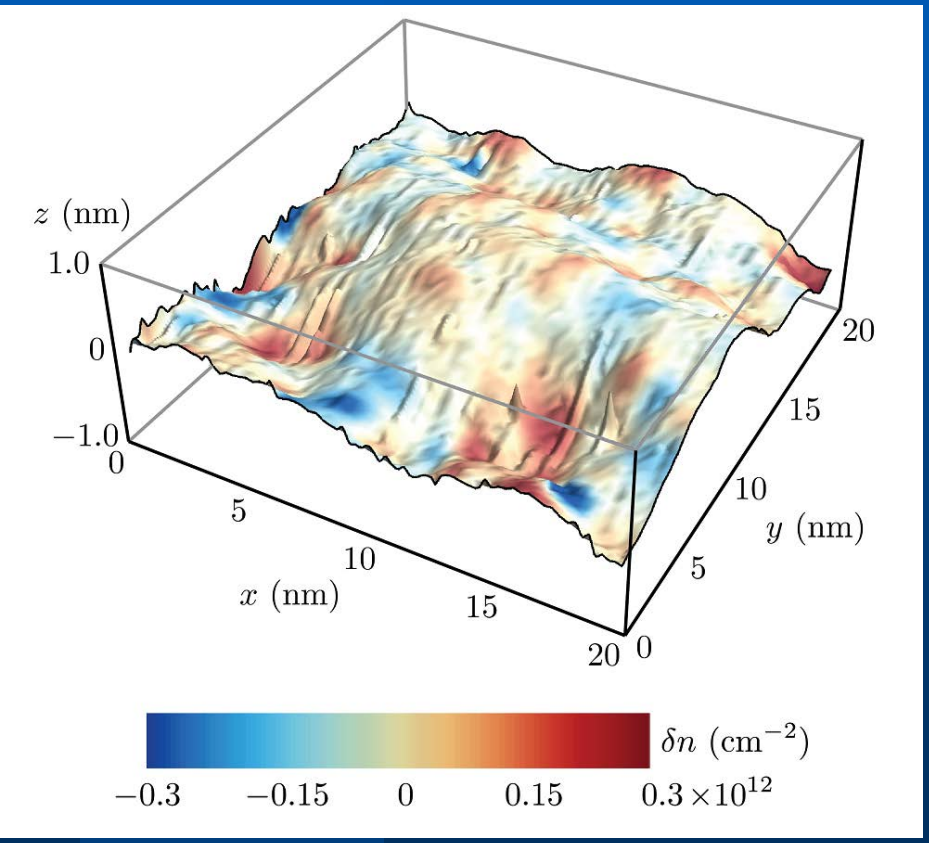

FIG. 9. (Color online) One-dimensional plots of the self-consistent density profiles (as functions of $x$ in nm for $y=21.1$ nm) for different values of doping: $\bar{n}_c \simeq 0.8\times10^{12}$ cm$^{-2}$ (circles), $\bar{n}_c \simeq 3.96\times10^{12}$ cm$^{-2}$ (triangles), and $\bar{n}_c \simeq 3.17\times10^{13}$ cm$^{-2}$ (squares). The data reported in this figure have been obtained by setting $g_1=3$ eV and $\alpha_{ee}=2.2$. The inset shows $\delta n(\boldsymbol{r})$ (in units of $10^{12}$ cm$^{-2}$) at a given point $\boldsymbol{r}$ in space as a function of the average carrier density $\bar{n}_c$ (in units of $10^{12}$ cm$^{-2}$).

# Ripples and puddles II

## Graphene on $SiO_2$

Gibertini, Tomadin, Guinea, MIK & Polini PR B 85, 201405 (2012)

Experimental STM data: V.Geringer et al (M.Morgenstern group)



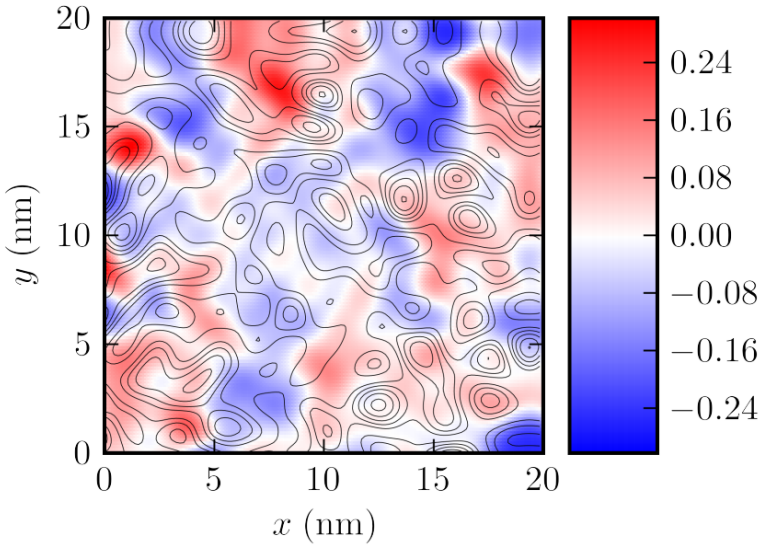




FIG. 3: (Color online) Fully self-consistent induced carrier-density profile $\delta n(\boldsymbol{r})$ (in units of $10^{12}$ cm$^{-2}$) in the corrugated graphene sheet shown in Fig. 1. The data reported in this figure have been obtained by setting $g_1 = 3$ eV, $\alpha_{ee} = 0.9$, and an average carrier density $\bar{n}_c \approx 2.5 \times 10^{11}$ cm$^{-2}$. The thin solid lines are contour plots of the curvature $\nabla^2_{\boldsymbol{r}} h(\boldsymbol{r})$. Note that there is no simple correspondence between topographic out-of-plane corrugations and carrier-density inhomogeneity.

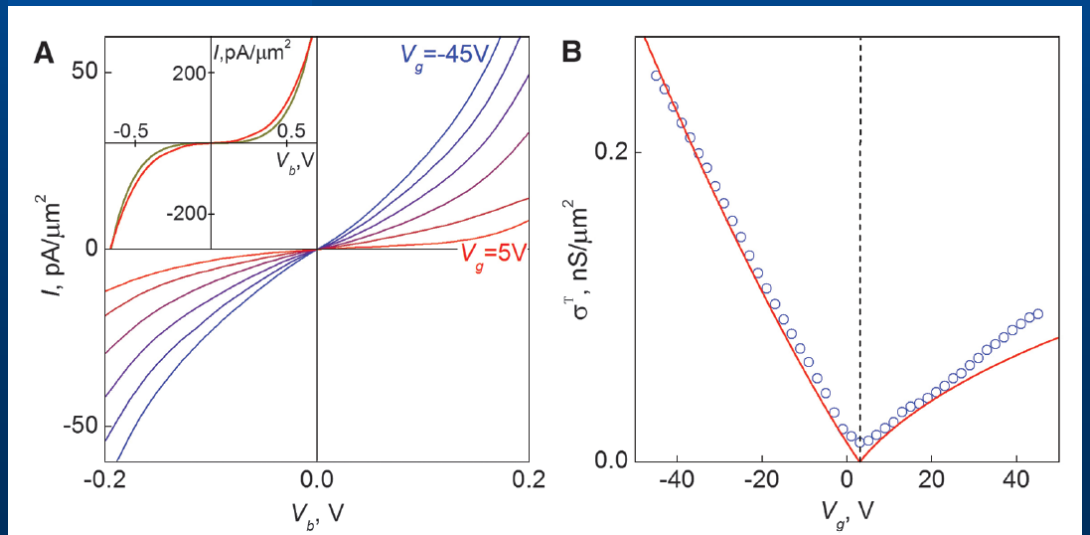
# The role of Klein tunneling

Without Klein tunneling graphene near the neutrality point will be insulator and, anyway, will have a high enough mobility

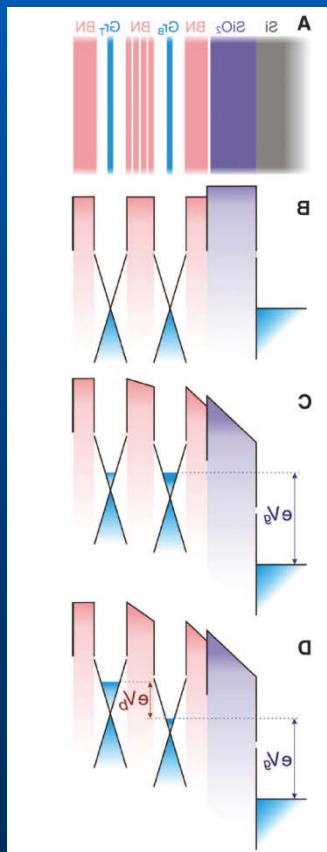
At the same time: a problem with transistors, one needs to use a complicated and indirect ways

Crucial phenomenon for physics and electronic applications of graphene

**Field-Effect Tunneling Transistor Based on Vertical Graphene Heterostructures**
L. Britnell, *et al.*
*Science* **335**, 947 (2012);




Tunneling transistor with vertical geometry



# *Semiclassical theory*

Tudorovkiy, Reijnders, MIK, Phys. Scr. T 146, 014010 (2012);
Reijnders, Tudorovskiy, MIK, Ann. Phys. (NY) 333, 155 (2013)

