Parametric Furstenberg theorem on random products of $SL(2,\mathbb{R})$ matrices*

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Random products of matrices appear naturally in many different settings, in particular in smooth dynamical systems, probability theory, spectral theory, mathematical physics. The crucial result is Furstenberg's Theorem [2, 3] on positivity of Lyapunov exponents. It claims that generically the exponential rate of growth (Lyapunov exponent) of product of random matrices is well defined and positive. In the talk we will discuss the random products of $SL(2,\mathbb{R})$ matrices that depend on a parameter. This is motivated, in particular, by the study of discrete Schrödinger operators with random potentials. In that case the Schrödinger cocycle is given by the random products of transfer matrices, and energy serves as a natural parameter. From spectral point of view it is natural to fix the potential first, and then vary the energy. As a more general setting, one can consider random products of matrices depending on a parameter, and study existence and properties of Lyapunov exponent for a typical fixed sequence when the parameter varies. We will show, for example, that in the non-uniformly hyperbolic regime almost surely upper Lyapunov exponent is positive (and coincides with the one prescribed by Furstenberg Theorem) for all parameters, but lower Lyapunov exponent vanishes for a topologically generic parameter. These result explain the difficulties one encounters in the classical proofs of Anderson localization for random Schrödinger operators.

Let us now provide the formal statement of the result. Let $\mathcal{A} \subset SL(2,\mathbb{R})$ be a precompact set of matrices (i.e. for some $M > 0$ we have $\|A\| \leq M$ for all $A \in \mathcal{A}$). Let Ω be a topological space, and μ some Borel probability measure with $\operatorname{supp}\mu = \Omega$. Let $J \subset \mathbb{R}$ be a compact interval of parameters. Suppose a continuous map $F\colon \Omega \times J \to \mathcal{A}$ is given with the following properties:

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• For each $\omega \in \Omega$ the matrix valued function $F_a(\omega)$ is a C^1 function of a, with C^1-norm uniformly bounded in ω;

• Monotonicity: there exists $\delta > 0$ such that $\frac{d}{da} arg(F_a(\omega)\bar{v}) > \delta > 0$ for all $a \in J, \omega \in \Omega, \bar{v} \in \mathbb{R}^2 \backslash \{0\}$.

• For each $a \in J$ the collection of matrices $\{F_a(\omega) \mid \omega \in \Omega\}$ is not uniformly hyperbolic.

• Furstenberg condition: denote by μ_a the measure $\mu_a = F_a(\mu)$. Let G_a be the smallest closed subgroup which contains the support of μ_a. Suppose that for each $a \in J$ the subgroup G_a is not compact, and there is no proper linear subspace $L \subset \mathbb{R}^2$ (or a pair of subspaces L_1, L_2) such that $A(L) = L$ (resp., $A(L_1 \cup L_2) = L_1 \cup L_2$) for all $A \in G_a$.

Under the conditions above, consider random sequences $\bar{\omega} = \omega_1 \omega_2 \ldots$, where $\omega_i \in \Omega$ are chosen independently with respect to the measure μ. For a given $\bar{\omega} \in \Omega$ set $T_{n,a,\bar{\omega}} = F_a(\omega_1)F_a(\omega_2)\ldots F_a(\omega_n)$.

Let us denote by $\lambda_F(a)$ the Furstenberg value of the Lyapunov exponent. Notice that due to Furstenberg Theorem for each $a \in J$ there exists a subset Ω_a with $\mu(\Omega_a) = 1$ such that for $\bar{\omega} \in \Omega_a$ one has

$$\lim_{n \to \infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = \lambda_F(a).$$

In turns out that the full measure set Ω_a of "typical" sequences cannot be chosen independent of the parameter $a \in J$. Moreover, the following statement holds:

Theorem. *In the notations above the following holds μ-almost surely:*

- *For all $a \in J$ we have*

$$\limsup_{n \to +\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = \lambda_F(a) > 0.$$

- *We have*

$$\dim_H \left\{ a \in J \mid \liminf_{n \to +\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| < \lambda_F(a) \right\} = 0.$$

- *The set*

$$\left\{ a \in J \mid \liminf_{n \to +\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = 0 \right\}$$

  is a dense G_δ subset of J.

It is interesting to compare this result with the results in [1].

References

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