Lyapunov unstable global attractors

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A global attractor is a set that attracts most points of the phase space. One of the definitions of a global attractor was presented by J. Milnor in [1]: *the likely limit set* is the smallest closed subset of the phase space that contains the ω-limit set for Lebesgue-a.e. point. This definition will be our primary example; however, the list of other, nonequivalent definitions includes *minimal* and *statistical* attractors and *the generic limit set*. The results below apply to any definition of global attractor provided that the attractor thus defined

- exists and is unique for any mapping in the class under consideration (say, for any diffeomorphism of a given smooth closed manifold),
- is contained in the nonwandering set,
- is closed,
- contains every hyperbolic sink.

Assume from now on that such a definition of global attractor is fixed.

Recall that a subset of the phase space is called *Lyapunov stable* if for any its neighborhood there is a smaller neighborhood such that the orbits which start at the latter never quit the first one. This is a nice property which topological attractors, such as sinks, have. Yet it is easy to give an example of a diffeomorphism whose global attractor is Lyapunov unstable: consider a diffeomorphism of a circle with a single semi-stable fixed point.

It turns out that the Lyapunov instability of global attractors is a locally topologically generic phenomenon closely related to *the Newhouse phenomenon*, i.e., to existence of open subsets in the space of diffeomorphisms where the maps exhibiting a homoclinic tangency associated with a continuation of a single hyperbolic saddle are dense and where coexistence of infinitely many sinks is generic.

Theorem. *Suppose that in an open set of diffeomorphisms there is a persistent homoclinic tangency associated with a sectionally dissipative periodic saddle. Then for a topologically generic diffeomorphism in this set the global attractor is Lyapunov unstable.*

The argument that proves this result can be applied with minor modifications to the case of P. Berger's locally topologically generic finite-parameter families of diffeomorphisms with robustly infinitely many sinks (see [2]). This yields the following result.

Theorem. *There exist locally topologically generic finite-parameter families of diffeomorphisms where the unit ball in the parameter space corresponds to diffeomorphisms with Lyapunov unstable global attractors.*

For any C^2-smooth diffeomorphism, as well as for a C^1-generic one, if this diffeomorphism satisfies axiom A, its likely limit set is Lyapunov stable (same for the minimal and statistical attractors and the generic limit set). When hyperbolicity is violated strongly enough, we have the contrary, as the following result shows.

Theorem. *If a topologically generic C^1-diffeomorphism of a closed manifold has a homoclinic class that does not admit a dominated splitting, the global attractor is unstable for this diffeomorphism or for its inverse.*

References

1. *Milnor J.* On the concept of attractor // Commun. Math. Phys. 1985. V. 99. P. 177–195.
2. *Berger P.* Generic family with robustly infinitely many sinks // Invent. Math. 2016. V. 205, N 1. P. 121–172; arXiv: 1411.6441.
3. *Shilin I.* Locally topologically generic diffeomorphisms with Lyapunov unstable Milnor attractors: E-print. arXiv: 1604.02437.