Metric entropy, KAM theory and Dual lens maps

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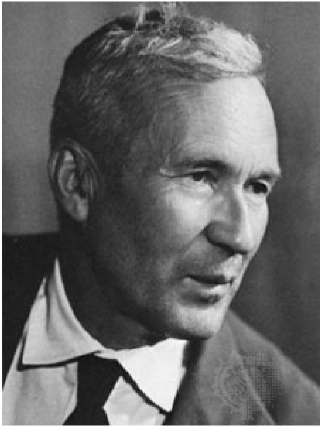


A.N.Kolmogorov
(1903-1987)

Can a
KAM system ('54)
has positive
Kolmogorov-Sinai
entropy ('58,'59)
?

Hamiltonian systems and geodesic flows

- Let (Ω, ω) be an $2n$-dimensional symplectic manifold. Let H be a smooth function on Ω. We can define the **Hamiltonian vector field** X_H as the unique solution to the equation

$$\omega(X_H, V) = dH(V)$$

  for any smooth vector field V on Ω. The flow Φ_H^t on M defined by $\dot{\Phi}_H^t = X_H$ is called **the Hamiltonian flow on Ω with Hamiltonian H**.
- A typical example of Hamiltonian systems is geodesic flows. Let (M, φ) be a Finsler manifold, i.e. φ is a quadratically convex norm on each tangent space. The cotangent bundle T^*M has a natural symplectic form ω.

Hamiltonian systems and geodesic flows

We define the dual norm on cotangent bundle T^*M by

$$\varphi^*(\alpha) := \sup_{v \in UT_xM} \{\alpha(v)\}, \text{ for } \alpha \in T_x^*M.$$

The geodesic flow on (M, φ) is defined to be the Hamiltonian flow on T^*M with Hamiltonian $(\varphi^*)^2/2$. One can just regard the geodesic flow on a Finsler manifold as a standard 2-homogeneous Hamiltonian flow on the cotangent bundle T^*M.
A Finsler metric φ is **reversible** if the norm/Hamiltonian is symmetric with respect to the cotangent vectors.

- Example: geodesic flow on Euclidean torus $\mathbb{T}^n$, geodesic flow on standard sphere $S^{n-1} \subseteq \mathbb{R}^n$.

Introduction to KAM Theory

- A Hamiltonian system on (Ω, ω) with degree of freedom $2n$ is **completely integrable** if there exists n integrals with pairwise vanishing Poisson brackets. If in addition some level set is compact, this level set will be foliated by invariant tori, on which the dynamics are conjugate to linear flows.
- Examples: geodesic flow on flat torus, Solar systems without mutual gravitational force between planets, geodesic flow on standard sphere.
- Action-angle coordinates: for a completely integrable Hamiltonian system, we can find a coordinate system $(\mathbf{I}, \mathbf{\Phi})$ on Ω such that the Hamiltonian is $H = H(\mathbf{I})$.
  In this coordinate the flow is given by the following equations:

$$\dot{\mathbf{I}} = 0, \dot{\mathbf{\Phi}} = \partial H / \partial \mathbf{I}.$$

Introduction to KAM Theory

- A completely integrable system $H = H(\mathbf{I})$ is called **nondegenrate** if the Hessian $\det \partial^2 H/\partial \mathbf{I}^2 \neq 0$. The geodesic flow on a flat torus is nondegenrate but that on a standard sphere is not.
- Kolmogorov, Arnol'd and Moser showed that, after perturbing a nondegenrate completely integrable system, the resulting flow still contains a set of overwhelming measure of invariant tori (a.k.a. **KAM tori**), on which the dynamics are conjugate to linear flows.
- These resulting systems are called **KAM-nondegenrate nearly integrable systems**, or **KAM systems** for short.
- What happens outside KAM tori?

