Shadowing and density of minimal points

Danila Cherkashin, Sergey Kryzhevich

Geneva University, Geneva, Switzerland,
and Saint Petersburg State University, St. Petersburg, Russia
Saint Petersburg State University, St. Petersburg, Russia,
and University of Nova Gorica, Nova Gorica, Slovenia

kryzhevicz@gmail.com

We study structural stability and shadowing in dynamical systems by means of Topological Dynamics. Consider a homeomorphism T of a compact metric space (X, ρ) to itself. Let $\mathrm{M}(X,T) \subset \mathrm{R}(X,T) \subset \Omega(X,T) \subset \mathrm{CR}(X,T)$ be sets of minimal, recurrent, non-wandering and chain recurrent points of T. Let NW be the class of dynamical systems with $X = \Omega(X,T)$.

Definition 1. Let $d > 0$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ is a *d-pseudotrajectory* if $\rho(x_{k+1}, T(x_k)) \leq d$ for all $k \in \mathbb{N}$.

Definition 2. An ε-network Y is *almost invariant* if for every $n \in \mathbb{Z}$ the set $T^n(Y)$ is an ε-network. Denote by Q the class of systems (X,T) that have finite almost invariant ε-networks for every $\varepsilon > 0$.

Definition 3. Let W be the class of dynamical systems (X,T) such that for any $\varepsilon > 0$ there exists a $d > 0$: for any d-pseudotrajectory $\{x_k\}$ there exist points $y^1, \dots, y^N$ such that x_k is ε close to one of points $T^k(y^i)$ for all $k \in \mathbb{N}$. Then the system (X,T) is said to satisfy the *multishadowing* property.

Proposition. *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any δ-pseudotrajectory $p = \{p_k\}$ of the map T*

$$\liminf_{N\to\infty} \# K_\varepsilon \bigcap [0,N]/N > 1 - \varepsilon.$$

Here $K_\varepsilon = \{k \geq 0 \colon p_k \in U_\varepsilon(\mathrm{R}(X,T))\}$ where $U_\varepsilon(\mathrm{R}(X,T))$ is the ε-neighborhood of all recurrent points in X.

Theorem. 1. $\mathrm{Q} = \mathrm{W} \bigcap \mathrm{NW}$.

2. *For any homeomorphism from the class* Q *there exists a probability invariant measure, supported on all X.*

3. $(X,T) \in \mathrm{W}$ *if and only if* $\mathrm{CR}(X,T) = \overline{\mathrm{M}(X,T)}$.

This fact implies some interesting corollaries, e.g. the following.

Corollary. *Let X be a C^1 smooth compact manifold. There exists a residual subset $\mathrm{Z} \subset \mathrm{Diff}^1(X)$ such that for any $T \in \mathrm{Z}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathrm{dist}_{C^0}(S, T) < \delta$ implies $\Omega(X, S) \subset U_\varepsilon(\Omega(X, T))$.*

We also discuss some other corollaries of the formulated theorem.