Some problems on pseudo-Anosov homeomorphisms

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The research of pseudo-Anosov and generalized pseudo-Anosov homeomorphisms of surfaces is one of the areas that has grown from the fundamental works of D.V. Anosov on the theory of dynamical systems. The concept of pseudo-Anosov homeomorphism was introduced by W. Thurston in his development of the J. Nielsen theory of classification up to isotopy surfaces homeomorphisms (see. [1]). In Thurston theory the pseudo-Anosov homeomorphisms appear as natural representatives of "mysterious" third Nielsen class. Thus, the dynamics helped topology in solving of the old difficult problem. At the same time, Thurston's results were useful for dynamics as they are closely related to the smooth cascades on surfaces possessing hyperbolic strange attractors which are the important objects of the theory of smooth dynamical systems.

Dmitry Victorovich was keenly interested in the emerging theory and its connections. Shortly before his illness the author of the report had a chance to discuss with him a program of a research in this area. In this talk we will speak on the problems in accordance with this program. The author is also going to report his comprehension about the progress in resolving of some of these problems. Below there are formulations of these problems in exactly the same form in which they were discussed with Dmitry Victorovich.

Begin with few remarks on the terminology used (see [2] for more details).

Pseudo-Anosov homeomorphisms (PA) stand out among *generalized pseudo-Anosov* (GPA) by absence of valence 1 singularities (thorns) in its invariant foliations. *Singular type* of GPA-homeomorphism is the sequence $\{b_d \colon d \in \mathbb{N}\}$, where b_d is the number of valence d singularities. Almost all elements of this sequence are 0 and we may assume that $b_2 = 0$.

1. Is the Thurston theorem on the classification up to isotopy of homeomorphisms true for non-orientable surfaces?

2. Can be extended to the nonorientable case Bestvina–Handel algorithm [3] for constructing train-track by automorphism of the fundamental group of the surface and defining the singular type of GPA-homeomorphism if such is defined by this automorphism according to Dehn–Nielsen theorem?

3. Describe the singular types which are formally admitted by Euler–Poincaré formula and actually realized by (generalized) pseudo-Anosov homeomorphisms of non-orientable surfaces with orientable and nonorientable invariant foliations. In the case of orientable surfaces the answer is given in [4].

4. Is it true that there are no pseudo-Anosov homeomorphisms of nonorientable surface of genus 3?

5. Is it true that there are no GPA-homeomorphisms of the Klein bottle of singular type $\{b_3 = b_1 = 1\}$? Note: this singular type is not realized in the case of the torus.

6. Whether exists PA-homeomorphism of singular type $\{b_3 = b_5 = 1\}$ on non-orientable surface of genus 4? Note: this singular type is not realized in the case of an orientable surface of genus 2.

7. Give an example of PA-homeomorphism for each non-orientable surface of odd genus > 5 (for $g = 5$ and each even genus, such examples are known [2]).

8. Is it true that for each surface (orientable or not) there is a GPA-homeomorphism with the topological entropy smaller than prescribed value? (With the decreasing of entropy increase the number of thorns.) The answer is yes for the sphere and the projective plane [2].

9. Evaluate the minimum of dilation of PA-homeomorphisms of the non-orientable surface depending on its genus. For orientable surfaces, such evaluations are known (see [5] for example), but whether it is possible to improve them?

10. Evaluate the minimum of dilation for GPA-homeomorphisms with a fixed number of valence 1 singularities depending on the type of surface and the number of thorns.

11. Whether is it possible to find the exact value of the minimum for at least surfaces of small genus, and with a small number of thorns? For Anosov diffeomorphisms of torus, and GPA-homeomorphisms with no more than with 4 thorns it is simple. Some other results see in [6].

12. Is it true that the simplest (i.e. with minimal dilatation) PA-homeomorphism (GPA) of fixed surface and with a fixed number of thorns is unique up to topological conjugacy?

13. Is it true that for any PA-homeomorphism of orientable surface of genus 2 there exists an invariant leaf?

14. Is it true that for a fixed singular type there are only a finite number of conjugacy classes of GPA-homeomorphisms having no invariant leaf?

15. Evaluate the growth rate of the number of conjugacy classes of PA-homeomorphisms for given surface (GPA-homeomorphisms with a given number of thorns) depending on dilation.

References

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