5. *Васильев Ф.П., Потапов М.М., Артемьева Л.А.* Регуляризованный экстраградиентный метод в многокритериальных задачах управления с неточными данными // Диф. уравнения. 2016. Т. 52, №11. С. 1555–1567.
6. *Fliege J., Svaiter B.F.* Steepest descent methods for multicriteria optimization // Math. Methods Oper. Res. 2000. V. 51. P. 479–494.
7. *Grana Drummond L.M., Iusem A.N.* A projected gradient method for vector optimization problems // Comput. Optim. Appl. 2004. V. 28. P. 5–29.

Analogue of Pontryagin's maximum principle for multiple integrals minimization problems

M. I. Zelikin

Moscow State University, Moscow, Russia

mzelikin@mtu-net.ru

Let $\mathfrak{N}$ be a domain on a smooth n-dimensional manifold. Let $\rho\colon \xi \to \mathfrak{N}$ be a ν-dimensional vector bundle over the base $\mathfrak{N}$; the fibre of this bundle over a point $t \in \mathfrak{N}$, i.e., the full preimage of the point t under the mapping ρ, is a ν-dimensional linear space. Local coordinates on $\mathfrak{N}$ will be denoted by $t = (t^1, \dots, t^n)$, and those on fibres, by $x = (x^1, \dots, x^\nu)$. The common convention on summation over repeated indices is used, with the Latin indices, related to coordinates on the base, running from 1 to n, and the Greek ones, related to coordinates on fibres, running from 1 to ν. Multi-indices will be denoted by capital letters. The symbol I means the whole set from 1 to n. Consider

$$\mathcal{F} = \int_V f\left(t, x, \frac{Dx}{Dt}\right) dt^I. \tag{1}$$

Let us denote by $J_1(\xi)$ the bundle of 1-jets over ξ, and let

$$q_i^\alpha := \frac{\partial x^\alpha}{\partial t^i} = g_i^\alpha(t, x) \tag{2}$$

be a section of $J_1(\xi)$.

Problem 1. Given $y(\cdot) \in C^1(\partial\mathfrak{N})$, find the strong minimum of the functional over C^1-manifolds $x(\cdot)\colon \mathfrak{N} \to \xi$ defined in $\mathfrak{N}$ subject to the boundary conditions $x(t)|_{\partial\mathfrak{N}} = y(t)$.

The natural necessary condition of minimum of the functional is the non-negativity of the second variation. Thorough investigation of the second variation for multiple integrals is due to A. Clebsch. He studied the Dirichlet functional

$$\delta^2\mathcal{F} = \int_V \left[\frac{\partial^2 \hat{f}}{\partial\left(\frac{\partial x^\alpha}{\partial t^i}\right)\partial\left(\frac{\partial x^\beta}{\partial t^j}\right)} \frac{\partial h^\alpha}{\partial t^i}\frac{\partial h^\beta}{\partial t^j} + 2\frac{\partial^2 \hat{f}}{\partial\left(\frac{\partial x^\alpha}{\partial t^i}\right)\partial x_\beta}\frac{\partial h^\alpha}{\partial t^i}h^\beta + \frac{\partial^2 \hat{f}}{\partial x^\alpha \partial x^\beta} h^\alpha h^\beta \right] dt^I.$$

The hat over a function (say, $\hat{f}$) means that one substitutes the extremal $\hat{x}(t)$ into all its arguments.

By using ideas of many-dimensional Riccati techniques, Clebsch presupposed the existence of a solution to a partial differential Riccati-type equation. Now we know that it is the many-dimensional counterpart of the condition of absence of conjugate points. Using this solution, he transformed variables and reduced the functional to its principal part, that is, to the quadratic form of the converted first derivatives of the desired functions.

