Typical points in chaotic dynamics*

Michael Blank

Institute for Information Transmission Problems, RAS, Moscow, Russia, National Research University Higher School of Economics, Moscow, Russia

blank@iitp.ru

The classical Birkhoff ergodic theorem in its most popular version says that the time average along a single typical realization of a Markov process is equal to the space average with respect to the ergodic invariant distribution. This result is one of the cornerstones of the entire ergodic theory and its numerous applications. In this talk I'll address two questions related to this subject: how large is the set of typical realizations, in particular in the case when there are no invariant distributions, and how this is connected to properties of the so called natural measures (limits of images of "good" measures under the action of the system).

Our main results concern with necessary and sufficient conditions under which for a given reference measure (e.g. Lebesgue measure), whose support might be much larger than the support of the invariant one, the set of typical initial points is of full measure. It turns out that one of the main assumptions here is the ergodicity of the natural measure. To deal with the situation when the invariant measure does not exist we extend the notion of ergodicity to measures being non invariant.

To give an example of a system without invariant distributions satisfying our setup, consider the following deterministic Markov process: a family of maps from the unit disc $X := \{(\phi, R) \colon \ 0 \le \phi < 2\pi, \ 0 \le R \le 1\}$ into itself defined in the polar coordinates (ϕ, R) by the relation:

$$T(\phi, R) := \begin{cases} (\phi + 2\pi\alpha + \beta(R - r) \bmod 2\pi, \gamma(R - r) + r) & \text{if } r(R - r) \neq 0, \\ (\phi + 2\pi\alpha \bmod 2\pi, (1 + r)/2) & \text{otherwise} \end{cases}$$

with the parameters $\alpha, \beta, \gamma, r \in (0, 1)$. One can show that for any probability measure μ absolutely continuous with respect to the Lebesgue measure, the sequence of measures $\frac{1}{n}\sum_{k=0}^{n-1} T_*^k \mu$ (Cesaro averages of images of the measure μ under the action of T) converges weakly to a certain limit measure $\tilde{\mu}$ on the circle $\{R = r\}$, but this measure is no longer invariant. Depending on the choice of the rotation parameters $\alpha, \beta \in (0, 1)$ properties

*Supported by the Russian Science Foundation (project no. 14-50-00150).

of the set of $\tilde{\mu}$-typical points turn out to be very different. In particular, if the parameter α is irrational, then the limit measure is unique and the set of $\tilde{\mu}$-typical points coincides with the entire unit disk.

Questions discussed above turn out to be especially actual in the case of large systems, when even in the presence of ergodic invariant measures, their supports cover only a small part of the phase space.

To formulate the results we need a few definitions.

Let $(X, \mathcal{B}, m)$ be a compact measurable space with a probabilistic reference measure m on it (e.g. a unit cube X equipped with the Borel σ-algebra $\mathcal{B}$ and the Lebesgue measure m). Only probability measures will be considered and the reference measure is supposed to be positive on open sets.

For a given measure ν denote by $\mathcal{M}(\nu)$ the set of measures μ on X absolutely continuous with respect to ν (notation $\mu \ll \nu$). The map T induces the transfer-operator T_* acting on measures according to the formula $T_*\mu(A) := \mu(T^{-1}A)$, $A \in \mathcal{B}$. A measure μ is said to be *wandering* if its images under the action of the transfer-operator are mutually singular.

Denote by S the support of the limit measure $\tilde{\mu}$ and by μ_S the conditional measure on S constructed from a measure μ.

Depending on the properties of the transfer-operator there are three different situations. The first of them is the regular case: the transfer-operator T_* is continuous at the limit measure $\tilde{\mu}$. The following result gives necessary and sufficient conditions that the set of $\tilde{\mu}$-typical points

$$Z_{\tilde{\mu}} := \{x \in X\colon \quad \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \xrightarrow{n\to\infty} \int f d\tilde{\mu} \quad \forall f \in C^0(X)\}$$

is of full reference measure m.