One-dimensional potential

$$\left[\nu\begin{pmatrix} 0 & \hat{p}_x - ip_y \\ \hat{p}_x + ip_y & 0 \end{pmatrix} + u(x/l) - E\right]\Psi = 0$$

$$\tilde{x} = x/l,\ \tilde{p}_x = -ihd/d\tilde{x},\ \tilde{p}_y = p_y/p_0,\ h = \hbar/p_0 l,\ \tilde{u} = u/\nu p_0$$

$$\tilde{E} = E/\nu p_0$$

Skipping tildas: the Hamiltonian

$$\hat{H} = \begin{pmatrix} 0 & \hat{p}_x - ip_y \\ \hat{p}_x + ip_y & 0 \end{pmatrix} + u(x)$$

# Semiclassical theory II

Reduction to exact Schrödinger equations for complex potential

$$\left(\hat{p}_x^2 + p_y^2 - v(x)^2 - ih\sigma_x v'(x)\right)\Psi = 0$$

$$v(x) = u(x) - E$$

$$\Psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \eta_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \eta_2$$

$$\left(h^2 \frac{d^2}{dx^2} + v(x)^2 - p_y^2 \pm ihv'(x)\right) \eta_{1,2} = 0$$

Schrödinger equation with complex potential

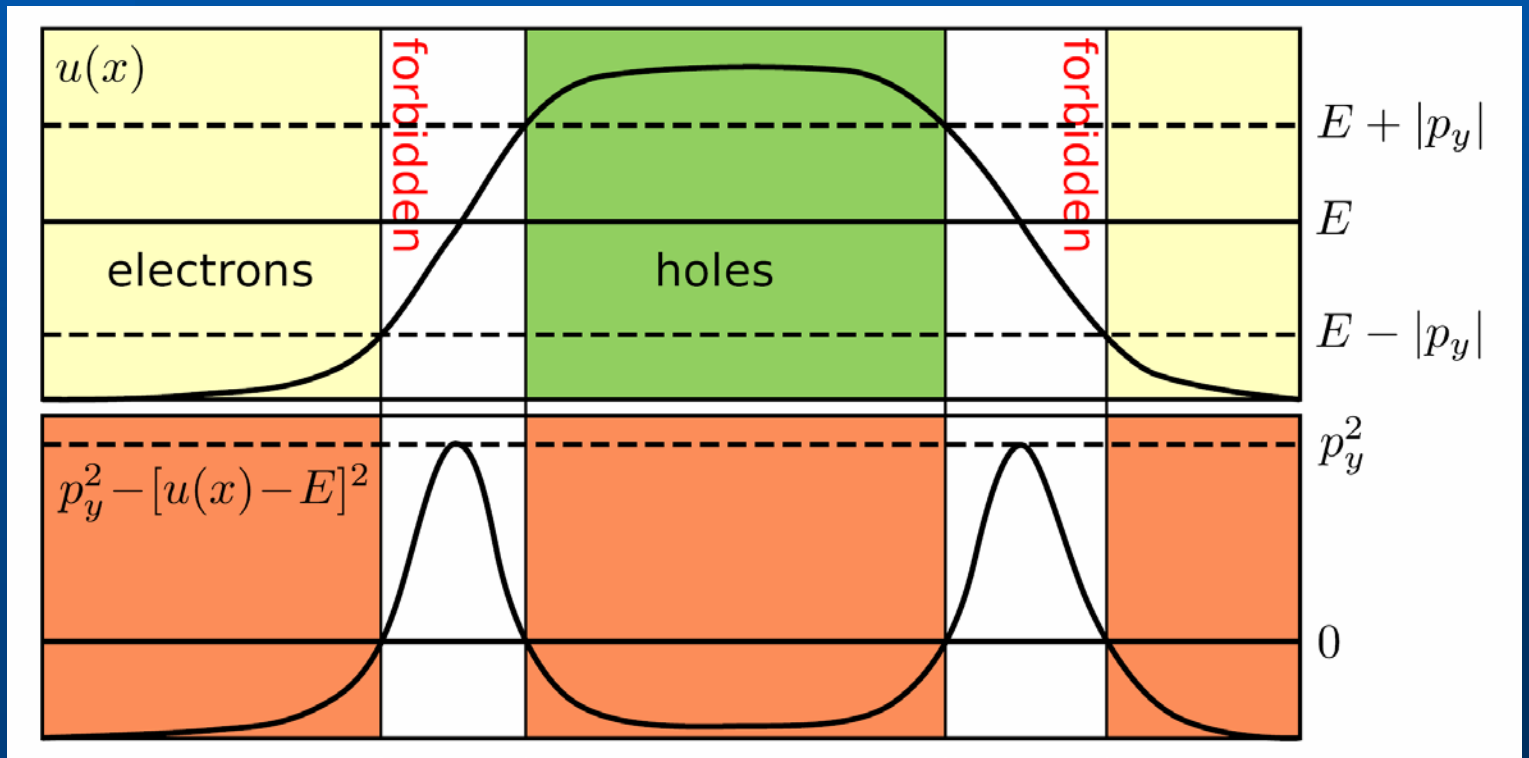
# Semiclasical theory III

Classical mechanics: $E = \pm|\boldsymbol{p}| + u(x)$

Effective Hamiltonian $\mathcal{L}(p_x, x) = p_x^2 - v^2(x) = -p_y^2$

$v(x) = u(x) - E$

The case of Klein tunneling $E < U_{\max},\ |p_y| < U_{\max} - E$



# p-n-p junction

Exact requirement: 100%transmission for normally incident wave
One needs to use comparison equation, exact solution for parabolic potential

Transmission probability

$$t_{npn} = \frac{e^{-K_{np}/h} e^{-K_{pn}/h} e^{-iL/h}}{1 - \sqrt{1 - e^{-K_{np}/h}} \sqrt{1 - e^{-K_{pn}/h}} e^{-2iL/h + i\pi - i\theta_{np} - i\theta_{pn}}}$$

$$K_{np} = \int_{x_1}^{x_2} dx \sqrt{p_y^2 - v^2(x)}$$

$X_{1,2}$ are turning points $v^2(x_0) - p_y^2 = 0$

$$L = \int_{x_2}^{x_3} dx' \sqrt{v^2(x') - p_y^2}$$

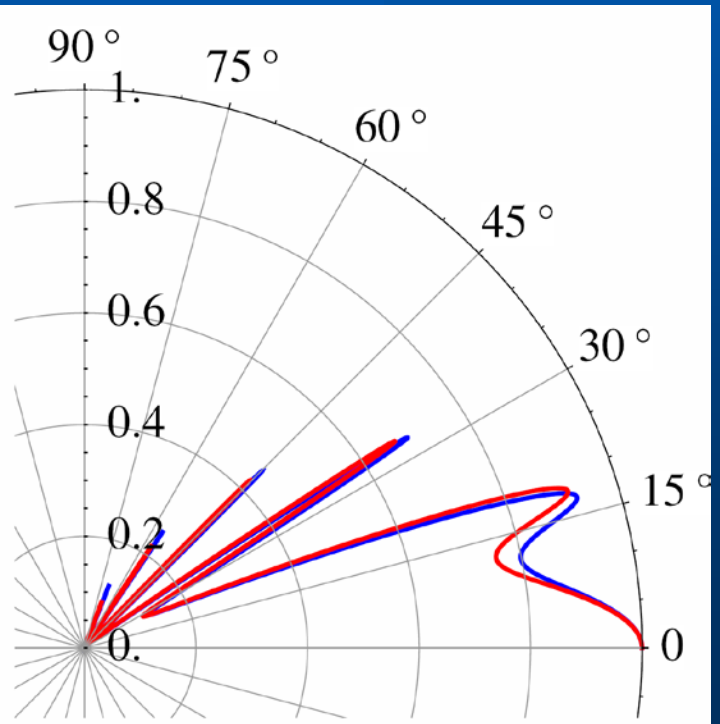
$$\theta = \mathrm{Arg}\left[\Gamma\left(1 + i\frac{K}{\pi h}\right)\right] - \frac{\pi}{4} + \frac{K}{\pi h} - \frac{K}{\pi h}\ln\left(\frac{K}{\pi h}\right)$$

# Fabri-Perot resonances

Magic angles with 100% transmission survives only for symmetric barriers (except normal incidence)

$$|t_{\text{res}}| = \frac{1}{\cosh(K_{np}/h - K_{pn}/h)}$$

$$K_{np}/h \gg 1,\ K_{pn} \gg 1$$




$$u(x/l_1) = \frac{U_{max}}{2}\left[1 + \tanh\left(10\frac{x}{l_1} - 5\right)\right]$$

The angular dependence of the transmission coefficient for a particle of energy 80 meV incident on an n-p-n junction of height 200 meV. The barrier width $l_2 = 250$ nm and the n-p and p-n regions have characteristic lengths $l_1 = 150$ nm and $l_3 = 50$ nm, respectively. The blue line shows the numerical results for 99 steps, while the red line shows the uniform approximation (5.77).