Kolmogorov-Sinai Entropy

Let (X, μ) be a probability space and $T : (X, \mu) \to (X, \mu)$ be a measure preserving automorphism. For any measuarble partition ξ of X, define

$$F(\xi) = -\sum_{C \in \eta} \mu(C) \log \mu(C).$$

For any $n \in \mathbb{N}$ we define a partition ξ^T_{-n} by

$$\xi^T_{-n} := \eta \vee T^{-1}(\xi) \vee \cdots \vee T^{-n+1}(\xi),$$

where $\xi_1 \vee \xi_2 := \{C \cap D : C \in \xi_1, D \in \xi_2\}$ for measurable partitions ξ_1, ξ_2. We define

$$h_\mu(T, \xi) := \lim_{n \to \infty} \frac{F(\xi^T_{-n})}{n}.$$

Such limit exists since $F(\xi^T_{-m-n}) < F(\xi^T_{-m}) + F(\xi^T_{-n})$.

Kolmogorov-Sinai Entropy

- Example: $(X, \mu) = (S^1 = \mathbb{R}/\mathbb{Z}, Leb)$. The map T is given by

$$T : S^1 \to S^1, T(x) = 2x(mod 1).$$

  The partition is the trivial partition. Then $h_\mu(T, \xi) = \log 2$.
- The **Kolmogorov-Sinai entropy** (or metric entropy) of T is given by

$$h_\mu(T) := \sup_{\xi : h_\mu(T,\xi) < \infty} h_\mu(T, \xi).$$

- The metric entropy of a Hamiltonian flow Φ_H^t is defined to be $h_\mu(\Phi_H^1)$ where $\mu = \omega^n$ is the volume form.
- Anosov proved that the geodesic flows on negative curved manifolds have positive metric entropy.

Topological Entropy

- Let (X, d) be a metric space and $T : X \to X$ be a continuous map. For any ϵ, n we define

$$B_T(x, n, \epsilon) := \{y \in X : d(T^i x, T^i y) < \epsilon, \forall\, 0 \leq i \leq n-1.$$

A set $E \subseteq X$ is called (n, ϵ)-spanning if $X \subseteq \cup_{x \in E} B_T(x, n, \epsilon)$. Let $S_d(T, \epsilon, n)$ be the minimal cardinality of an (n, ϵ)-spanning set. Define the **topological entropy** of T by

$$h_d(T) := \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log S_d(T, \epsilon, n).$$

- Example: (S^1, T), $Tx = 2x$, $h_d(T) = \log 2$.
- Variational principle: $h_d(T) = \sup_\mu h_\mu(T)$.
- Geodesic flow on flat torus has zero topological entropy, hence zero metric entropy.

Genericity of Positive Topological Entropy

What happens outside the KAM tori?

- A C^2-generic Hamiltonian flow has positive topological entropy. (Newhouse, 77')
- A C^2-generic Riemannian metric on S^2 has positive topological entropy (Contreras-Paternain, '02).
- A C^2-generic Riemannian metric on any closed manifold has positive topological entropy (Contreras, '10).

What about the metric entropy?

- Positive metric entropy $\Rightarrow$ Positive topological entropy, but not vice versa (Bolsinov-Taimanov, '00)

In 2016, D. Burago and S. Ivanov constructed a reversible Finsler metric $S^n (n \geq 4)$ that is C^∞ close to the standard metric, and its geodesic flow has positive metric entropy. Moreover the flow is **entropy non-expansive**, i.e. positive entropy comes from a small neighborhood of an orbit.
However, the geodesic flow on S^n is degenerate. Hence it does not fit the KAM model.

Main Theorems

Theorem 1 (C.)

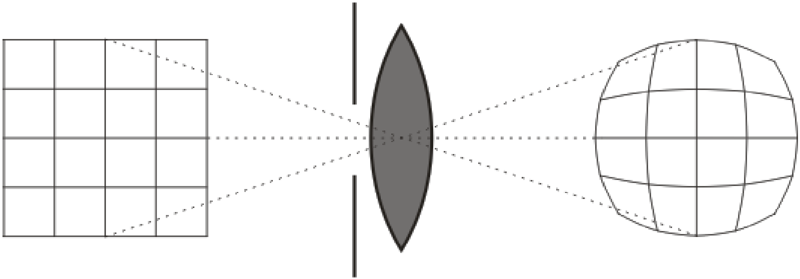
For all $n \geq 3$ and $\epsilon > 0$ there exists a reversible Finsler metric on $\mathbb{T}^n$ which is ϵ-close to the Euclidean metric in the C^∞-sense and such that the associated geodesic flow has positive metric entropy.