Theorem 1. *The property $m(Z_{\tilde{\mu}}) \cdot m_S(Z_{\tilde{\mu}}) = 1$ is equivalent to the following three assumptions*:

(i) $\frac{1}{n}\sum_{k=0}^{n-1} T_*^k \mu \xrightarrow{n\to\infty} \tilde{\mu} \quad \forall \mu \in \mathcal{M}(m) \cup \mathcal{M}(m_S)$,

(ii) *the limit measure $\tilde{\mu}$ is ergodic,*

(iii) *there are no wandering measures in $\mathcal{M}(m_S)$.*

The situation when the transfer-operator T_* is discontinuous at the limit measure $\tilde{\mu}$ (see example above) we refer to as an irregular case. In this situation there might be no invariant measures and thus the Birkhoff ergodic theorem is no longer applicable. To overcome this difficulty we (motivated by the fact that in the regular setting the set of typical points is of full ergodic invariant measure) say that a measure μ is *weakly ergodic* if $\mu(Z_\mu) = 1$.

Theorem 2. *The assumptions*

(i) $\frac{1}{n}\sum_{k=0}^{n-1} T_*^k \mu \xrightarrow{n\to\infty} \tilde{\mu} \quad \forall \mu \in \mathcal{M}(m) \cup \mathcal{M}(m_S)$,

(ii) *the limit measure $\tilde{\mu}$ is weakly ergodic,*

(iii) *there are no wandering measures in $\mathcal{M}(m_S)$,*

imply that $m(Z_{\tilde{\mu}}) \cdot m_S(Z_{\tilde{\mu}}) = 1$.

The third situation under study corresponds to the so called self-consistent dynamical systems. By a *self-consistent dynamical system* we mean a skew product map $\mathcal{T}(x, \mu) := (T_\mu x, (T_\mu)_* \mu)$ acting in the direct product space $X \times \mathcal{M}$. Here $\{T_\mu\}$ is a family of maps acting from X into itself and indexed by measures μ from $\mathcal{M}$, the space of probability measures on X. This construction is a deterministic counterpart of the so called nonlinear Markov chains. In this setting we also give sufficient conditions under which the set of typical points for the limit measure is of full reference measure.

Let us mention a few known results related to the questions under study. The classical ergodicity assumption may be justified by the well known Oxtoby–Ulam result [1], according to which a generic volume preserving homeomorphism of a compact manifold is ergodic. In turn ergodicity by the Birkhoff ergodic theorem means that almost all points (with respect to the volume measure) are typical in this case. In a recent paper [2] this reasoning was extended to generic continuous maps *without* the assumption of the volume preservation and it has been shown that for a generic map the Birkhoff average converges almost everywhere, but the limit value may depend sensitively on the initial point. This disproves a conjecture by D. Ruelle [3] who expected that generically those averages should diverge, which he called *historical behavior*. A different approach to this question together with a comprehensive review of corresponding results may be found in [4].

We already mentioned that the questions we consider are closely related to the connections between the *natural* and *observable* versions of the so called Sinai–Ruelle–Bowen measures. We refer a reader to [5], where this question has been raised in the first time and to [6, 7] where further clarifications were obtained.

References

1. *Oxtoby J.C., Ulam S.M.* Measure-preserving homeomorphisms and metrical transitivity // Ann. Math. (2). 1941. V. 42. P. 874–920.

2. *Abdenur F., Andersson M.* Ergodic theory of generic continuous maps // Commun. Math. Phys. 2013. V. 318. P. 831–855.
3. *Ruelle D.* Historical behaviour in smooth dynamical systems // Global analysis of dynamical systems / Ed. by H. Broer, B. Krauskopf, G. Vegter. London: Inst. Phys., 2001. P. 63–66.
4. *Araujo V., Pinheiro V.* Abundance of wild historic behavior, ergodic decomposition and generalized physical measures: E-print. arXiv: 1609.05356v1 [math.DS].
5. *Blank M., Bunimovich L.* Multicomponent dynamical systems: SRB measures and phase transitions // Nonlinearity. 2003. V. 16, N 1. P. 387–401.
6. *Misiurewicz M.* Ergodic natural measures // Contemp. Math. 2005. V. 385. P. 1–6.
7. *Jarvenpaa E., Tolonen T.* Relations between natural and observable measures // Nonlinearity. 2005. V. 18, N 2. P. 897–912.