ITERATED MONODROMY GROUPS AND INVARIANT TREES

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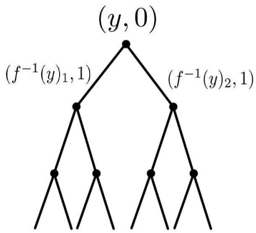
Definition of the iterated monodromy group.

Definition 1. A post-critically finite orientation preserving branched covering f with $deg(f) \geq 2$ is called *Thurston map*.

We can relate to it a special group which acts on some rooted tree. This rooted tree is a tree of preimages. The point $(y, 0)$ is the root (the second number means the level) and is connected with its preimages $(f^{-1}(y)_1, 1)$ and $(f^{-1}(y)_2, 1)$ by two edges. As we have branched coverings of degree 2, there always will be two preimages. Thus the root will be connected by concatenations of edges with all its preimages of any order.



To describe the group, we label all the vertices by finite words in the alphabet $\{0, 1\}$. The vertex $(y, 0)$ is labeled by empty word, the vertices $(f^{-1}(y)_1, 1)$ and $(f^{-1}(y)_2, 1)$ are labelled by 0 and 1 respectively. Let the path from y to $f^{-1}(y)_1$ be l_0 and the path from y to $f^{-1}(y)_2$ be l_1. At the next steps we can consider the lift of the paths l_0 and l_1 originating at $f^{-1}(y)_1$, $f^{-1}(y)_2$. We will add at the end of the word the element 0 if we take the lift of l_0, and 1 otherwise.

Computation of iterated monodromy group gives us an ability to describe the action of any element of the fundamental group of $S^2 - P$, where P is postcritical set of our branched covering, on any word from alphabet $\{0, 1\}$.

Let the $\mathfrak{M}$ be a set of homotopy classes of paths starting in y and terminating in preimage $f^{-n}(y)$ of order n of y (one of the vertexes of the tree of preimages). There is a structure of *bimodule*, defined for the $\mathfrak{M}$ on the fundamental group $\pi_1(S^2 - P)$. That means that both right and left actions of the $\pi_1(S^2 - P)$ are defined on $\mathfrak{M}$. Left action is just a simple composition of $\gamma \in \pi_1(S^2 - P)$ and $\alpha \in \mathfrak{M}$:

$$\gamma.\alpha = \gamma\alpha$$

It is well defined when we choose the basepoint as y so the loop γ has the same end with the path α.

In the case of the right action we take the path α first, so it finishes in $f^{-n}(y)$. To define composition correctly we take the preimage of γ under the covering starting in $f^{-n}(y)$. So:

$$\alpha.\gamma = \alpha f^{-n}(\gamma)$$

In this case in terms of binary words these actions are equal.

The group of automorphisms of a tree acting on the right as described above is called *the iterated monodromy group (IMG).*

For the description of the iterated monodromy group for any generator $\gamma \in \pi_1(S^2 - P)$ we should find elements $\gamma_i \in \pi_1(S^2 - P)$ and symbols $u_i \in \{0, 1\}$, $i \in \{0, 1\}$ such that $0.\gamma = \gamma_0 u_o$ and $1.\gamma = \gamma_1 u_1$.

Definition 2. The map F acting on the fundamental group of the sphere without the postcritical set $F\colon \pi_1(S^2 - P(f)) \to \pi_1(S^2 - P(f))$ is defined by the following property: $\alpha.\gamma = F(\gamma)\tilde{\alpha}$.

Getting the IMG by invariant tree. Assume that we have a post-critically finite branched covering f with post-critical set $P(F)$. Let it have an invariant tree Γ with vertices set V such that $P(f) \subset V$. Let us choose some point a that does not belong to the tree as a basepoint. We will call the loops starting at the basepoint and crossing Γ only once at the edge *the legitimate loop.*

Theorem. *Homotopy classes of the legitimate loops and the trivial loop form a closed set under the action of F.*

References

1. *Nekrashevych V.* Self-similar groups.