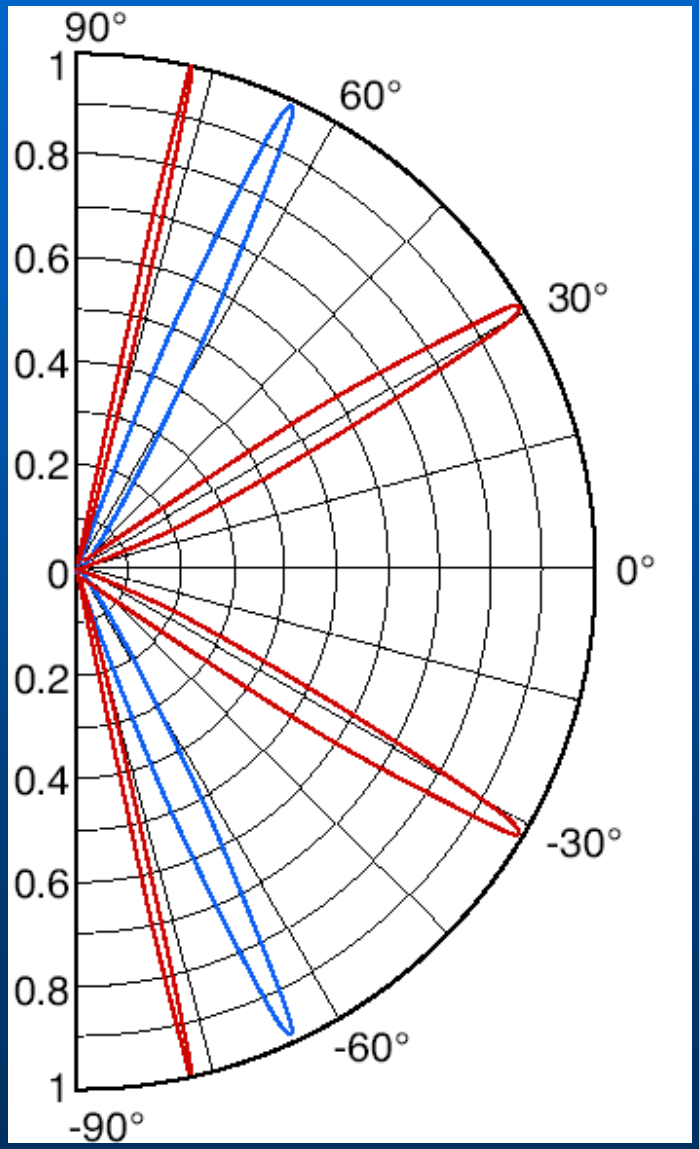
Very nice agreement with numerics

# *Chiral tunneling - bilayer*

Problem: graphene transistor can hardly be locked!

Possible solution: use bilayer graphene: chiral fermions with parabolic spectrum – no analogue in particle physics!

Transmission for bilayer; parameters are the same as for previous slide



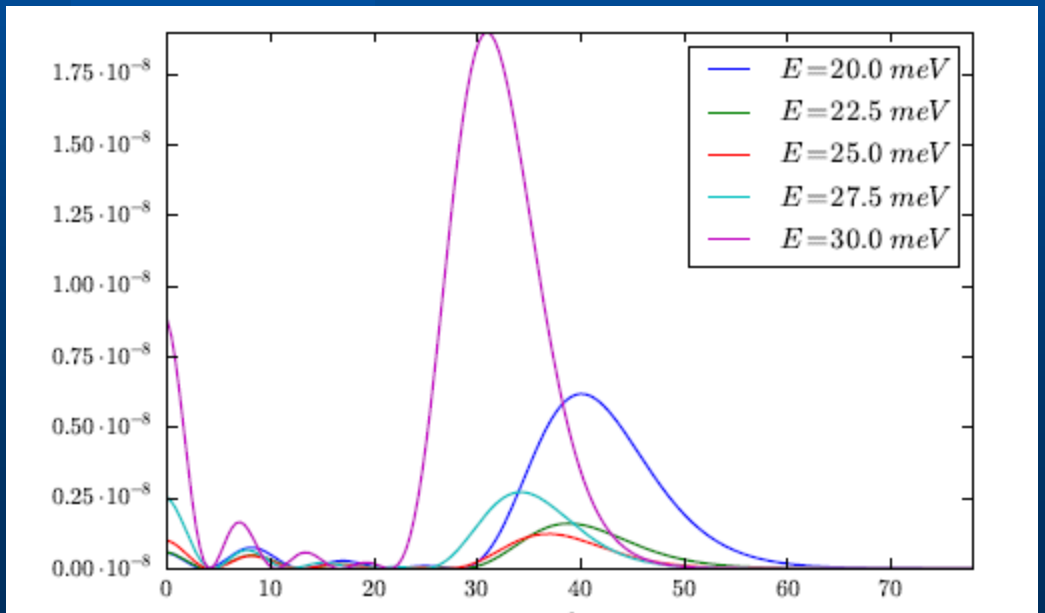
# *Chiral tunneling – bilayer II*

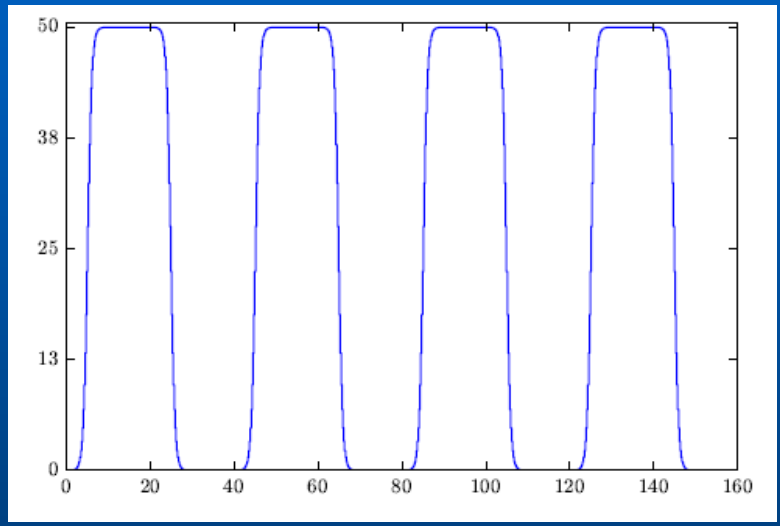
Kleptsyn, Okunev, Schurov, Zubov & MIK, PRB **92**, 165407 (2015)

For symmetric potential V(x)=V(-x): one real equation for magic angles but not necessarily real solutions

Magic angles are not protected

Example: fast-oscillating potential




Very small transmission probability

From analysis of ODE to graphene transistor?!

# *Relativistic collapse for supercritical charges*

**Coulomb potential**

$$V_0(\mathbf{r}) = \frac{Ze^2}{\epsilon r}$$

following Shytov, MIK & Levitov, PRL 99, 236801; 246803 (2007)

**Naive arguments: Radius of atom *R*, momentum *ħ/R*. Nonrelativistic case:**

***E(R) ~ ħ² /mR² – Ze²/R***

**Minimum gives a size of atom.**

**Relativistic case: *E(R) ~ ħc*/R – Ze²/R***

Either no bound state or fall on the center.

**Vacuum reconstruction at *Z* > 170**

# Supercritical charges II

Superheavy nuclei

I. Pomeranchuk and Y. Smorodinsky, J. Phys. USSR **9**, 97 (1945)

Graphene:
$v \approx c/300$,
$\alpha_{eff} \approx 1$

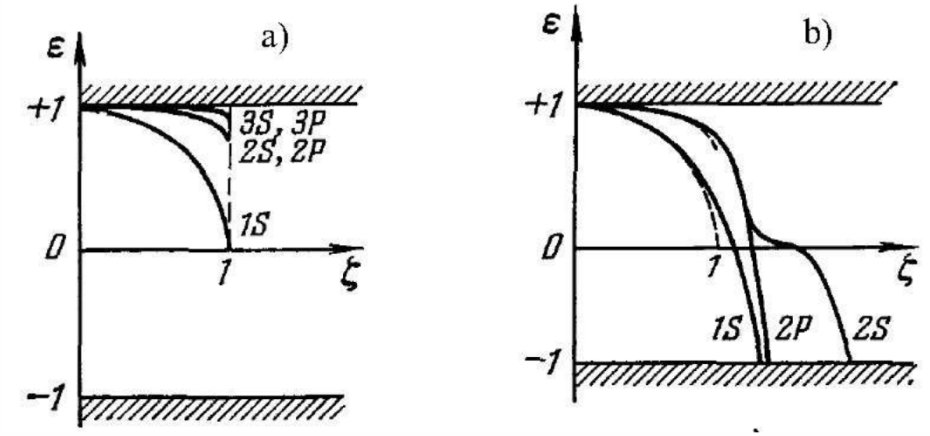
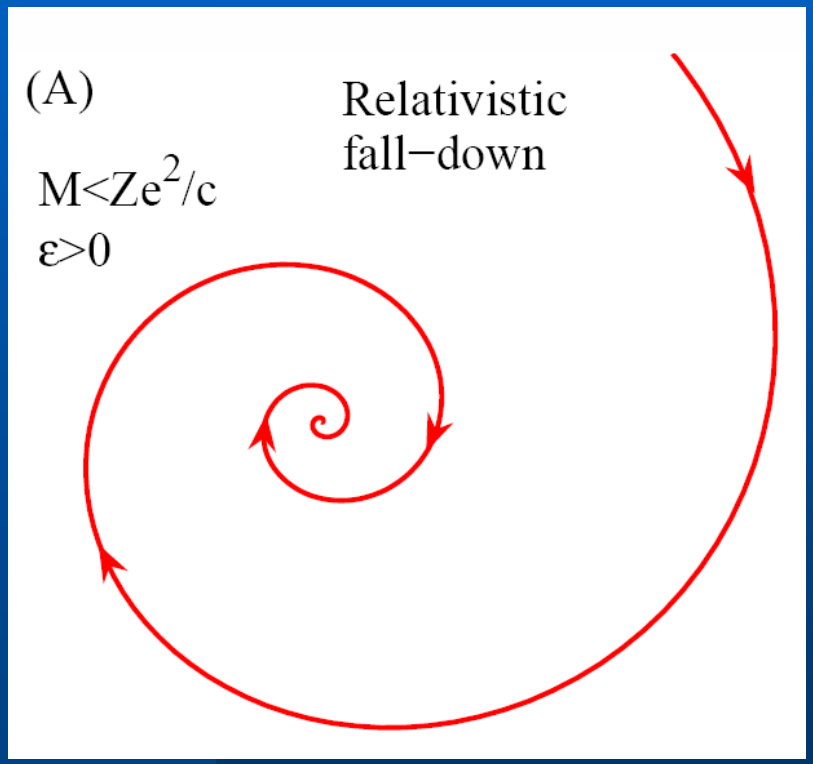



FIG. 1: a) Energy levels of superheavy atoms obtained from Dirac equation for Coulomb potential $-Ze^2/r$, plotted as a function of $\zeta = Z\alpha$, where $Z$ is nuclear charge, and $\alpha = e^2/\hbar c$ is the fine structure constant. Energy is in the units of $mc^2$. (b) Energy levels for Coulomb potential regularized on the nuclear radius. As $Z$ increases, the discrete levels approach the continuum of negative-energy states and dive into it one by one at supercritical $Z > 170$ (from Ref.[23]).

