On the development of Anosov's topological ideas in fixed point and coincidence theory

Tatiana N. Fomenko

Moscow State University, Moscow, Russia

tn-fomenko@yandex.ru

The report is a survey (not pretending to the completeness) of results in fixed point and coincidence theory inspired by Anosov's famous theorem on Nielsen numbers for self-mappings of a nilmanifold (that is a homogeneous space of a simply connected nilpotent group by a discrete cocompact subgroup).

In 1985 D.V. Anosov proved [1] the following remarkable theorem. For definitions see, for example, [2, 3].

Theorem. *For any self-mapping $f\colon X \to X$ of a nilmanifold X (mappings are always supposed to be continuous), its Nielsen number $N(f)$ coincides, up to the sign, with its Lefschetz number $\Lambda(f)$.*

E. Fadell and S. Husseini [4] independently gave another proof of Anosov's theorem. Earlier, such fact was known only for mappings of n-tori [5]. Later, D.L. Gonçalves [6] extended this fact to coincidence Nielsen numbers for pairs of mappings of n-tori.

Anosov's main idea consists of the fact that an n-torus should be considered not as a Lie group, but as a nilmanifold. It is known that the Nielsen number is an object difficult to compute, while the Lefschetz number is computable in many cases. So, the significance of Anosov's theorem is that the Nielsen number can be calculated in the indicated situation. It is also well-known that the computing of Nielsen number is important because in many cases $N(f)$ coincides with the least number of fixed points in the homotopy class of a given mapping f. The same is true for the coincidence Nielsen number $N(f, g)$ of a pair of mappings f, g (see [9, 10]).

Let us list a selection of results generalizing and developing the Anosov theorem. It should be mentioned that in the fixed point theory it was generalized by C.K. McCord [7] for mappings of solvmanifolds (a solvmanifold is a quotient space of a connected solvable Lie group by a closed subgroup). Later this was done by E. Keppelman and C. McCord [8] for mappings of so called exponentially solvmanifolds as well.

There are also some more generalizations of the Anosov theorem for coincidence Nielsen numbers. J. Jezierski [11] obtained the equality $N(f,g) = |\Lambda(f,g)|$ (here $\Lambda(f,g)$ is the coincidence Lefschetz number) for any mappings $f,g\colon X \to X$, where X is a nilmanifold. R. Brooks and P. Wong [12] tried to prove the same result for mappings $f,g\colon M \to M$, where M is a compact connected nilmanifold, converting the fixed point and coincidence problems for self-mappings of nilmanifolds into a root problem. Unfortunately, an error slipped in this paper. B. Jiang took notice of it, and later P. Wong corrected the arguments.

Following the technique of [12] for nilmanifolds, C. McCord [7](II) proved the equality $N(f,g) = |\Lambda(f,g)|$ for the case when $X \neq Y$, but $\dim X = \dim Y$. Earlier, in 1992, C. McCord in [7](I) claimed the inequality $N(f,g) \geq |\Lambda(f,g)|$ for mappings of solvmanifolds of the same dimension. But, there was a gap in his arguments, partly corrected in [7](II). So, the last inequality was kept as a conjecture.

Using the fiberwise technique, P. Wong in [13] investigated the coincidence problem for mappings f,g of solvmanifolds of different dimensions. In particular, he proved the inequality $N(f,g) \geq |\Lambda(f,g)|$ for the case of mappings f,g between two oriented solvmanifolds of the same dimensions and eliminated the gaps in [7](I,II).

D. Gonçalves [14] proved that the following 3 conditions are equivalent for any 2 mappings f,g between nilmanifolds.

1. $R(f,g) < \infty$;
2. $Coin(f_\sharp, g_\sharp) = 1$, $f_\sharp, g_\sharp\colon \pi_1(X) \to \pi_1(Y)$;
3. $N(f,g) \neq 0$.

In addition, he proved that under any of these conditions $N(f,g) = R(f,g) = |\Lambda(f,g)|$, where $R(f,g)$ stands for the Reidemeister number of a pair of mappings (f,g).

Now, let us list some results by P. Penninckx (see for definitions and details [15, 16]). In his Ph.D. dissertation he proved statements generalizing the Anosov's theorem.

As it was noticed by D.V. Anosov himself in [1], his theorem cannot be generalized to arbitrary infra-nilmanifolds. Nevertheless, it aroused interest in proving fixed point theory results for infra-nilmanifolds, under suitable conditions. P. Penninckx generalizes the Anosov theorem to a large and important class of infra-nilmanifolds. More exactly, let G be a connected, simply connected, nilpotent Lie group. The automorphism group $Aut(G)$ of G acts naturally on G. The semi-direct product $Aff(G) = G \rtimes Aut(G)$ acts on G by $(d,D)(g) = dD(g)$ for all $(d,D) \in Aff(G), g \in G$. Let C be a maximal compact subgroup of $Aut(G)$. If a cocompact discrete subgroup Γ of $G \rtimes C$ is torsion free, and $\Gamma \subset Aff(G)$, then the quotient space $\Gamma \setminus G$ is a manifold. Such manifolds are called infra-nilmanifolds. In particular, if $\Gamma \subset G$, the quotient $\Gamma \setminus G$ is called a nilmanifold. The finite group $F = \{A \in Aut(G) | \exists a \in G\colon (a, A) \in \Gamma\}$ is called the holonomy group of Γ. P. Penninckx proved that if F is a finite group, the Anosov theorem holds for any infra-nilmanifold with holonomy group F if and only if F has no index two subgroup. In addition, for some results from fixed point theory, P. Penninckx proves a counterpart in coincidence theory. For the results from fixed point theory which can not be generalized to coincidence theory, he gives counterexamples. For example, P. Penninckx proves that if M1 and M2 are infra-nilmanifolds of equal dimension with the same odd order cyclic holonomy group, the equality $N(f, g) = |\Lambda(f, g)|$ is true for any pair of mappings $f, g\colon M_1 \to M_2$.

There are a lot of other works which have appeared under the influence of Anosov's topological ideas which had been realized in his theorem mentioned above. It served as a starting point for the booming development of Nielsen fixed point and coincidence theory and generated a continuing flow of interesting new results by many authors in this field.

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