Estimates of correlations in dynamical systems: from Hölder continuous to general observables

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Let μ be a Borel measure on a metric space M and $T\colon M \to M$ be a measure preserving transformation, i.e. $\mu(A) = \mu(T^{-1}A)$ for all Borel sets $A \subseteq M$. For observables $f \in L_p(M,\mu)$ and $g \in L_q(M,\mu)$ with Hölder conjugate $p, q \in [1,\infty]$, we denote as $c_n(f,g)$ the pair correlations, i.e.

$$c_n(f,g) = \int_M f(x)g(T^n x)\,d\mu(x) - \int_M f(x)\,d\mu(x)\int_M g(x)\,d\mu(x),\ \ n \in \mathbb{N}.$$

As is well known, for mixing measure preserving transformation the pair correlations $c_n(f,g)$ for $f, g \in L_2(M,\mu)$ may decay to zero as $n \to \infty$ with arbitrary slow rate, and furthermore such behavior is typical in L_2 (see [1]). Nevertheless for general observables f and g it is interesting to know how slow the pair correlations $c_n(f,g)$ decay. We present in this talk the approach to estimating of the pair correlations for general observables via approximation by observables with already known information on the decay of pair correlations.

More precisely, let $\mathfrak{F}_p \subseteq L_p(M)$ and $\mathfrak{G}_q \subseteq L_q(M)$ be densely embedded Banach spaces of complex valued functions on M. Assume that for all $f \in \mathfrak{F}_p$ and $g \in \mathfrak{G}_q$

$$|c_n(f,g)| \le A\|f\|_{\mathfrak{F}_p}\|g\|_{\mathfrak{G}_q}\Phi(n), \quad n \in \mathbb{N}, \tag{\#}$$

with some constant $A > 0$ and $\Phi \searrow 0$ as $n \to \infty$.

There are a lot of the dynamical systems with estimates of the pair correlations like (#), among which there are the classical transitive Anosov diffeomorphisms with sets $\mathfrak{F}_p$ and $\mathfrak{G}_q$ of Hölder continuous functions in stable and unstable directions respectively (see [2]).

For $f \in L_p(M)$ and $t \ge 0$ denote as $\tau_f(t)$ the best $\mathfrak{F}_p$-approximation of order t for function f, i.e.

$$\tau_f(t) = \inf\{\|f-h\|_p\colon\ h \in \mathfrak{F}_p,\ \ \|h\|_{\mathfrak{F}_p} \le t\}.$$

Let us define new Banach spaces of functions associated with different estimates of the best approximations. Let Ξ be the set of all decreasing to zero functions $\Theta\colon \mathbb{R}_+ \to \mathbb{R}_+$, i.e. $\Theta(t_1) \geq \Theta(t_2)$ for $0 \leq t_1 \leq t_2$ and $\Theta(t) \to 0$ as $t \to +\infty$. Let $\Xi^0 \subset \Xi$ be the set of functions which equal to zero beginning at some point.

Definition. For $\Theta \in \Xi$ denote as $\mathfrak{F}_p(\Theta)$ the set of all functions $f \in L_p(M)$ such that the best $\mathfrak{F}_p$-approximation satisfies the estimate

$$\tau_f(ct) \leq c\Theta(t)$$

for all $t \geq 0$ and some constant $c \geq 0$. The set of such constants is denoted as $C(\Theta, f)$. Let

$$\|f\|_{\mathfrak{F}_p(\Theta)} = \inf_{c \in C(\Theta, f)} c$$

be the norm in the space $\mathfrak{F}_p(\Theta)$.

Let us formulate the main result.

Theorem. *Assume that the estimate (#) holds true. Let $\Theta_1, \Theta_2 \in \Xi$, then for any $f \in \mathfrak{F}_p(\Theta_1)$, $g \in \mathfrak{G}_q(\Theta_2)$ for all $n \geq n_0$*

$$|c_n(f,g)| \leq A' \|f\|_{\mathfrak{F}_p(\Theta_1)} \|g\|_{\mathfrak{G}_q(\Theta_2)} \Phi'(n) \tag{\#\#}$$

for some $n_0 \in \mathbb{N}$, constant $A' > 0$, and $\Phi' \searrow 0$ as $n \to \infty$.

In the case $\Theta_1 \vee \Theta_2 \notin \Xi^0$, we have $\Phi'(n) = \Phi(n)\upsilon(\Phi(n))$ with the function $\upsilon\colon \mathbb{R}_+ \to \mathbb{R}_+$ which is the inverse of

$$\frac{1}{t}(\Theta_1 \vee \Theta_2)(\sqrt{t}), \ t > 0,$$

and n_0 is the smallest integer satisfying the estimate

$$\Phi(n_0)\upsilon(\Phi(n_0)) \leq 1.$$

In the case $\Theta_1 \vee \Theta_2 \in \Xi^0$, we have $\Phi'(n) = \Phi(n)$ and $n_0 = 1$.

As we see the estimate (##) looks like (#) and evidently extends it to all observables.

As application of the main result one can obtain, following the approach [3], the CLT for Anosov diffeomorphisms with some new observables, for example for the characteristic functions of the sets with power behavior of the measure near the boundary.

References

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