Our proof strategy

1°. Define "pseudo-width" of a clause.

2°. If the size of a proof is small then we can "eliminate" from it all clauses of large pseudo-width.

3°. Show that the pseudo-width of any refutation of fun-PHP_n^m must be large.

Goals are very similar to [BW] but the implementation is very different.

Definition of pseudo-width

Remove negations:

$$\bar{x}_{ij} \rightsquigarrow x_{i1} \vee x_{i2} \vee \dots \vee x_{i,j-1} \vee x_{i,j+1} \vee \dots \vee x_{in}$$

C is a positive clause, or, equivalently, a bipartite graph:

$$C = \bigvee_{(i,j) \in G} x_{ij}.$$

$d_1(C), \dots, d_m(C)$ are degrees of all pigeons in C.

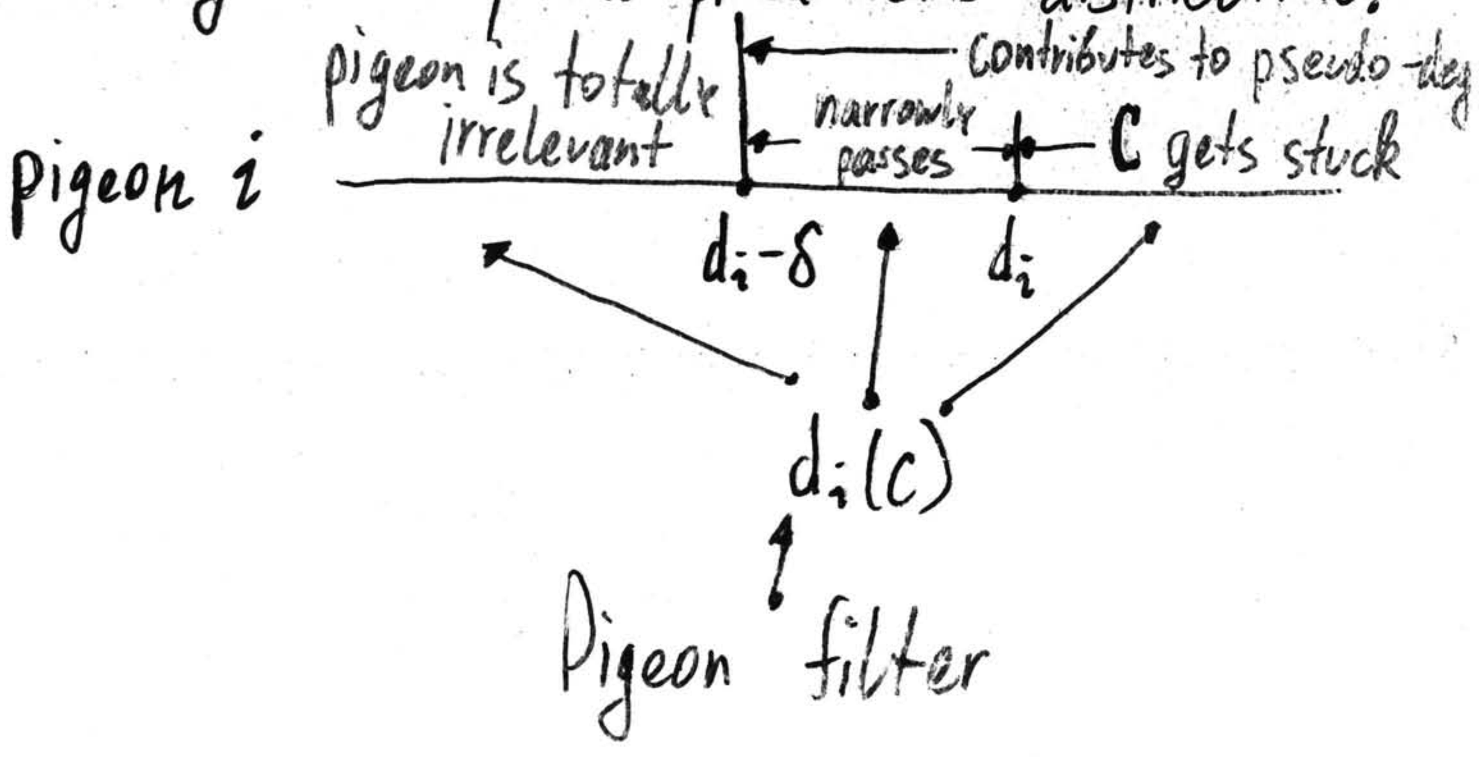
It is easier to tell apart clauses C of large pseudo-width from those of small pseudo-width.

