Main subspaces of the space of C^1-smooth skew products of interval maps*

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We consider the space $\widetilde{T}^1_*(I)$ of C^1-smooth skew products of maps of an interval with invariant boundary of a closed rectangle $I = I_1 \times I_2$ (I_1, I_2 are closed intervals) such that quotient map $f\colon I_1 \to I_1$ of a skew product $F \in \widetilde{T}^1_*(I)$ is Ω-stable[1] in the space of C^1-smooth maps of interval I_1 into itself with invariant boundary of interval I_1. Here

$$F(x,\, y) = (f(x),\, g_x(y)) \;\; g_x(y) = g(x,\, y) \;\text{ for any point }\; (x,\, y) \in I.$$

In addition, we suppose that a map $F \in \widetilde{T}^1_*(I)$ has quotient of type $\succ 2^\infty$, i. e. f has a periodic orbit with a (least) period $\notin \{2^i\}_{i\geq 0}$.

Our considerations are based on use of the special multifunctions such as the Ω-function, auxiliary and suitable functions (for details see, e.g., [2]).

We formulate Decomposition Theorem [2] for the above space of skew products. This Theorem makes it possible to present the considering space as the union of nonempty pairwise disjoint subspaces $\widetilde{T}^1_{*,\,j}(I)$ for $j = 1, 2, 3, 4$.

These main subspaces are distinguished in accordance with continuity property of auxiliary multifunctions (as $\widetilde{T}^1_{*,\,1}(I)$) or in accordance with continuity property of suitable (but not auxiliary) multifunctions (as $\widetilde{T}^1_{*,\,2}(I)$); or vise versa, in accordance with discontinuity property of suitable multifunctions in combination with continuity property of the Ω-function (as $\widetilde{T}^1_{*,\,3}(I)$) or in combination with discontinuity property of the Ω-function (as $\widetilde{T}^1_{*,\,4}(I)$). Thus, Decomposition Theorem gives implicit description of subspaces $\widetilde{T}^1_{*,\,j}(I)$ for $1 \leq j \leq 4$.

We distinguish important nonempty subsets of the above subspaces and investigate approximate properties of skew products from distinguished subsets. Explicit description of these subsets of skew products is based, in the

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[1] The paper [1] contains deep analysis of the concept of Ω-stability for maps on manifolds with boundary.

first place, on the concept of stability as a whole in C^1-norm of family of fibers maps [3] and, in the second place, on the concept of dense stability as a whole in C^1-norm of family of fibers maps.

To give the Definition we need some information on iterations of a skew product F and their special presentation. So, for any $n \geq 1$ and any point $(x, y) \in I$ the following equality holds

$$F^n(x, y) = (f^n(x), g_{x,n}(y)), \text{ where } g_{x,n}(y) = g_{f^{n-1}(x)} \circ \ldots \circ g_x(y).$$

Any iteration of a skew product F can be presented as composition of two maps $F_n\colon I \to I$, where $F_n(x, y) = (id(x), g_{x,n}(y))$, and $F_{n,1}\colon I \to I$, where $F_{n,1}(x, y) = (f^n(x), id(y))$ (here $id(x)$ and $id(y)$ are identity maps of intervals I_1 and I_2 respectively). Let us give the exact formula:

$$F^n = F_{n,1} \circ F_n.$$

Definition. We say that family of fibers maps of a skew product $F \in \widetilde{T}^1_*(I)$ with quotient of type $\succ 2^\infty$ *is stable as a whole in C^1-norm* if for any $\delta > 0$ there is a neighborhood $B^1_\varepsilon(F)$ of a map F in the space $\widetilde{T}^1_*(I)$ such that for every map $\Phi \in B^1_\varepsilon(F)$ and for return times l^*_i ($i \geq i^*$ for some $i^* \geq i_*$) of trajectories from f-nonwandering set $\Omega(f)$ there exists δ-closed to the identity map in C^0-norm homeomorphism $H^{\langle l^*_i \rangle}\colon \overline{\eta}^F_{l^*_i} \to \overline{\eta}^\Phi_{l^*_i}$ of skew products class satisfying: maps $F_{l^*_i\,|\Omega(f)\times I_2}$ and $\Phi_{l^*_i\,|\Omega(\varphi)\times I_2}$ are Ω-conjugate with respect to $H^{\langle l^*_i \rangle}$, where $\overline{\eta}^{(\cdot)}{}_{l^*_i}$ is the graph in I of the suitable function corresponding to l^*_i-th iteration of a map.

The property of *dense stability as a whole in C^1-norm* of family of fibers maps means breakdown of the above property on the nonempty closed nowhere dense subset of f-nonwandering set.