It seems that Clebsch believed that for multiple integrals there is a direct analogue of the necessary Legendre condition: The non-negativity of the second variation implies the non-negativity of the principal part of the quadratic form defined on all $n \times \nu$ matrices

$$q_i^\alpha = \frac{\partial x^\alpha}{\partial t^i}.$$

Half a century after the work of Clebsch, J. Hadamard showed that it is not true. Hadamard proved the following necessary optimality condition:

Theorem 1 (Hadamard)**.** *Let the functional*

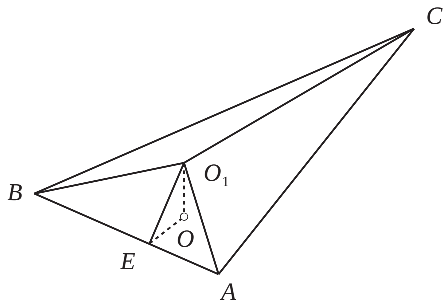
$$\delta^2\mathcal{F} = \int_V \left[a_{\alpha\beta}^{ij}(t)\frac{\partial x^\alpha}{\partial t^i}\frac{\partial x^\beta}{\partial t^j} + 2c_{\alpha\beta}^i(t)\frac{\partial x^\alpha}{\partial t^i}x^\beta + b_{\alpha\beta}(t)x^\alpha x^\beta \right] dt^I \qquad (3)$$

be non-negative for $x(\cdot)$ satisfying the boundary condition

$$x|_{\partial V} = 0.$$

Then for all values of $t \in V$, the quadratic form $a_{\alpha\beta}^{ij}(t)q_i^\alpha q_j^\beta$ takes non-negative values on all $n \times \nu$ matrices of the form $q_i^\alpha = \xi^\alpha \eta_i$ (that is, on all matrices of rank 1).

The theorem can be reformulated as follows: The biquadratic form $a_{\alpha\beta}^{ij}(t)\xi^\alpha\xi^\beta\eta_i\eta_j$ is non-negative for all $t \in V$ and $\xi \in \mathbb{R}^\nu$, $\eta \in (\mathbb{R}^n)^*$.



Scheme of the variation (δx).

The gap between the necessary and sufficient conditions was essentially diminished by Van Hove. In the late 1940s he proved that the natural amplification of the Hadamard–Legendre condition

$$\frac{\partial^2 \hat{f}}{\partial\left(\frac{\partial x^\alpha}{\partial t^i}\right)\partial\left(\frac{\partial x^\beta}{\partial t^j}\right)}\xi^\alpha\xi^\beta\eta^i\eta^j \geq \varepsilon|\xi|^2|\eta|^2 \tag{4}$$

is a locally sufficient condition of C^1-minimum. The term "locally sufficient" means that the domain of integration is sufficiently small.

We denote by $\mathcal{U}$ the set of matrices of rank 1.

The role of Pontryagin's function will be played by

$$H = -f + q^i_\alpha \frac{\partial x^\alpha}{\partial t^i}.$$

Theorem 2 (maximum principle)**.** *Suppose f is a smooth function. Suppose that $\hat{x}(\cdot)$ provides a strong minimum for the functional in Problem 1.*

Then there exists a solution to the conjugate system of variational equations such that Pontryagin's function H attains its maximum value on the set of slopes $\frac{Dx}{Dt}$ defined by the rank 1 matrices $\mathcal{U}$:

$$\max_{(\xi^\alpha\eta^i)\in\mathcal{U}}\left[-f\left(t, \hat{x}, \frac{D\hat{x}}{Dt} + (\xi^\alpha\eta_i)\right) + \frac{\partial f}{\partial\left(\frac{Dx}{Dt}\right)}(\xi^\alpha\eta_i)\right] = -\hat{f} + \frac{\partial \hat{f}}{\partial\left(\frac{Dx}{Dt}\right)}\left(\frac{D\hat{x}}{Dt}\right). \tag{5}$$

Here we can only demonstrate the scheme of the used variation (see the figure).

Example 1. As an example, consider a problem associated with elasticity theory.