δ is a parameter ($\delta = n^{2/3}/2$).

$$W(C) = \sum_{i=1}^{m} 2^{d_i(C)/\delta}$$

C is large if $W(C)$ exceeds a certain threshold and small otherwise.

$(d_1, \ldots, d_m)$ is a degree sequence picked according to a special probabilistic distribution.



Pigeon filter lemma. If the size of a proof is small ($< \exp(\varepsilon \cdot n^{1/3})$) then with high probability we have:

1°. C is large $\Rightarrow$ many ($\geq \underbrace{n^{1/3}}_{= w_0}$) pigeons get stuck.

2°. C is small $\Rightarrow$ relatively few ($\leq \underbrace{10 n^{1/3}}_{= 10 w_0}$) pigeons contribute to pseudo-degre (i.e., get stuck or pass narrowly).

$$\boxed{\text{Fix once and for all } (d_1, \dots, d_m) \text{ with these properties}}$$

<u>Definition</u>. (d, w_0)-axiom is any positive clause C such that for w_0 pigeons $i_1, \dots, i_{w_0}$ $d_i(C) = d_i$ and for all others $d_i(C) = 0$.

$$\boxed{\text{Every large clause } C \text{ in the proof contains a } (d, w_0)\text{-axiom}}$$

<u>Definition</u>. Pseudo-width of a clause C is the number of pigeons i for which $d_i(C) \geq d_i - \delta$

$$\boxed{\text{Every small clause } C \text{ has pseudo-width } \leq 10 w_0}$$

Reduction of pseudo-width

We have transformed our original proof into a proof of small ($\le 10 w_0$) pseudo-width but with not too many additional (d, w_0)-axioms.

Bounding pseudo-width

First version (April 01) – plain method similar to [BW99] but unfortunately not good for the functional version of PHP_n^m.

Second method (July 01) – uses ad hoc algebraic trick.

Linear evaluations

L is a linear space. For a positive clause C we define a subspace $\phi(C) \leq L$.

<u>Main Property</u>. If C is obtained from C_0, C_1 via a single application of the inference rule and pseudo-width of C_0, C_1 is at most $10 w_0$ then $\phi(C) \leq \phi(C_0) + \phi(C_1)$.

When we proceed along a proof of small pseudo-width, the subspace spanned by $\phi(C)$ remains the same

C is a (d, w_0)-axiom $\Rightarrow$ $\dim(\phi(C))$ is small

$$\phi(0) = L.$$

V. Open Problems

- Does the (moderately, very) weak pigeonhole principle have polynomial size bounded-depth Frege proofs?
- Does the (moderately, very) weak pigeonhole principle have quasi-polynomial size $R(O(1))$-proofs? (yes: $R(\text{poly-log})$, no: $R(2)$)?
- $$\exp(\Omega(n^{1/3})) \leq S_R(\text{fun-}PHP^m_n) \leq \exp(\tilde{O}(n^{1/2}))$$

  What is the right value of this $\varepsilon \in [1/3, 1/2]$.

- Resolution lower bounds for onto-PHP^m_n
- Degree lower bounds for Polynomial Calculus proofs of onto-PHP^m_n in characteristic $p | m-n$ ([Beame, Cook, Edmonds, Impagliazzo, Pitassi 95; Beame, Riis 97] for the weaker Nullstellensatz proof system).