Theorem 2 (Burago-C.-Ivanov)

For all $n \geq 4$ and $\epsilon > 0$ there exists a reversible Finsler metric on $\mathbb{T}^n$ which is ϵ-close to the Euclidean metric in the C^∞-sense and such that its geodesic flow has positive metric entropy and is entropy non-expansive.

Both theorems imply positive metric entropy can appear outside KAM tori.

Lens map



Source: smus.com

Lens map

A Finsler metric φ on a disc $D = D^n$ is called **simple** if it satisfies:

- Every pair of points in D is connected by a unique geodesic.
- Geodesics depend smoothly on their endpoints.
- The boundary is strictly convex, that is, geodesics never touch it at their interior points.

Once (D, φ) is simple, denote by U_{in}, U_{out} the set of inward, outward pointing unit tangent vectors with base points in ∂D.
We define a **lens map** $\beta : U_{in} \to U_{out}$ by associating a vector $v \in U_{in}$ with the first intersection of U_{out} and the orbit of v under the geodesic flow. If φ is reversible then $-\beta(-\beta(v)) = v$.

Dual lens map

Let $\mathscr{L} : TD \to T^*D$ be the Legendre transform with Lagrangian $\varphi^2/2$ and $U^*_{in} := \mathscr{L}(U_{in}), U^*_{out} := \mathscr{L}(U_{out})$. The symplectic structures on U^*_{in} and U^*_{out} are induced by that on T^*D. We define the **dual lens map** $\sigma : U^*_{in} \to U^*_{out}$ by $\sigma := \mathscr{L} \circ \beta \circ \mathscr{L}^{-1}$.
Note that σ is symplectic. If φ is reversible then σ is **symmetric** in the sense $-\sigma(-\sigma(\alpha)) = \alpha$ for all $\alpha \in U^*_{in}$.

Theorem 3 (Burago-Ivanov, '16)

For any $n \geq 3$ any W the complement of a compact set in U^*_{in}, every sufficiently small symplectic perturbation $\tilde{\sigma}$ of σ such that $\tilde{\sigma}|_W = \sigma|_W$ can realized as the dual lens map of a simple metric $\tilde{\varphi}$ which coincides with φ in some neighborhood of ∂D.
Moreover, $\tilde{\varphi} \to \varphi$ in C^∞ as $\tilde{\sigma} \to \sigma$. If $\tilde{\sigma}$ is symmetric, then $\tilde{\varphi}$ can be chosen to be reversible.

Perturbation on S^4

Lemma 4

For all $n \geq 3$, there exists a symplectomorphism $\theta : D^{2n} \to D^{2n}$ which is arbitrarily close to the identity in C^∞, coincides with the identity map near the boundary, and has positive metric entropy.

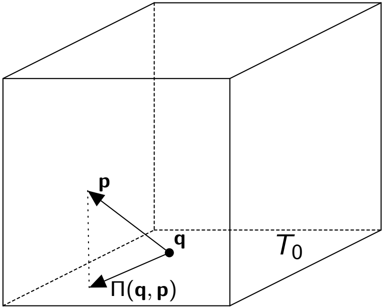
Idea of proof.

Let Σ be a Riemannian surface with genus 2. Denote $I := (-2, 2)^{n-2}$ and $N := \Sigma \times I$. We construct a Hamiltonian flow on T^*N containing an invariant subset B^*N on which the motion is a product of the geodesic flow on Σ with trivial flow on I. We take $\theta := \Phi_H^t$ for small t.

$\square$

Perturbation on tori

For $n \geq 3$, φ_0 — the Euclidean metric on $\mathbb{T}^n$. We regard $\mathbb{T}^n$ as the cube $[-1,1]^n$ with sides identified. $UT^*\mathbb{T}^n$ — unit cotangent bundle with $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. $T_0 := [-1,1]^{n-1} \times \{-1\}$ — "bottom face" of $\mathbb{T}^n$. Consider the Poincaré map R_0 of the geodesic flow onto the section $\Gamma_0 := \{\mathbf{q} \in T_0, p_n > 0\}$.



