Fast–slow partially hyperbolic systems and pathological foliations*

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In [2–4] the first two authors studied the following class of partially hyperbolic systems of the fast–slow type on $\mathbb{T}^2$

$$F_\varepsilon(x,\theta) = (f(x,\theta), \theta + \varepsilon\omega(x,\theta)) \mod 1, \tag{1}$$

with $\varepsilon > 0$, small, $F_\varepsilon \in \mathcal{C}^5(\mathbb{T}^2, \mathbb{T}^2)$, and $\inf_{x,\theta} \partial_x f(x,\theta) \geq \lambda > 1$, $\|\omega\|_{\mathcal{C}^4} = 1$. As usual it is important to specify the type of initial conditions under which we like to study the dynamical systems $(\mathbb{T}^2, F_\varepsilon)$. It is well known that, in order to be able to obtain meaningful results for long times, they must be random. More precisely, if we define $(x_n, \theta_n) = F_\varepsilon^n(x_0, \theta_0)$, then we would like to consider, at least, the initial condition $\theta_0 \in \mathbb{T}^1$ fixed, while $x_0 \in \mathbb{T}^1$ is distributed according to a probability measure with smooth density w.r.t. Lebesgue. Then (x_n, θ_n) can be viewed as a (Markov) random process.

Even though (1) is arguably the simplest possible model problem for a fast–slow partially hyperbolic system, its exact properties are not understood in full generality. If we want to develop a general theory for fast–slow partially hyperbolic systems, it is then important to see were do we stand and what are the open problems for the above basic model.

In [1], we studied the statistical properties of (1) in a much greater detail. In my presentation, however, I will only focus on the geometrical implica-

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tions of these properties. In particular, in [1] we showed that, contrary to naive intuition, it is possible that the central Lyapunov exponent χ_c is positive, despite having a statistical sink. We also proved below that F_ε has an invariant foliation made of smooth compact leaves tangent to the central distribution. If $\chi_c > 0$, these leaves have to expand in average but at the same time their length is uniformly bounded.

The reason why this is not contradictory is that the center foliation fails to be absolutely continuous. This means that, despite each leaf being individually smooth, the foliation as a whole is very wild. This situation is strange but known to happen, see the papers of Ruelle, Shub and Wilkinson [7, 6] where they presented an open set of volume preserving partially hyperbolic systems with non absolutely continuous central foliation for a perturbation of the product of an Anosov map by an identity map on the circle. This behaviour was later observed in many other partially hyperbolic systems.

Proposition 1. *There exists a C^1-open set $\mathcal{U}_\varepsilon$ such that for any $F \in \mathcal{U}_\varepsilon$ we have $\chi_c > 0$.*

In particular, if $F \in \mathcal{U}_\varepsilon$ has a physical measure, then it must have positive Lyapunov exponents.

Theorem 1. *For every map F from $\mathcal{U}_\varepsilon$*

1. *the central distribution E^c is uniquely integrable to a C^1 foliation $\mathcal{W}^c$;*
2. *if, in addition, F has j-pinching, $j \geq 1$, then $\mathcal{W}^c$ is C^j;*
3. *every leaf $W \in \mathcal{W}^c$ is a diffeomorphic circle of uniformly bounded length;*
4. *if F has a physical measure, then $\mathcal{W}^c$ is not absolutely continuous.*

Remark 1. According to Tsujii [5], the existence of the physical measure is generic and hence it holds generically for $F \in \mathcal{U}_\varepsilon$. Accordingly, the above Theorem implies that generically $\mathcal{W}^c$ is not absolutely continuous.

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