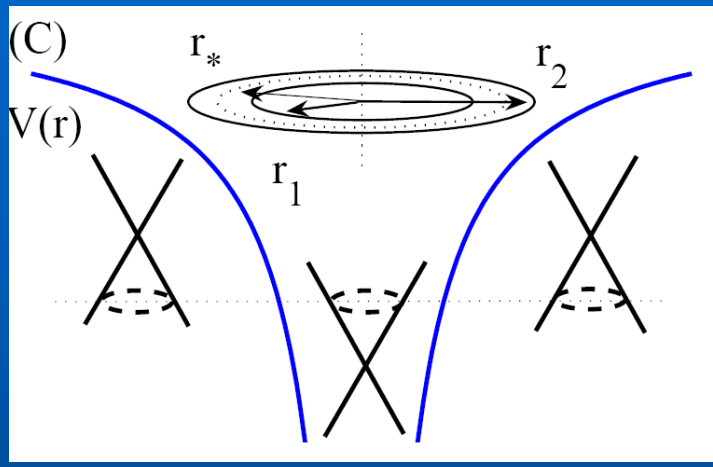
[23] Y. B. Zeldovich and V. S. Popov, Usp. Fiz. Nauk **105**, 403

# Supercritical charges III

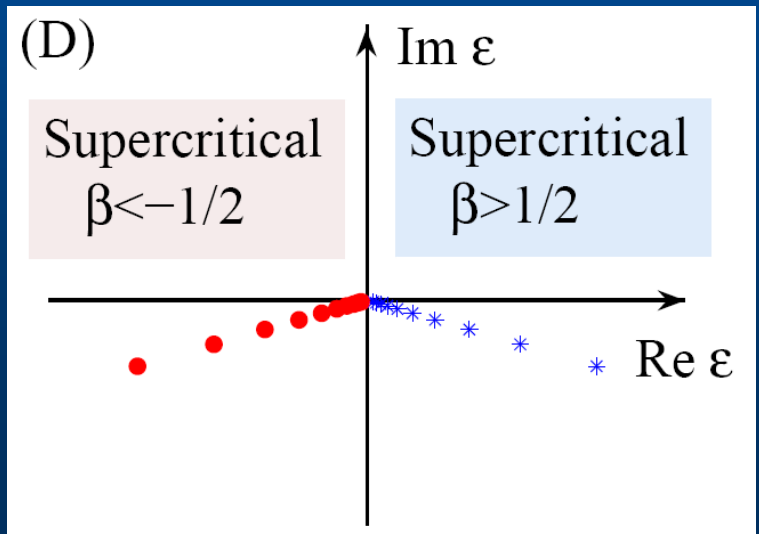
$\beta = Ze^2/\hbar v_F \varepsilon > ½$