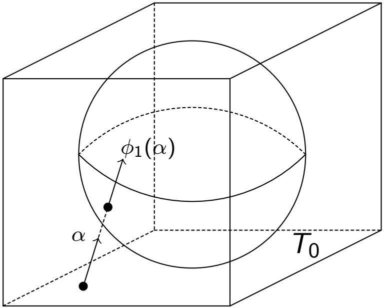
Γ_0 is bijective to B^*T_0 via the projection $\Pi : \Gamma_0 \to B^*T_0$. Let $R_1 := \Pi \circ R_0 \circ \Pi^{-1} : B^*T_0 \to B^*T_0$. Then by simple calculation:

$$R_1(\mathbf{q}, \mathbf{p}) = \left(\mathbf{q} + \frac{\mathbf{p}}{\sqrt{1 - |\mathbf{p}|^2}}, \mathbf{p} \right)$$

Perturbation on tori

Claim: For any sufficiently small compact neighborhood K of $\mathbf{q} = \mathbf{p} = \mathbf{0}$ and any perturbation $\tilde{R}_1$ of R_1 with $\tilde{R}_1|_{K^c} = R_1|_{K^c}$, there exists a Finsler metric $\tilde{\varphi}$ on $\mathbb{T}^n$ with Poincaré map

$$\tilde{R}_0 := \Pi^{-1} \circ \tilde{R}_1 \circ \Pi.$$



Indeed, let $D^n := \{\mathbf{q} \in \mathbb{R}^n : \sum_{i=1}^n q_i^2 \leq 1\}$ be an n-disc and $\sigma : U^*_{in} \to U^*_{out}$ be the dual lens map. For each $(\mathbf{q}, \mathbf{p}) \in K$, let $\alpha := \Pi^{-1}(\mathbf{q}, \mathbf{p})$, then define $\phi_1(\alpha)$ as the first intersection of orbit of α with U^*_{in}. Perturbing the Poincaré map is basically perturbing the dual lens map.

So we have only to perturb R_1.

Perturbation of R_1 when $n = 3$

Recall that when $n = 3$, $\dim T_0 = 2$, and $R_1 : B^* T_0 \to B^* T_0$ is given by

$$R_1(q_1, q_2, p_1, p_2) = \left(q_1 + \frac{p_1}{\sqrt{1 - p_1^2 - p_2^2}}, q_2 + \frac{p_2}{\sqrt{1 - p_1^2 - p_2^2}}, p_1, p_2 \right).$$

(1) $R_1 = \Phi^1_{\tilde{H}_0}$ for $\tilde{H}_0 := 1 - \sqrt{1 - p_1^2 - p_2^2}$. In order to perturb R_1 to get positive metric entropy, we have only to do so for the Hamiltonian flow $\Phi^t_{\tilde{H}_0}$, or "equivalently" for $\Phi^t_{H_0}$ with

$$H_0(q_1, q_2, p_1, p_2) := \frac{p_1^2 + p_2^2}{2}.$$

Note that $\tilde{H}_0 = 1 - \sqrt{1 - 2H_0}$.

Perturbation of R_1 when $n = 3$

(2) Construct a smooth function $g : \mathbb{T}^2 \to (0, 1]$ with the following properties:
(i) The support of Dg can be arbitrarily small.
(ii) There exists a $\delta_0 > 0$ such that the metric given by

$$ds^2 = (g(q_1, q_2)^2 + \delta)(dq_1^2 + dq_2^2)$$

has positive metric entropy for all $-\delta_0 < \delta < \delta_0$.
The idea of construction is to build up a Donnay-Burns-Gerber cap with negatively curved neck.

(3) Let $\xi : \mathbb{R}_{\geq 0} \to [0, 1]$ be a smooth function with $\xi \equiv 1$ on $[0, 1/3]$ and $\xi \equiv 0$ on $[2/3, 1]$. Define

$$H_\epsilon := H_0 + \epsilon(1 - g(q_1, q_2)^2)\xi(p_1^2 + p_2^2).$$

$H_\epsilon < \frac{1}{6} \Rightarrow p_1^2 + p_2^2 < \frac{1}{3}$, hence $\xi \equiv 1$ when H_ϵ is small.

