Some examples of KAM-nondegenerate nearly integrable systems with positive metric entropy

Dong Chen

Pennsylvania State University, Department of Mathematics, University Park, PA, 16802, USA

dxc360@psu.edu

The celebrated KAM Theory says that if one makes a small perturbation of a non-degenerate completely integrable system, we still have a huge measure of invariant tori with quasi-periodic dynamics in the perturbed system. These invariant tori are known as KAM tori. What happens outside KAM tori draws lots of attention. In this talk I will present two types of C^∞ small Lagrangian perturbation of the geodesic flow on a flat torus. Both resulting flows have positive metric entropy. From this result we get positive metric entropy outside some KAM tori. What is special in the second type is that positive metric entropy comes from an arbitrarily small tubular neighborhood of one trajectory [1]. This is a joint work with Burago and Ivanov.

In [3] we prove the following theorem:

Theorem 1. *The Euclidean metric φ_0 of $\mathbb{T}^n (n \geq 3)$ can be perturbed in the class of reversible Finsler metrics so that the resulting metric φ_ϵ has positive metric entropy of its geodesic flow. φ_ϵ converges to φ_0 in C^∞ as $\epsilon \to 0$.*

In higher dimensions we are able to construct perturbation with entropy non-expansive geodesic flows [1]:

Theorem 2. *The Euclidean metric φ_0 of $\mathbb{T}^n (n \geq 4)$ can be perturbed in the class of reversible Finsler metrics so that the resulting geodesic flow has positive metric entropy and is entropy non-expansive. Such perturbations can be made C^∞ small.*

In order to prove Theorems 1 and 2 we use notions and definitions from [2]:

Let D be an n-dimensional disc and φ a Finsler metric on D.

Definition 1. φ is called *simple* if it satisfies the following three conditions:

(1) Every pair of points in D is connected by a unique geodesic.

(2) Geodesics depend smoothly on their endpoints.

(3) The boundary is strictly convex, that is, geodesics never touch it at their interior points.

Denote by U_{in} the set of unit tangent vectors with base points at the boundary ∂D and pointing inwards. And U_{out} denotes the unit tangent vectors at the boundary, pointing outwards. For a vector $v \in U_{in}$, we can look at the geodesic with initial velocity v. Once it hits the boundary again, we get its velocity vector $\beta(v) \in U_{out}$. This defines a map $\beta \colon U_{in} \to U_{out}$, which is called *the lens map of* φ. If φ is reversible, then the lens map is reversible in the following sense: $-\beta(-\beta(v)) = v$ for every $v \in U_{in}$.

We denote by UT^*D the unit sphere bundle with respect to the dual norm φ^*. Let $\mathscr{L} \colon TD \to T^*D$ be the Legendre transform of the Lagrangian $\varphi^2/2$. It maps UTD to UT^*D. For a tangent vector $v \in UT_xD$, its Legendre transform $\mathscr{L}(v)$ is the unique covector $\alpha \in U_x^*D$ such that $\alpha(v) = 1$.

Then consider subsets $U_{in}^* = \mathscr{L}(U_{in})$ and $U_{out}^* = \mathscr{L}(U_{out})$ of UT^*D. *The dual lens map of* φ is the map $\sigma \colon U_{in}^* \to U_{out}^*$ given by $\sigma := \mathscr{L} \circ \beta \circ \mathscr{L}^{-1}$ where β is the lens map of φ. If φ is reversible then σ is symmetric in the sense that $-\sigma(-\sigma(\alpha)) = \alpha$ for all $\alpha \in U_{in}^*$.

Note that U_{in}^* and U_{out}^* are $(2n-2)$-dimensional submanifolds of T^*D. The restriction of the canonical symplectic 2-form of T^*D to U_{in}^* and U_{out}^* determines the symplectic structure. And the dual lens map σ is symplectic. In [2], Burago and Ivanov proved the following theorem, which says that under certain natural restrictions, a symplectic perturbation of σ is the dual lens map of some metric that is closed to φ:

Theorem 3. *Assume that $n \geq 3$. Let φ be a simple metric on $D = D^n$ and σ its dual lens map. Let W be the complement of a compact set in U_{in}^*. Then every sufficiently small symplectic perturbation $\tilde{\sigma}$ of σ such that $\tilde{\sigma}|_W = \sigma|_W$ can be realized by the dual lens map of a simple metric $\tilde{\varphi}$ which coincides with φ in some neighborhood of ∂D.*

The choice of $\tilde{\varphi}$ can be made in such a way that $\tilde{\varphi}$ converges to φ whenever $\tilde{\sigma}$ converges to σ (in C^∞). In addition, if φ is a reversible Finsler metric and $\tilde{\sigma}$ is symmetric then $\tilde{\varphi}$ can be chosen reversible as well.

In the proof of Theorem 2 we use the following lemma from [2]:

Lemma 1. *There exists a symplectomorphism $\theta \colon D^6 \to D^6$ which is arbitrarily close to the identity in C^∞, coincides with the identity map near the boundary, and has positive metric entropy.*

References

1. *Burago D.Y., Chen D., Ivanov S.V.* An example of entropy non-expansive KAM-nondegenerate nearly integrable system, in progress.
2. *Burago D.Y., Ivanov S.V.* Boundary distance, lens maps and entropy of geodesic flows of Finsler metrics // J. Geom. Topol. 2016. V. 20. P. 469–490.
3. *Chen D.* Positive metric entropy arises in some nondegenerate nearly integrable systems: E-print. arXiv: 1604.07483v1 [math.DS].