Towards the theory of solutions of the flow equations of incompressible liquid

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The Euler system of equations

$$\vec{U}_t + (\vec{U}\cdot\vec{\nabla})\vec{U} + \vec{\nabla}P = 0, \quad (\vec{\nabla}\cdot\vec{U}) = 0, \tag{1}$$

where $\vec{U} = [U(\vec{x},t), V(\vec{x},t), W(\vec{x},t)]$ and $P = P(\vec{x},t)$ are the velocity and, respectively, the pressure of liquid, after change of the variables $x = x + \alpha t,\ y = y + \beta t,\ z = z + \gamma t$ and the functions $U(x,y,z,t) = \alpha + U(x-\alpha t, y-\beta t, z-\gamma t),\ V(x,y,z,t) = \beta + V(x-\alpha t, y-\beta t, z-\gamma t),\ W(x,y,z,t) = \gamma + U(x-\alpha t, y-\beta t, z-\gamma t),\ P(x,y,z,t) = P_0 + P(x-\alpha t, y-\beta t, z-\gamma t)$, takes the form

$$P_x + UU_x + VU_y + WU_z = 0,\ P_y + UV_x + VV_y + WV_z = 0,$$

$$P_z = UW_x + VW_y + WW_z = 0, \quad U_x + V_y + W_z = 0. \tag{2}$$

To integrate the system (2) we transform it into the system of equations (in new variables)

$$VWP_x + UWP_y + UVP_z - V^2(WU)_y - W^2(VU)_z - U^2(WV)_x = 0,$$

$$(UU_x)_y + (VU_y)_y + (WU_z)_y - (UV_x)_x - (VV_x)_x - (WV_z)_x = 0,$$

$$(UU_x)_z + (VU_y)_z + (WU_z)_z - (UW_x)_x - (VW_y)_x - (W_zW)_x = 0,$$

$$(UV_x)_z + (VV_y)_z + (WV_z)_z - (UW_x)_y - (VW_y)_y - (WW_z)_y = 0, \tag{3}$$

using the relations $UU_x = P_x - VU_y - WU_z,\ VV_y = (P_y - UV_x - WV_z),\ WW_z = (P_z - UW_x - VW_y)$, and conditions of their compatibility.

As a result we find that the simplest non singular solution (1-soliton) of the system (3) has the form

$$W(\vec{x},t) = c + \frac{e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}}{1+e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}},$$

$$V(\vec{x},t) = b + \frac{e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}}{1+e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}},$$

$$U(\vec{x},t) = a - \frac{(\beta+\delta)\,e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}}{\alpha\,\left(1+e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}\right)}$$

where (a,b,c) and α,β,δ are the parameters.

More complicate solution (2-soliton) of the system (3) is

$$W(\vec{x},t) = c + \frac{e^{-x+at-(x-at)\beta+\beta\,(y-bt)+z-ct} + e^{-x+at-(x-at)\beta+\beta\,(y-bt)+2\,z-2\,ct-y+bt}}{\left(1+e^{-x+at-(x-at)\beta+\beta\,(y-bt)+z-ct}\right)\left(1+e^{-y+bt+z-ct}\right)},$$

$$V(\vec{x},t) = b + \frac{e^{-x+at-(x-at)\beta+\beta\,(y-bt)+z-ct} + e^{-x+at-(x-at)\beta+\beta\,(y-bt)+2\,z-2\,ct-y+bt}}{\left(1+e^{-x+at-(x-at)\beta+\beta\,(y-bt)+z-ct}\right)\left(1+e^{-y+bt+z-ct}\right)},$$

$$U(\vec{x},t) = a + \frac{e^{-x+at-(x-at)\beta+\beta\,(y-bt)+z-ct} + e^{-x+at-(x-at)\beta+\beta\,(y-bt)+2\,z-2\,ct-y+bt}}{\left(1+e^{-x+at-(x-at)\beta+\beta\,(y-bt)+z-ct}\right)\left(1+e^{-y+bt+z-ct}\right)}.$$

In given examples of the flows the condition $P(\vec{x},t) = const$ is satisfied.

As the flows with condition $P(x,y,z,t) \neq const$ may be considered following example

$$W(x,y,z,t) = c - 1/2\,\sin(x-at) - \sin(y-bt),$$

$$V(x,y,z,t) = b - 1/2\,\cos(x-at) + \cos(z-ct),$$

$$U(x,y,z,t) = a + \cos(y-bt) - \sin(z-ct),$$

$$P(x,y,z,t) = P_0 - 1/4\,\cos(-y+bt+x-at)$$

$$+1/4\,\cos(y-bt+x-at) + 1/4\,\cos(x-at-z+ct)$$

$$+1/4\,\cos(x-at+z-ct) + 1/2\,\sin(y-bt+z-ct) - 1/2\,\sin(y-bt-z+ct).$$

More general consideration of the system (3) allow as to formulate the

Theorem 1. *Non singular and non stationary solution of the system* (1) *has the form*

$$U(x,y,z,t) = a + A\sin(z-ct) + C\cos(y-bt),$$

$$V(x,y,z,t) = b + B\sin(x-at) + A\cos(z-ct),$$

$$W(x,y,z,t) = c + C\sin(y-bt) + B\cos(x-at),$$

$$P(x,y,z,t) = 1/2\,CB\sin(-y+bt+x-at) - 1/2\,CB\sin(y-bt+x-at)$$

$$-1/2\,BA\sin(-z+ct+x-at) - 1/2\,BA\sin(z-ct+x-at)$$

$$+1/2\,AC\sin(-z+ct+y-bt) - 1/2\,AC\sin(z-ct+y-bt) + F_2(t), \quad (4)$$

which is a generalization of the famous stationary $ABC-flow$.

Theorem 2. *Non singular and non stationary solution of the system* (3) *has the form*

$$U(x,y,z,t) = a + 1/2\,C_1\,\sin(y-bt) + 1/2\,E_1\,\cos(y-bt) + 1/2\,F_1\,\cos(z-ct)$$

$$+1/2\,H_1\,\sin(z-ct),$$

$$V(x,y,z,t) = b + 1/2\,F_1\,\sin(z-ct) - 1/2\,H_1\,\cos(z-ct) - A_1\,\sin(x-at)$$

$$+B_3\,\cos(x-at),$$

$$W(x,y,z,t) = c + A_3\,\cos(x-at) + B_3\,\sin(x-at)$$

$$+1/2\,C_1\,\cos(y-bt) - 1/2\,E_1\,\sin(y-bt),$$

Remark. To integrate the Navier–Stokes system of equations

$$\vec{U}_t + (\vec{U}\cdot\vec{\nabla})\vec{U} + \vec{\nabla}P = \mu\Delta\vec{U}, \quad (\vec{\nabla}\cdot\vec{U}) = 0, \quad (5)$$

the approach described above can be applied, where the condition

$$VWP_x + UWP_y + UVP_z - V^2(WU)_y - W^2(VU)_z - U^2(WV)_x +$$

$$+\mu(VW\Delta U + UW\Delta V + UV\Delta W) = 0,$$

is used, which is consequence of the condition of incompressibility of liquid $(\vec{\nabla}\cdot\vec{U}) = 0$.

References

1. *Dryuma V.* On solving equations of flows of incompressible liquids // Abstr. Int. Conf. “Mathematics and Information Technologies: Research and Education (MITRE-2016)”, Chisinau, June 23–26, 2016. P. 28–29.