On a model of Josephson effect, dynamical systems on two-torus and Heun equations*

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We study a family of two-parametric nonlinear equations that arises in the problem of modeling the overdamped Josephson junction in superconductivity. This family is parametrized by the third parameter: the frequency. It originates from quantum mechanics but also arises in several problems of classical mechanics and geometry. It is equivalent to a special three-parametric dynamical systems on two-torus that also arises in the theory of slow-fast systems. We fix a frequency and consider the rotation number of dynamical system as a function of two remaining parameters. The *phase-lock areas* are the level sets of the rotation number that have non-empty interiors. An important problem is the description of the dynamics of the structure of phase-lock area portrait, as the frequency tends to either zero, or infinity. We present a series of results and conjectures on the geometry of phase-lock areas obtained in collaboration with V.M. Buchstaber [5, 6] and in the previous joint papers of V.M. Buchstaber and S.I. Tertychnyi [2–4]. These results were obtained via complexification, which allowed to use the methods of investigation of equations in the complex domain. This was realized via reduction of the nonlinear equations under question to appropriate second order linear ordinary differential equations on functions of complex variable: a certain subfamily of double confluent Heun equations. It appears that the phase-lock areas exist only for integer rotation numbers (the *quantization effect of rotation number*: a joint result of V.M. Buchstaber, O.V. Karpov and S.I. Tertychnyi [1]). A series of joint results of V.M. Buchstaber and S. Tertychnyi relates the geometry of phase-lock areas to the existence of entire (or polynomial) solutions of the double confluent Heun equations. A joint result with V.M. Buchstaber [5] implies a conjecture due to himself and S. Tertychnyi [3, 4] about the description of the parameter values corresponding to the adjacencies of the phase-lock areas (equivalently, to the double confluent Heun equations having entire solutions). We

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also present a recent joint result with V.M. Buchstaber [6] describing the double confluent Heun equations having monodromy eigensolution with a given eigenvalue. The latter result implies a description of boundaries of phase-lock areas as solutions of explicit analytic functional equations. Its proof is based on the uniqueness theorem for existence of entire solution E of a non-homogeneous version of double confluent Heun equation. The uniqueness theorem is equivalent to the statement that the sequence of three-term linear relations on the Taylor coefficients of E (equivalent to the differential equation) has a unique converging solution. This is proved by using ideas from hyperbolic theory: the converging solution corresponds to the unstable manifold of appropriate fiberwise contracting skew product dynamical system.

References

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