Dynamical constructions on the space of finitely generated groups and the concept of random group

Rostislav Grigorchuk

Texas A&M University, College Station, USA

grigorch@math.tamu.edu

Let $\mathbb{X}_m, m \geq 2$ be a space of *marked* m-generated groups introduced by the speaker in 1984 [1]. It consists of pairs (G, S), where G is a group and $S = \{s_1, s_2, \dots, s_m\}$ is ordered generating set of cardinality m. Space $\mathbb{X}_m$ is supplied by a local topology: two marked groups $(G_1, S_1), (G_2, S_2)$ are close in this topology if the Cayley graphs $\Gamma(G_1, S_1), \Gamma(G_2, S_2)$ are isomorphic on the large neighborhood of the identity element. Each of the spaces $\mathbb{X}_m$ is a compact metrizable totally disconnected space. $\mathbb{X}_m$ naturally embeds into $\mathbb{X}_{m+1}$ and the union $\mathbb{X}_\infty = \bigcup_{m \geq 2} \mathbb{X}_m$ can be considered as a space of finitely generated groups. Each of the spaces $\mathbb{X}_m, \mathbb{X}_\infty$ has a scattered part consisting of isolated points of different Cantor–Bendixson rank and a condensation part (perfect kernel) denoted $\mathbb{X}_m^*$ and $\mathbb{X}_\infty^*$, respectively. It is interesting problem to identify groups that belong to each of these parts and compute the Cantor–Bendixson ranks $r(\mathbb{X}_m), r(\mathbb{X}_\infty)$. Perfect kernels $\mathbb{X}_m^*, 2 \leq m < \infty$ are homeomorphic to a Cantor set.

The group $\mathcal{N}_m$ of Nielsen transformations naturally acts by homeomorphisms on $\mathbb{X}_m$ (preserving condensation part $\mathbb{X}_m^*$), and the inductive limit $\mathcal{N}_\infty = \varinjlim \mathcal{N}_m$ acts by homeomorphisms on $\mathbb{X}_\infty$.

Problem 1. (a) *Is there is a continuous invariant Borel probability measure for the action* $(\mathcal{N}_m, \mathbb{X}_m), 2 \leq m \leq \infty$*?.*

(b) *If the answer to the part* (a) *is no, is there is a continuous quasi-invariant Borel probability measure for the action* $(\mathcal{N}_m, \mathbb{X}_m), 2 \leq m \leq \infty$*?.*

If such measure exists then it can be used for study of typical properties of groups in corresponding class (in the case $m = \infty$ it is the of all finitely generated groups). Suggested approach to the randomness is called by us *global* dynamical approach (it was suggested in [4]). An alternative *local* approach is described and used in [5].

There is a notion of random group due to Gromov [2] (based on his *density* model). It is far from to have a dynamical flavor as the randomness appears in the form of the quantity given by the limit behavior of the ratio m_i/n_i, where n_i is the cardinality of the set of finite group presentations with relations of the length i and m_i is the number of presentations that define a group with a certain group property $\mathcal{P}$. Some parameters are involved in the model and depending of their value typical group properties may vary.

Let $\mathcal{S}(G)$ and $\mathcal{N}(G)$ be sets of subgroups and normal subgroups of a countable group G. These sets can be considered as subsets of $\{0,1\}^{\mathbb{N}}$ and therefore can be supplied by induced topology of the product topology on $\{0,1\}^{\mathbb{N}}$. This makes them compact metrizable totally disconnected spaces. The questions about condensation and scattered parts of $\mathcal{S}(G)$ and of $\mathcal{N}(G)$ and about the Cantor–Bendixson ranks $r(\mathcal{S}(G)), r(\mathcal{N}(G))$ are interesting in many cases. A group G acts on $\mathcal{S}(G)$ by conjugation and a group of automorphisms $Aut(G)$ naturally acts on $\mathcal{S}(G)$ and $\mathcal{N}(G)$. Invariant probability measures for actions $(G, \mathcal{S}(G))$, $(Aut(G), \mathcal{S}(G))$ and $(Aut(G), \mathcal{N}(G))$ can be viewed as invariant random subgroup (IRS), characteristic random subgroup (CRS), or characteristically normal random subgroup (CNRS), and will be identified with them. Let $IRS(G)$, $CRS(G)$ and $CNRS(G)$ be the corresponding Choquets simplexes. We are interested in topological structure of them, number of critical points (they correspond to the ergodic measures), etc. A delta mass δ_N supported on one point $N \in \mathcal{N}(G)$ is a trivial example of IRS and therefore we focus on *continuous* random subgroups. Let F_m be a free group of finite or countably infinite rank m.