(4) Since $\epsilon > \max \epsilon(1 - g^2)$, by Maupertuis Principle, for each $\delta \in (-\delta_0, \delta_0)$, the Hamiltonian flow on the level set $\{H_\epsilon = \epsilon + \epsilon\delta\}$ has the same orbits as the geodesic flow on $\mathbb{T}^2$ with metric

$$ds^2 = \epsilon(g(q_1, q_2)^2 + \delta)(dx^2 + dy^2).$$

This geodesic flow has positive metric entropy, so does the flow on $\{H_\epsilon = \epsilon + \epsilon\delta\}$ for all $-\delta_0 < \delta < \delta_0$, which implies the Hamiltonian flow $\Phi^t_{H_\epsilon}$ has positive metric entropy.

(5) Define

$$\tilde{H}_\epsilon := 1 - \sqrt{1 - 2H_\epsilon}.$$

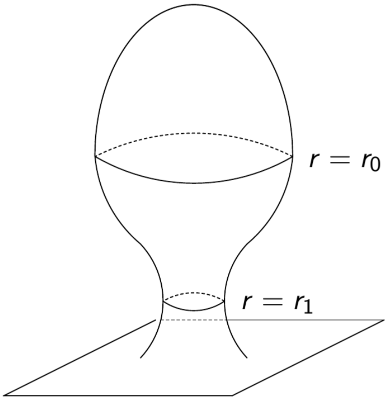
Then $\Phi^1_{\tilde{H}_\epsilon}$ has positive metric entropy since $\Phi^1_{H_\epsilon}$ does. Hence $\Phi^1_{\tilde{H}_\epsilon}$ is a C^∞-small perturbation of $R_1 = \Phi^1_{\tilde{H}_0}$ with positive metric entropy. This proves Theorem 1.

Non-ergodic Donnay-Burns-Gerber torus

We say a centrally symmetric cap $\mathscr{C} = \{r \leq r_1\} \subseteq \mathbb{R}^2$ is a **non-ergodic Donnay-Burns-Gerber (DBG) cap** if it satisfies:

- $\mathscr{C}$ has two paralell geodesics $r = r_0$ and $r = r_1$.
- The Gaussian curvature is positive on $\{r \leq r_0\}$, negative at $r = r_1$, and strictly decreasing from center to boundary.

If a torus contains a non-ergodic DBG cap and outside the cap the Gaussian curvature is nonpositive, then we call this torus a **non-ergodic DBG torus**.

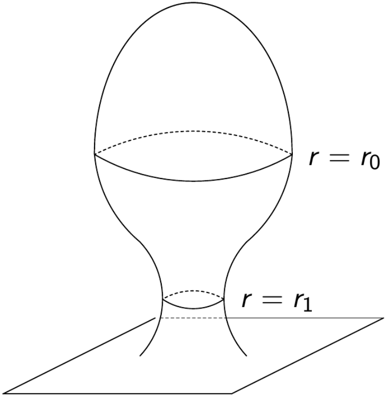


Construction of a conformal Non-ergodic DBG torus

Lemma 5

The geodesic flow on a non-ergodic DBG torus has positive metric entropy.

The proof is an imitation of the proof in Burns-Gerber. We use Wojkowski's cone field theory to prove there exists a positive measure set with nonzero maximum Lyapunov exponents.



Lemma 6

There is a C^∞ function $g : \mathbb{T}^2 \to (0,1]$ s.t. the metric given by

$$ds^2 = g(q_1, q_2)^2(dq_1^2 + dq_2^2)$$

is a non-ergodic DBG torus. Moreover, the support of Dg can be arbitrarily small.

Since the conditions of non-ergodic DBG are open, there exists a $\delta_0 > 0$ such that for all $-\delta_0 < \delta < \delta_0$, the metric given by

$$ds^2 = (g(q_1, q_2)^2 + \delta)(dq_1^2 + dq_2^2)$$

is also non-ergodic DBG.