Quasi-local states

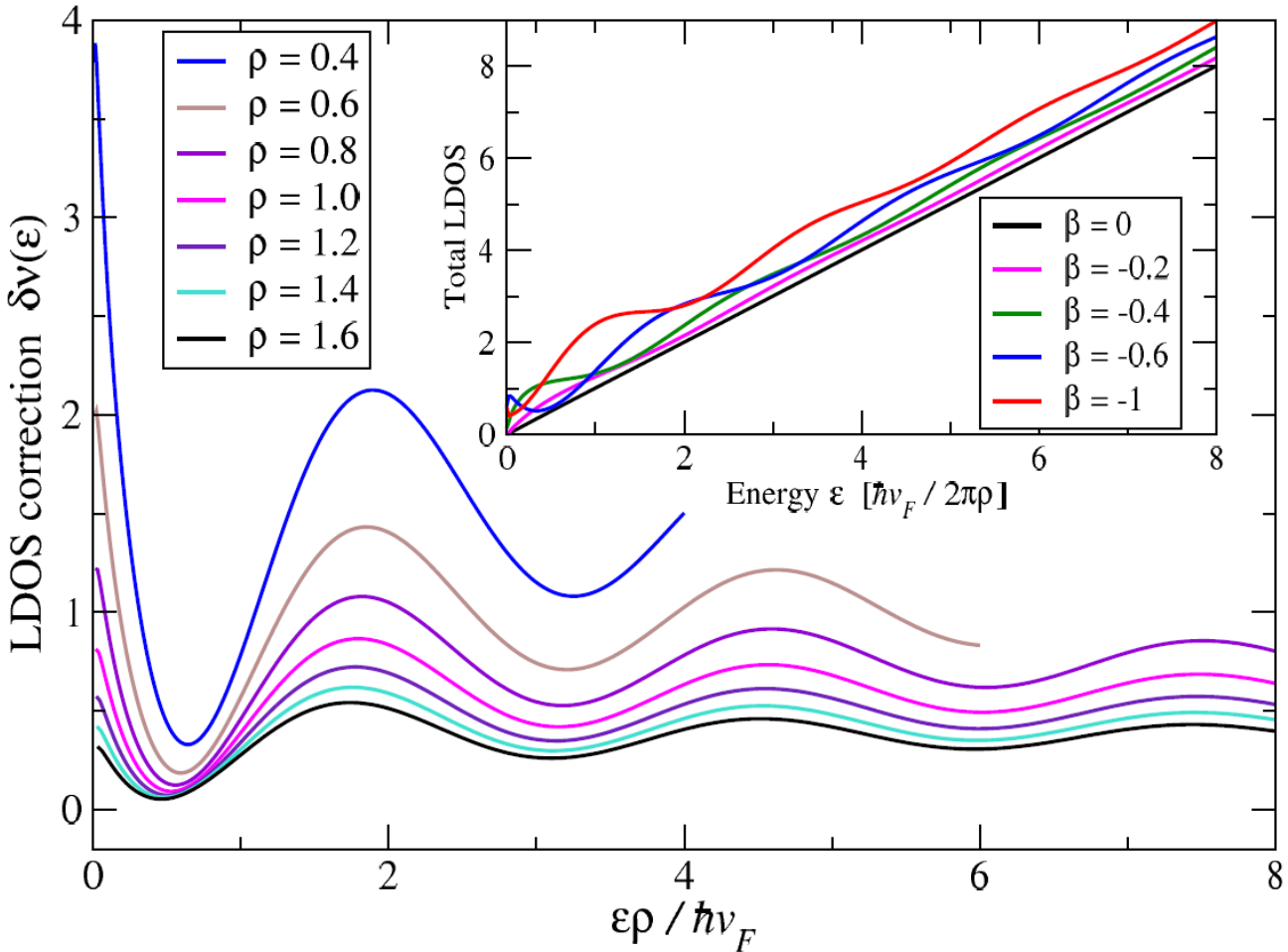



Klein tunneling



# Supercritical charges IV

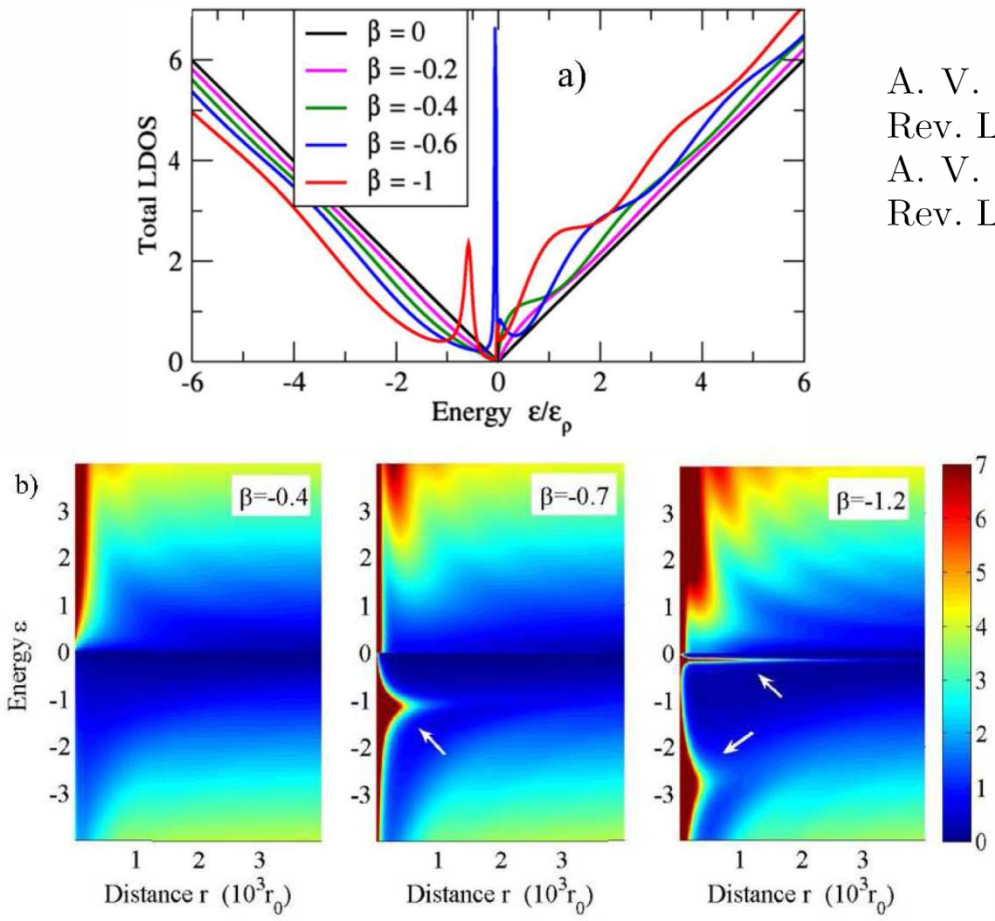
**Interference of scattered wave and wave described electron fall to the centre leads to oscillations of electron density**




*β = 0.6*

Inset: oscillations for different charges

# Supercritical charge V





A. V. Shytov, M. I. Katsnelson, and L. S. Levitov, Phys. Rev. Lett. **99**, 236801 (2007), arXiv:0705.4663

A. V. Shytov, M. I. Katsnelson, and L. S. Levitov, Phys. Rev. Lett. **99**, 246802 (2007), arxiv.org:0708.0837


FIG. 3: (a) Local density of states (12) calculated at a fixed distance $\rho = 10^3 r_0$ from the charged impurity, where $r_0$ is a short-distance parameter of the order of carbon lattice spacing (from Ref.[10]). Peaks in the LDOS, which appear at supercritical $\beta$ and move to more negative energies at increasing $|\beta|$, correspond to the resonant states. (b) Spatial map of the density of states, shown for several values of $\beta$, with resonances marked by white arrows (from Ref.[11]). Note that the spatial width of the resonances decreases at they move to lower energies, $\Delta\rho \propto 1/|\varepsilon|$, while the linewidth increases, $\gamma \propto |\varepsilon|$. The oscillatory structure at positive energies represents standing waves with maxima at $k\rho \approx (n + \frac{1}{2})\pi$, similar to those studied in carbon nanotubes [26]. Energy is given in the units of $\varepsilon_0 = 10^{-3}\hbar v/r_0 \approx 30\,\mathrm{mV}$ for $r_0 = 0.2\,\mathrm{nm}$.

# *Vacuum polarization effect and screening*

Dimensional analysis: induced charge density in undoped graphene

$$n(r) = A\delta(\mathbf{r}) + B/r^2$$

If *B* is not zero: logarithmically divergent induces charge and "nullification" of Coulomb potential – predicted by Thomas –Fermi theory

MIK, PR B 74, 201401 (2006)

Linear screening theory: constant dielectric function, screening charge focused at the coordinate origin (only first term)

# Screening II

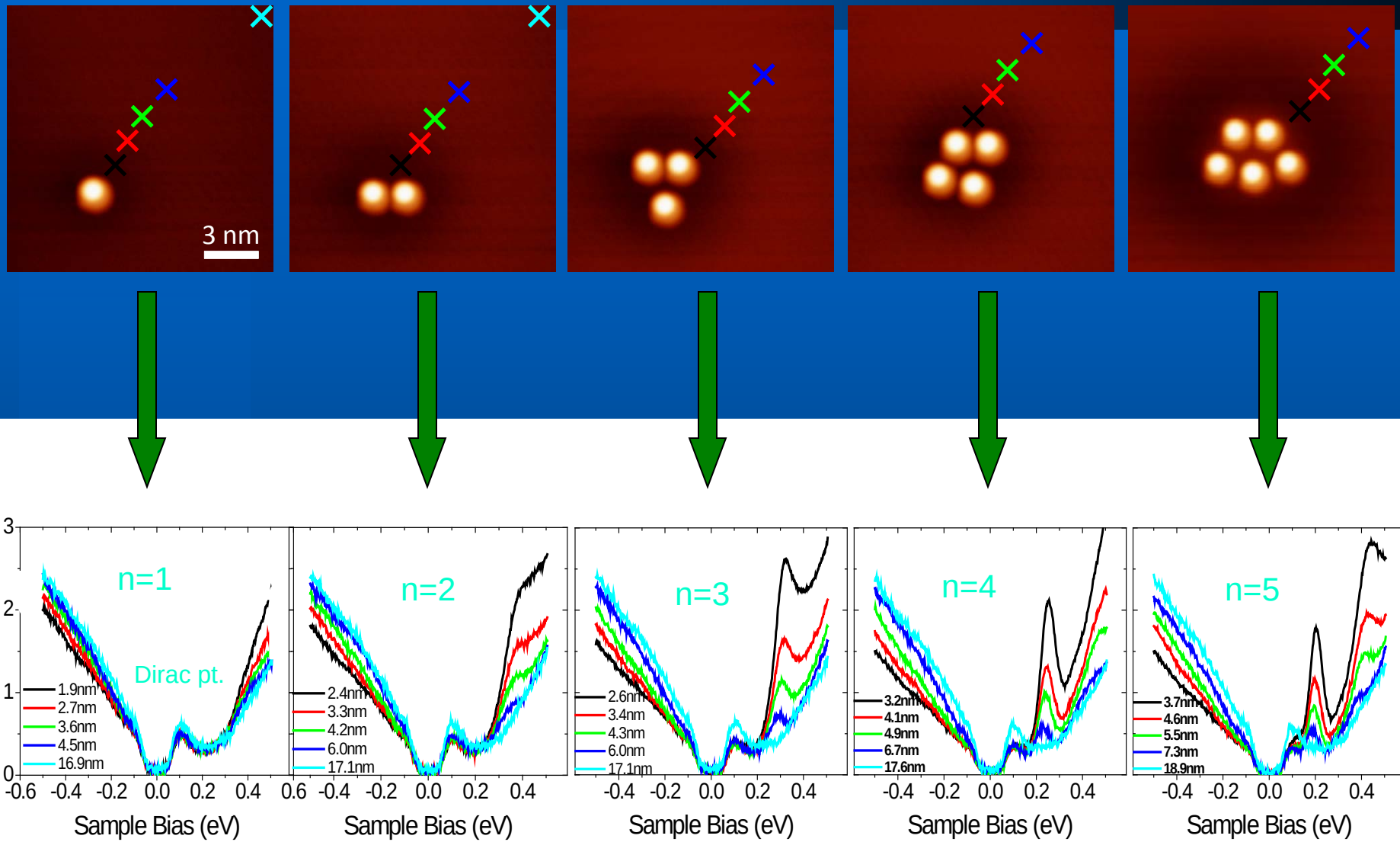
$$n_{\rm pol}(\rho) = -\frac{N\,\mathrm{sign}\,\beta}{2\pi^2\rho^2} \sum_{|m+\frac{1}{2}|<|\beta|} \sqrt{\beta^2 - \left(m+\tfrac{1}{2}\right)^2}$$

*N = 4* (two valleys, two spins)

Large *β*: replacing the sum by an integral recover the Thomas-Fermi result

RG analysis: supercritical charge is screened to *β* = ½ with a finite screening radius (similar to black hole horizon)

# *Exper.: Tuning Z by Building Artificial Nuclei from Ca Dimers*




Y. Wang et al, Science 340, 734 (2013)

# *Pseudomagnetic fields*

Nearest-neighbour approximation: changes of hopping integrals

$$\gamma = \gamma_0 + \left(\frac{\partial\gamma}{\partial\overline{u}_{ij}}\right)_0 \overline{u}_{ij}$$

$$H = v_F\sigma\left(-i\hbar\nabla - \frac{e}{c}\mathcal{A}\right)$$

"Vector potentials"

$$\mathcal{A}_x = \frac{c}{2ev_F}\left(\gamma_2 + \gamma_3 - 2\gamma_1\right),$$

$$\mathcal{A}_y = \frac{\sqrt{3}c}{2ev_F}\left(\gamma_3 - \gamma_2\right),$$

*K* and *K'* points are shifted in opposite directions; Umklapp processes restore time-reversal symmetry

Review: M. Vozmediano, MIK & F. Guinea, Phys. Rep. 496, 109 (2010)