The property of stability as a whole in C^1-norm of family of fibers maps of a skew product $F \in \widetilde{T}^1_*(I)$ selects C^1-smooth Ω-stable skew products (with respect to homeomorphisms of skew products class) [4].

Theorem 1. *A skew product $F \in \widetilde{T}^1_*(I)$ with quotient map of type $\succ 2^\infty$ is Ω-stable in C^1-norm if and only if its family of fibers maps is stable as a whole in C^1-norm. Moreover, Ω-stable skew products with quotient of type $\succ 2^\infty$ are contained in the subspace $\widetilde{T}^1_{*,1}(I)$.*

The claim of the following Theorem 2 means that Ω-stable skew products with quotient of type $\succ 2^\infty$ are not dense in $\widetilde{T}^1_{*,1}(I)$ (see [4]).

Theorem 2. *There exists a skew product $F \in \widetilde{T}^1_{*,1}(I)$ with quotient map of type $\succ 2^\infty$ such that some its neighborhood $B^1_\varepsilon(F)$ in the space $\widetilde{T}^1_{*,1}(I)$ does not contain Ω-stable skew products of maps of an interval.*

Let us consider skew products with densely stable as a whole in C^1-norm families of fibers maps.

Theorem 3. *For any $j = 1, 2, 3, 4$ there exists a map $F_j \in \widetilde{T}^1_{*,\,j}(I)$ with quotient of type $\succ 2^\infty$ and densely stable as a whole in C^1-norm family of fibers maps.*

The following Theorem 4 gives criterion of approximability in C^1-norm of skew products with quotient map of type $\succ 2^\infty$ and densely stable as a whole family of fibers maps by means of Ω-stable skew products.

Theorem 4. *Let $F \in \widetilde{T}^1_{*,\,j}(I)$ ($j = 1, 3$ or 4) be a skew product with quotient of type $\succ 2^\infty$ and densely stable as a whole in C^1-norm family of fibers maps.*

Then F admits approximation in C^1-norm by means of C^1-smooth Ω-stable skew products of maps of an interval with an arbitrary degree of accuracy if and only if for every locally maximal f-quasiminimal set $K(f)$ and $i \geq i^*$ there exists a connected component $C_{K(f),\,i}$ of the space of C^1-smooth Ω-stable maps of interval I_2 into itself satisfying the inclusion

$$\{g_{x,\,l_i^*}\}_{x\in K(f)} \subset \overline{C}_{K(f),\,i}.$$

Remark 1. At present, examples of skew products from subspace $\widetilde{T}^1_{*,\,2}(I)$ with quotient of type $\succ 2^\infty$ and densely stable as a whole in C^1-norm family of fibers maps, that satisfy conditions of Theorem 4, are unknown.

The subspace $\widetilde{T}^1_{*,\,4}(I)$ demonstrates new unusual properties.

Theorem 5. *The set of Ω-conjugacy classes of maps from the subspace $\widetilde{T}^1_{*,\,4}(I)$ is uncountable and has cardinality $\geq \aleph_1$, where $\aleph_1$ is cardinality of the set of ordinal numbers of second class.*

Moreover, $\widetilde{T}^1_{*,\,4}(I)$ contains the subset of skew products every of which can be approximated with any degree of accuracy in C^1-norm by means of skew products from the same subset with any depth of the center, which is ordinal number of second class.

Remark 2. Analogously results of paper [5], Theorem 5 demonstrates impossibility of complete dynamical description of skew products from the subspace $\widetilde{T}^1_{*,\,4}(I)$ based on the concept of Ω-conjugacy.

References

1. *Anosov D.V.* Structurally stable systems // Trudy Mat. Inst. Steklova. 1985. V. 169. P. 59–93.

2. *Efremova L.S.* A decomposition theorem for the space of C^1-smooth skew products with complicated dynamics of the quotient map // Mat. Sb. 2013. V. 204, N 11. P. 55–82.
3. *Efremova L.S.* Set-valued functions and dynamics of skew products of interval maps // Progress in Nonlinear Science: Int. Conf. Dedicated to the 100th Anniversary of A.A. Andronov, Nizhny Novgorod, Russia. July 2–6, 2001. Proc.: Math. Problems of Nonlinear Dynamics. 2002. V. 1. P. 219–224.
4. *Efremova L.S.* Stability as a whole of a family of fibers maps and Ω-stability of C^1-smooth skew products of maps of an interval // J. Phys.: Conf. Ser. 2016. V. 692, N 1. Art. 012010.
5. *Gonchenko S.V., Turaev D.V., Shil'nikov L.P.* Homoclinic tangencies of an arbitrary order in Newhouse domains // Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz., Moscow: VINITI, 1999. V. 67. P. 69–128.