Theorem 1. [6] *There is uncountably many continuous F_m-weakly mixing CRS's on F_m, $2 \leq m \leq \infty$.*

This is stronger than the results of L. Bowen [7], or E. Glasner–B. Weiss [8] about existence of uncountably many continuous ergodic IRS's on F_m.

Corollary 1. *Let G be acylindrically hyperbolic group. Then it has uncountably many continuous ergodic IRS's.*

This is because acylindrically hyperbolic group contains a normal subgroup isomorphic to a noncommutative free group. The class of acylindrically hyperbolic group includes such important subclasses as non-elementary Gromov hyperbolic groups, mapping class groups of oriented surfaces of negative Euler characteristic and groups of outer automorphisms $Out(F_m)$. The corresponding results are due to T. Delzant, F. Dahmani, V. Guirardel, and D. Osin.

In the proof of the above theorem we use existence of uncountably many characteristic subgroups in a free group $F_m, m \geq 2$, and description of ergodic CRS's on elementary abelian p-group of infinite rank $\mathcal{A}_p = \bigoplus_{\mathbb{N}} \mathbb{Z}_p$ (p prime). Recall that a simplex is called a Bauer simplex if the set of extreme points is dense.

Theorem 2 [6]. *The simplex $CRS(\mathcal{A}_p)$ is a Bauer simplex. The set of its extreme points is countable and can be enumerated as $\mu_n, n = 1, 2, \ldots$, where μ_n is a measure supported on subgroups of index p^n.*

In fact in [6] the CRS's are described on groups $\mathcal{A}_k = \bigoplus_{\mathbb{N}} \mathbb{Z}_k$ for arbitrary $k \in \mathbb{N}$ and on free abelian group $\mathbb{Z}^\infty$ of countably infinite rank. The case $k = p$ also can be deduced from the result of A. Gnedin and G. Olshanski [8].

By Nielsen theorem the group $\mathcal{N}_m$ is isomorphic to $Aut(F_m)$, and space $\mathbb{X}_m$ can be identified with space $\mathcal{N}(F_m)$, as each m-generated group is isomorphic to a quotient of a free group F_m. Therefore part (a) of Problem 1 is equivalent to the following problem: "Is there a continuous invariant Borel probability measure for the system $(Aut(F_m), \mathcal{N}(F_m))$?" (similarly, we can reformulate the part (b) of the problem). The CRS's given by Theorem 2 are supported on step 2 subnormal subgroups (normal subgroups of normal subgroups, in fact they are supported on normal subgroups of characteristic subgroups). Therefore to solve in affirmative way Problem 1 it is enough to construct a continuous CRS on F_m whose support is on the set of normal subgroups, i.e. to construct continuous CNRS. In the above discussion the system $(Aut(F_m), \mathcal{N}(F_m))$ can be replaced by the system $(Out(F_m), \mathcal{N}(F_m))$ as the group of inner automorphisms $Inn(F_m)$ acts trivially on $\mathcal{N}(F_m)$ and $Out(F_m) = Aut(F_m)/Inn(F_m)$. The groups $Aut(F_m)$ and $Out(F_m)$ are very popular objects of investigation in geometric group theory and their actions on Teichmuller type spaces are thoroughly studied. Perhaps this could help to solve the suggested problem.

A global categorical approach to the notion of typical group for study of typical properties of groups also works for described dynamical systems. A local categorical approach is used in [3] and [5]. Study of countable Borel equivalence relations $\sim_m, \sim_\infty$ given by the partition on orbits for systems $(\mathcal{N}_m, \mathbb{X}_m), 2 \leq m \leq \infty$ is another challenging problem. It is known that they are not *smooth* (i.e. not *tame* or not *measurable* in the language of Rokhlin), and are not *hyperfinite*). But it may happen that they are μ-hyperfinite and so Zimmer amenable, where μ is invariant or a quasi-invariant measure discussed above.

If (T, X) is a Cantor system where $X \subset \mathbb{X}_m$ is a Cantor set, T is a homeomorphism of X, and equivalence relation on X generated by T is a subrelation of $\sim_m$ (or of any other useful relation on $\mathbb{X}_m$), then any T-invariant probability measure μ on X can be used for study of typical properties of groups from X. This is what we mean by the local approach and it was used in [5] for study of typical growth of groups of intermediate growth from [1].

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