# Psedomagnetic fields II

Within elasticity theory (continuum limit)

$$\mathbf{A} = \frac{\beta}{a}\begin{pmatrix} u_{xx} - u_{yy} \\ -2u_{xy} \end{pmatrix}$$

$$\beta = -\partial \ln t / \partial \ln a \approx 2$$

Pseudomagnetic field

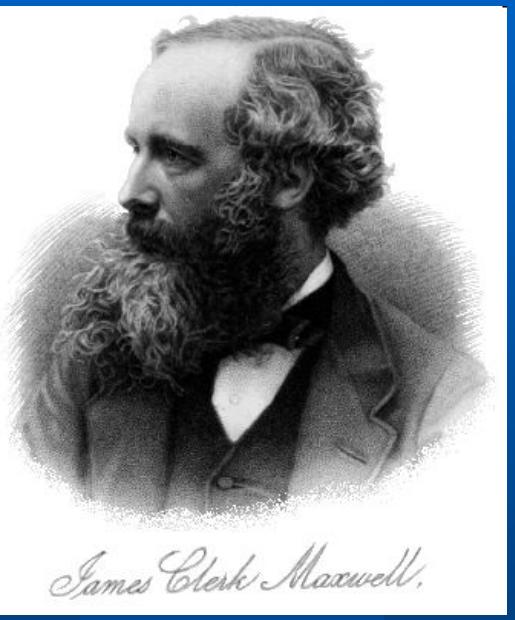
Shear deformations create vector potential

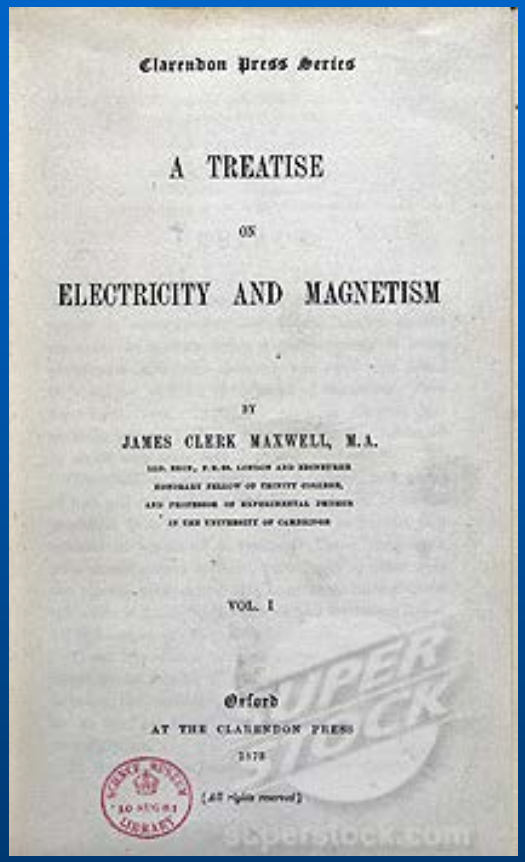
$$B_S = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{1}{r}\frac{\partial A_r}{\partial \theta} - \frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r}$$

Dilatation creates scalar (electrostatic) potential

$$V_1 = g_1(u_{xx} + u_{yy})$$

# *Gauge fields from mechanics: back to Maxwell*

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ON

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BY

JAMES CLERK MAXWELL, M.A.

VOL. I

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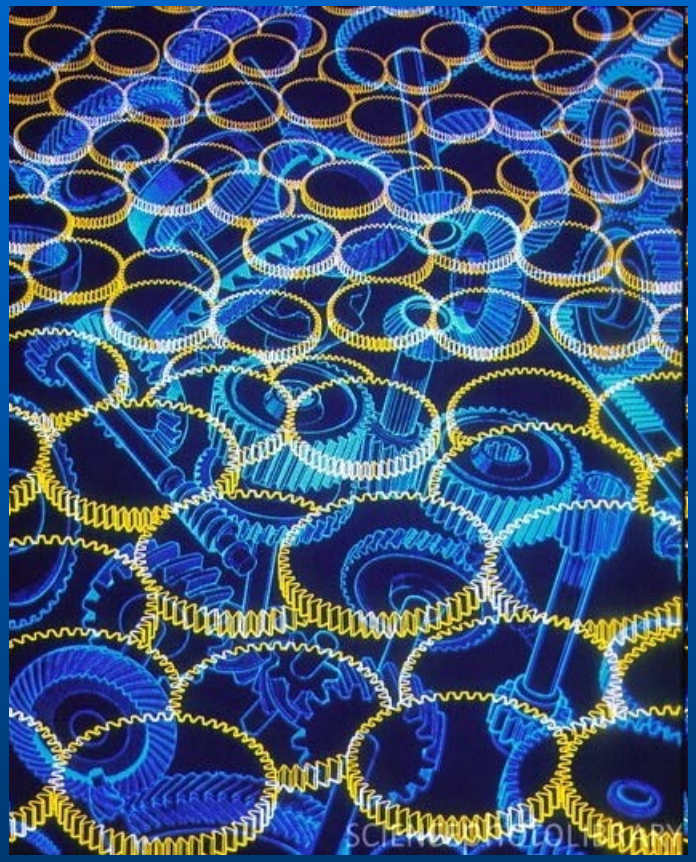


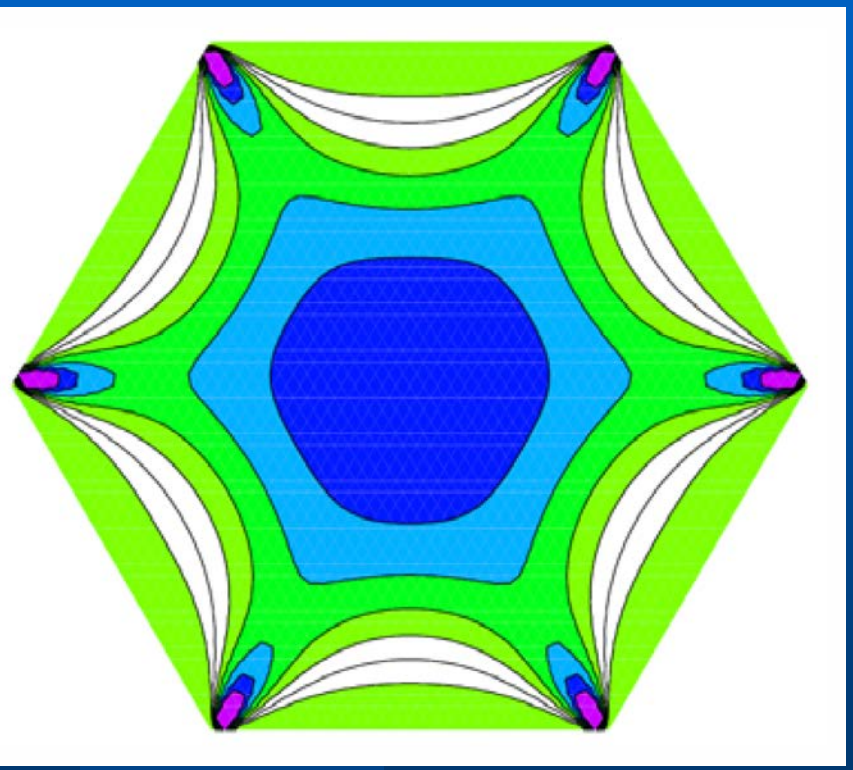
Electromagnetic fields as deformations in ether; gears and wheels

Review: Vozmediano, MIK & Guinea, Phys. Rep. 496, 109 (2010)

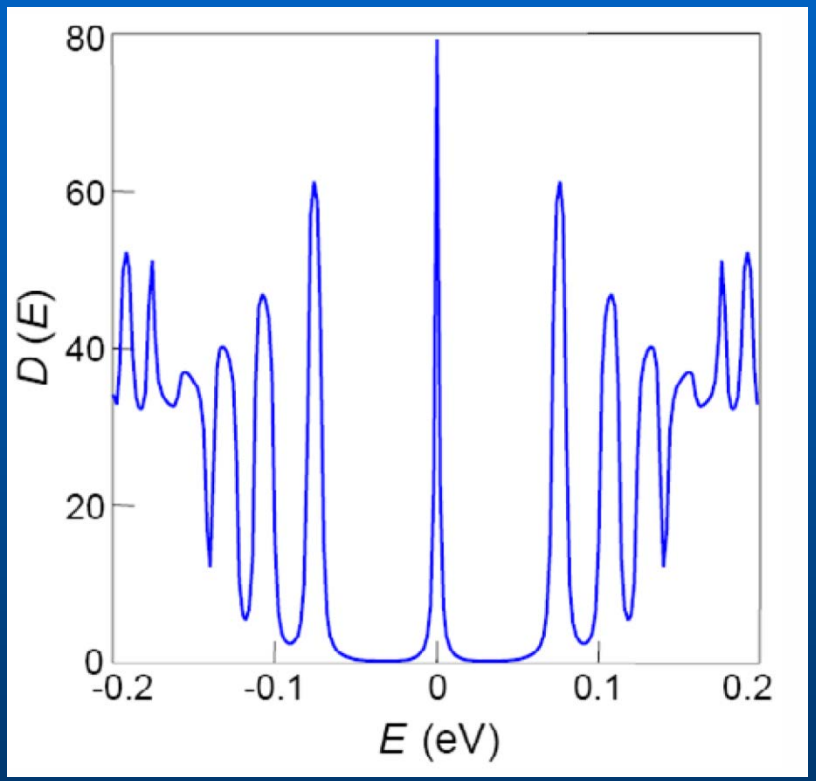
# *Zero-field QHE by strain engineering*

F. Guinea, MIK & A. Geim, Nature Phys. 6, 30 (2010)

Can we create uniform (or almost uniform) pseudomagnetic field?