Perturbation of R_1 when $n \geq 4$, entropy nonexpansive

I will show the proof for $n = 4$, dim $T_0 = 3$.

$$R_1 : B^* T_0 \to B^* T_0, \quad R_1(\mathbf{q}, \mathbf{p}) = \left(\mathbf{q} + \frac{\mathbf{p}}{\sqrt{1 - |\mathbf{p}|^2}}, \mathbf{p} \right)$$

(1) Define a function ψ on $[0, 1)$ by

$$\sqrt{1-t} = 1 - \frac{1}{2} t \psi^2(t), \text{ for } t > 0,$$

and $\psi(0) := \lim\limits_{t \to 0^+} \psi(t) = 1$. Then ψ is C^∞, positive and strictly increasing.

(2) Define $\mathbf{P} := \mathbf{p}\, \psi(|\mathbf{p}|^2)$. Then

$$R_1 = \Phi^1_{\tilde{H}_0}, \text{ for } \tilde{H}_0 = 1 - \sqrt{1 - |\mathbf{p}|^2} = \frac{|\mathbf{P}|^2}{2}.$$

$$\mathbf{P} := \mathbf{p}\,\psi(|\mathbf{p}|^2).$$

We make a coordinate change $\mathbf{q} \to \mathbf{Q}$ so that

$$\sum_{i=1}^{3} dQ_i \wedge dP_i = \sum_{i=1}^{3} dq_i \wedge dp_i.$$

(3) With this new coordinate we can define a perturbed Hamiltonian:

$$\tilde{H}_\epsilon(\mathbf{Q}, \mathbf{P}) := \tilde{H}_0 + \frac{\epsilon|\mathbf{Q}|^2}{2} B(\mathbf{Q}, \mathbf{P})$$

where B is some bump function. On low energy levels, $B \equiv 1$, so $\Phi^t_{\tilde{H}_\epsilon}$ is a rotation with period $2\pi/\sqrt{\epsilon}$.

(4) Choose appropriate $\epsilon > 0$ so that $N := 2\pi/\sqrt{\epsilon}$ is an integer. Let $T := \Phi^1_{\tilde{H}_\epsilon}$. By choosing small ϵ, $T \overset{C^\infty}{\approx} \Phi^1_{\tilde{H}_0} = R_1$ and $T^N = id$ on low level sets. Find a 6-disc D such that $D, T(D), \dots, T^{N-1}(D)$ are disjoint.

(5) For all $n \geq 3$, we constructed a symplectomorphism $\theta : D^{2n} \to D^{2n}$ which is arbitrarily close to the identity in C^∞, coincides with the identity map near the boundary, and has positive metric entropy.

(6) Extend the map $\theta : D \to D$ by identity to the whole $B^* T_0$. $(T \circ \theta)^N = \theta$ on D, hence $(T \circ \theta)^N$ has positive metric entropy, so does $T \circ \theta$.
$T \overset{C^\infty}{\approx} R_1$, $\theta \overset{C^\infty}{\approx} id$, so $T \circ \theta \overset{C^\infty}{\approx} R_1$. This proves Theorem 2.

Extension to general completely integrable systems, a 99% theorem

Theorem 7 (Burago-C.-Ivanov)

Let Φ_H^t be a completely integrable Hamiltonian flow on a symplectic manifold (M^{2n}, ω) with $n \geq 4$. For any $c \in \mathbb{R}$ and invariant torus $\mathbb{T} \subseteq H^{-1}(c)$ satisfying the following conditions:
(1) c is a regular value of H;
(2) The dynamic on $\mathbb{T}$ is periodic,
we can find a C^∞-small perturbation $\tilde{H}$ of H such that the Hamiltonian flow $\Phi_{\tilde{H}}^t$ on M has positive metric entropy. Moreover such perturbation can be made on an arbitrarily small neighborhood of any closed orbit in $\mathbb{T}$.

Further questions

- Can we lower the dimension?
- Can we get Riemannian metric instead of reversible Finsler metric?
- Is positive metric entropy property generic among nearly integrable systems?
- Etc.....

Thank you!