It is required to minimize the functional

$$\int_{\mathfrak{N}} \left[a(z_1^2 + z_4^2) + b(z_2^2 + z_3^2) + 2c \det z\right] dt^1 \wedge dt^2. \tag{6}$$

Here

$$z_1 = \frac{\partial x^1}{\partial t^1}, \qquad z_2 = \frac{\partial x^1}{\partial t^2}, \qquad z_3 = \frac{\partial x^2}{\partial t^1}, \qquad z_4 = \frac{\partial x^2}{\partial t^2}.$$

The term containing $\det z = z_1 z_4 - z_2 z_3$ defines the contraction–expansion degree of the material; the coefficient $2c$ is called the solid elasticity modulus. The coefficients a and b are connected with the Lame constants, that is, with the tensor of elasticity modulus. The slopes of the surface (variables z_i) serve as control variables.

The matrix of the quadratic form in the integrand is

$$\begin{pmatrix} a & 0 & 0 & c \\ 0 & b & -c & 0 \\ 0 & -c & b & 0 \\ c & 0 & 0 & a \end{pmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = a - c$, $\lambda_2 = a + c$, $\lambda_3 = b - c$, and $\lambda_4 = b + c$. Let, say, $a > b$. For $a > c > b$ one of the eigenvalues is negative. The quadratic form is non-convex. The maximum of Pontryagin's function

$$H = -\left\{ a\left(\frac{\partial x^1}{\partial t^1}\right)^2 + a\left(\frac{\partial x^1}{\partial t^4}\right)^2 + b\left(\frac{\partial x^2}{\partial t^1}\right)^2 + b\left(\frac{\partial x^1}{\partial t^2}\right)^2 + 2c\left(\frac{\partial x^1}{\partial t^1}\frac{\partial x^2}{\partial t^4} - \frac{\partial x^1}{\partial t^2}\frac{\partial x^2}{\partial t^1}\right)\right\} + q_\alpha^i \frac{\partial x^\alpha}{\partial t^i}$$

(which would be in line with the naive generalization of Pontryagin's maximum principle) is not attained on any extremal. Nevertheless, the restriction of this function to the level surface of the rank 1 matrices reduces H to a positive definite quadratic form.

Indeed, the first variation on extremals is zero. The variations of control $h = (h_1, h_2, h_3, h_4)$ that correspond to directions with the rank 1 matrices are equivalent to the degenerate matrices h. So, the main quadratic part of the expansion in h of the term $\det z$ on such variations (that is, $h_1 h_4 - h_2 h_3$) equals zero. There remains a positive definite quadratic form, which ensures our maximum principle. To test sufficient conditions, one should appeal to the theory of fields of extremals.

Example 2. Consider the problem of minimizing the functional

$$\int_{\mathfrak{N}} \left[(z_1)^3 + (z_2)^3\right] dt^1 \wedge dt^2. \tag{7}$$

Here

$$z_1 = \frac{\partial x^1}{\partial t^1}, \qquad z_2 = \frac{\partial x^2}{\partial t^2}, \qquad V = \{t_1^2 + t_2^2 \leq 1\}.$$

The boundary conditions are $x_1|_V = \cos\varphi$ and $x_2|_V = \sin\varphi$.

The Euler equations have the form

$$\frac{\partial}{\partial t_1}\left(3\left(\frac{\partial x_1}{\partial t_1}\right)^2\right) = 0, \qquad \frac{\partial}{\partial t_2}\left(3\left(\frac{\partial x_2}{\partial t_2}\right)^2\right) = 0. \tag{8}$$

It is easy to see that the unique solution to the Euler equations satisfying the boundary conditions is $x_1 = t_1$, $x_2 = t_2$. The second variation at the extremal on the rank 1 matrices equals $6\xi_1^2\eta_1^2 + 6\xi_2^2\eta_2^2$. It is strictly positive, so the Hadamard–Legendre condition of the weak minimum is fulfilled. However, Pontryagin's function on the rank 1 matrices equals $-(\xi_1^3\eta_1^3 + \xi_2^3\eta_2^3) + 6(\xi_1\eta_1 + \xi_2\eta_2)$. It reaches only a local maximum on the extremal $\{\xi_1\eta_1 = 1, \xi_2\eta_2 = 1\}$. The global maximum equals $+\infty$. By Theorem 2 we conclude that the minimum on the extremal is not strong.