If you keep trigonal symmetry, quasi-uniform pseudomagnetic field can be easily created



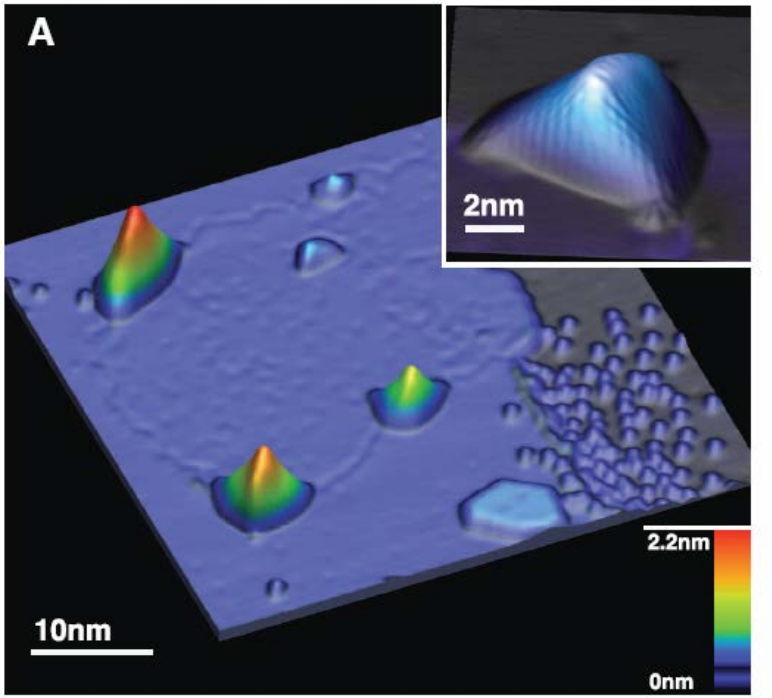

Normal stress applied to three edges size 1.4 μm, DOS in the center (0.5 μm)

# Experimental confirmation

## Strain-Induced Pseudo–Magnetic Fields Greater Than 300 Tesla in Graphene Nanobubbles

N. Levy,[1,2]*† S. A. Burke,[1]*‡ K. L. Meaker,[1] M. Panlasigui,[1] A. Zettl,[1,2] F. Guinea,[3] A. H. Castro Neto,[4] M. F. Crommie[1,2]§

30 JULY 2010 VOL 329 SCIENCE




Graphene on Pt(111)

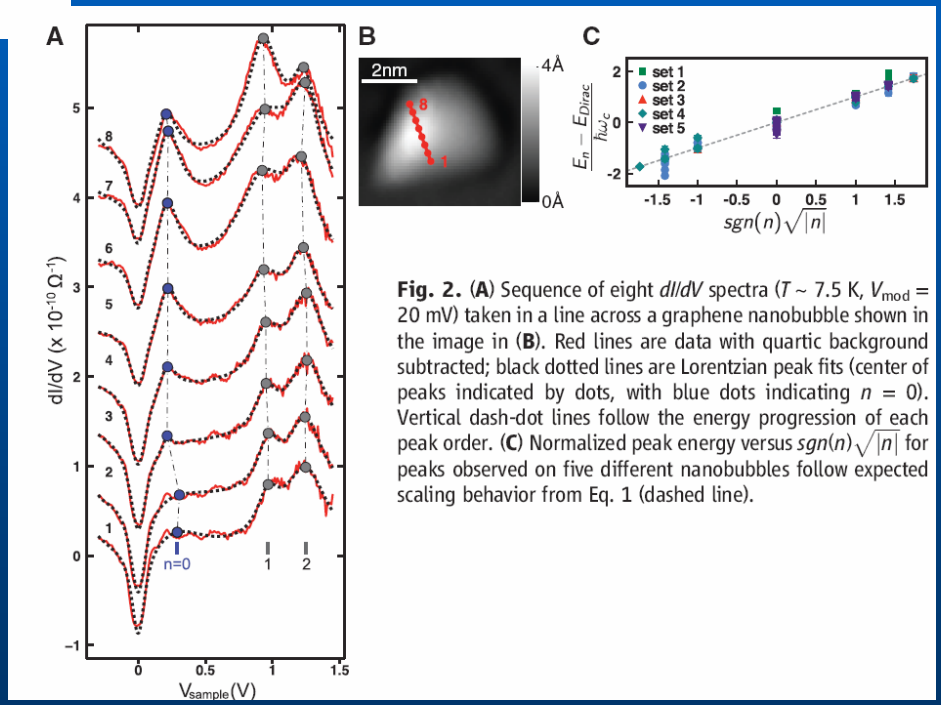



**Fig. 2.** (**A**) Sequence of eight *dI*/*dV* spectra ($T$ ~ 7.5 K, $V_{mod}$ = 20 mV) taken in a line across a graphene nanobubble shown in the image in (**B**). Red lines are data with quartic background subtracted; black dotted lines are Lorentzian peak fits (center of peaks indicated by dots, with blue dots indicating $n$ = 0). Vertical dash-dot lines follow the energy progression of each peak order. (**C**) Normalized peak energy versus $sgn(n)\sqrt{|n|}$ for peaks observed on five different nanobubbles follow expected scaling behavior from Eq. 1 (dashed line).

STM observation of pseudo-Landau levels

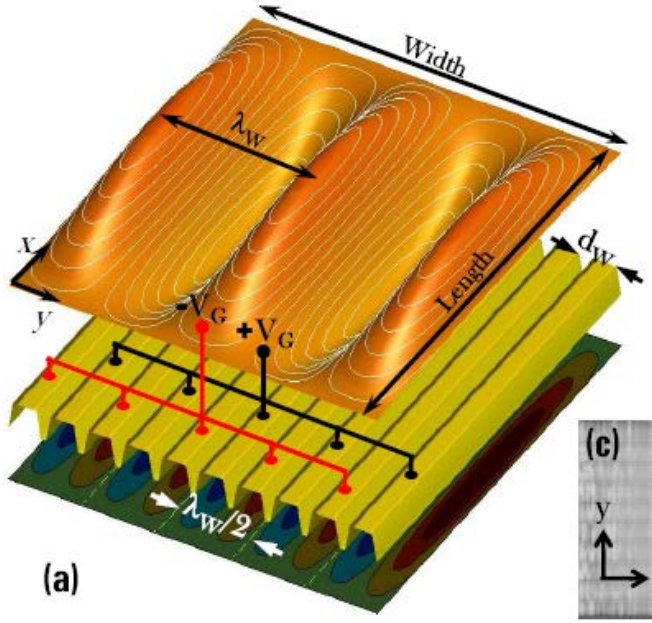
# Combination of strain and electric field: Haldane insulator state

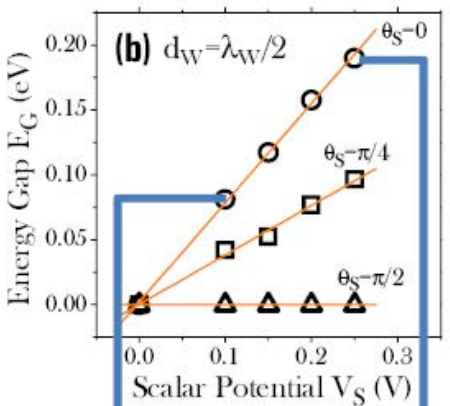
T. Low, F. Guinea & MIK, PRB 83, 195436 (2011)

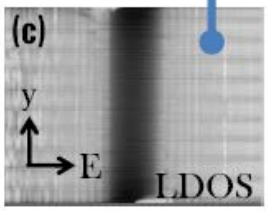
Without inversion center combination of vector and scalar potential leads to gap opening

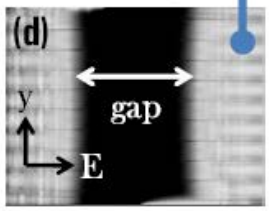
$$\Delta = -\mathrm{Tr}\left\{\sigma_z \frac{2}{v_F}\int d^2\vec{k}\frac{\mathrm{Im}\left(V_{-\vec{k}}\right)\left[(\vec{k}\vec{\sigma}),(\vec{A}_{\vec{k}}\vec{\sigma})\right]}{|\vec{k}|^2}\right\}$$

$$\propto \int d^2\vec{k}\frac{\mathrm{Im}(V_{-\vec{k}})\left(k_x A^y_{\vec{k}} - k_y A^x_{\vec{k}}\right)}{|\vec{k}|^2} \qquad (1)$$



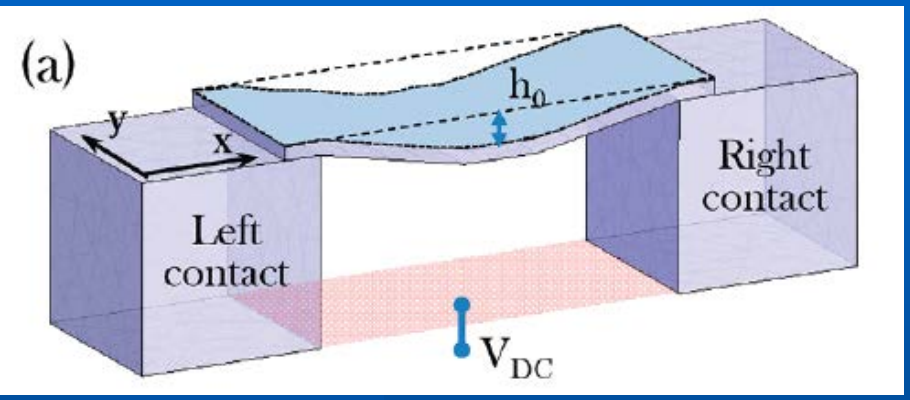




Wrinkles plus modulated scalar potential at different angles to the wrinkilng direction

# Quantum pumping

T. Low, Y. Jiang, MIK & F. Guinea Nano Lett. 12, 850 (2012)




Nanoelectromechanical resonator, periodic change of electric fields and pseudomagnetic fields (deformations) – a very efficient quantum pumping

Periodic electrostatic doping *plus* vertical deformation $a(t)$ created pseudomagnetic vector potential

$$V(t) = V_{dc} + V_{ac}\cos(\omega t)$$

$$\varepsilon_d = \hbar v_f (\pi C_T V_{dc}/e)^{1/2}$$

$$\delta\varepsilon_d = V_{ac}/V_{dc}$$

$$u_{xx} = 8h_0^2/3L^2$$

$$\delta u_{xx} = a/h_0$$

$$\mathcal{E}_{dg}(t) = \varepsilon_d\{1 + \delta\varepsilon_d \sin(\omega t)\}^{1/2}$$

$$\mathcal{U}_{xx}(t) = u_{xx}\{1 + \delta u_{xx} \sin(\omega t + \phi)\}^2 - \frac{\Delta L}{L}$$

# Quantum pumping II

## Scattering problem

$$\psi_j(x) = \begin{cases} \begin{pmatrix} 1 \\ \eta_l \end{pmatrix} e^{ik_{xl}x} + \mathcal{R}_{\upsilon} \begin{pmatrix} 1 \\ -\eta_l^{\dagger} \end{pmatrix} e^{-ik_{xl}x} \\ \alpha_l \begin{pmatrix} 1 \\ \eta_g \end{pmatrix} e^{ik_{xg}x} + \alpha_g \begin{pmatrix} 1 \\ -\eta_g^{\dagger} \end{pmatrix} e^{-ik_{xg}x} \\ \mathcal{T}_{\upsilon} \sqrt{\dfrac{k_{xl}k_{fr}}{k_{xr}k_{fl}}} \begin{pmatrix} 1 \\ \eta_r \end{pmatrix} e^{ik_{xr}x} \end{cases}$$

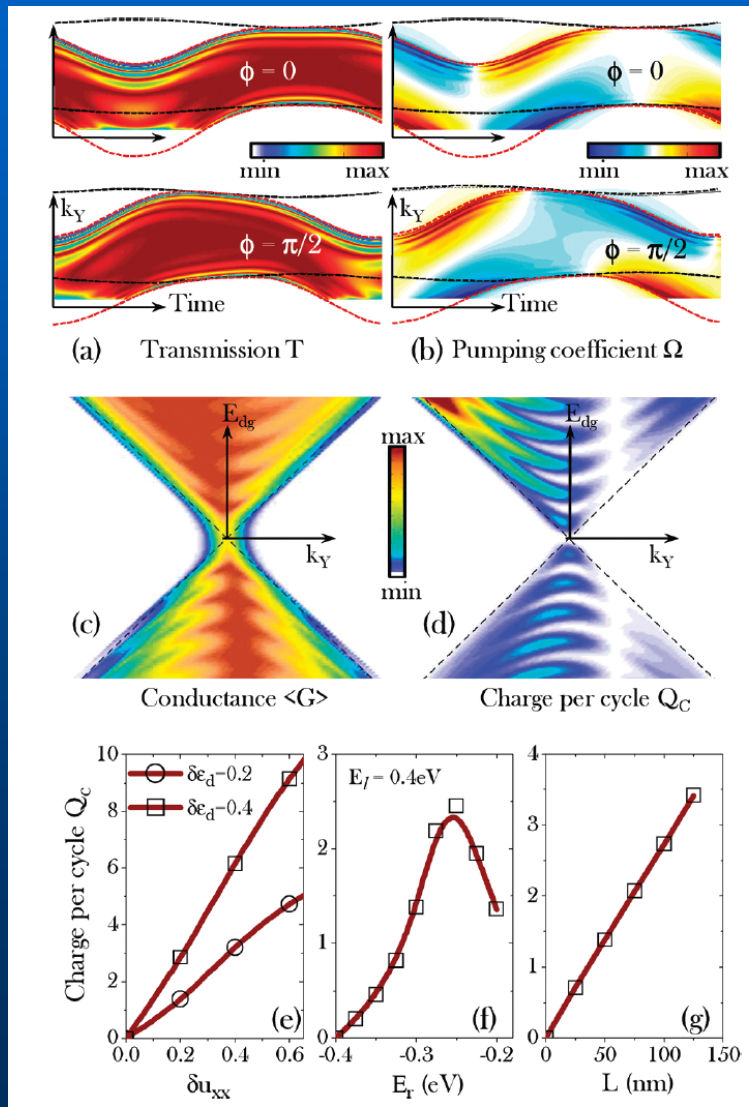
## Pumping current

$$\mathcal{I}_{\upsilon} = i\frac{e\omega}{4\pi^2} \sum_{k_y} \int_0^{2\pi/\omega} dt \int_{-\infty}^{\infty} d\varepsilon \frac{\partial f_0(\varepsilon)}{\partial \varepsilon} \Omega_{\upsilon}(k_y, t)$$

$$\Omega_{\upsilon} = (\partial \mathcal{T}_{\upsilon}/\partial t)\mathcal{T}_{\upsilon}^{\dagger} + (\partial \mathcal{R}_{\upsilon})/\partial t)\mathcal{R}_{\upsilon}^{\dagger}$$

Asymmetric leads (different doping)

## Results



# *Beyond the talk*

*Graphene on hBN: tunable incommensurate potential, very reach quantum and statistical mechanics*

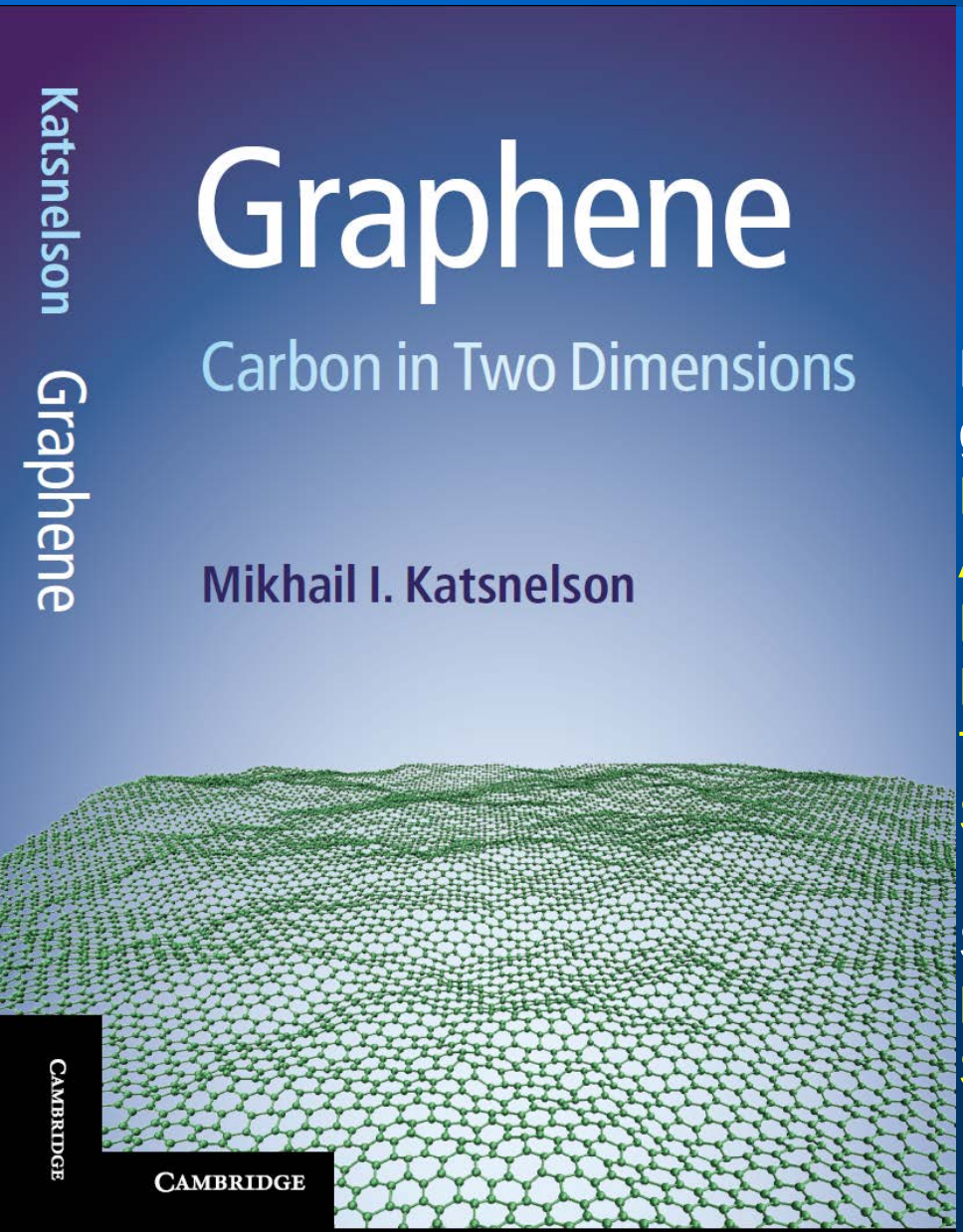
*Graphene as a prototype crystalline membrane: Ripples and all that*

*Many-body effects – we are just in the beginning*

*Other 2D materials (especially, black P which can be made anisotropic Dirac material)*

*….*

# *Further reading*

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Impossible to mention everyone...