$$= 2(A\dot h + Ch)\widetilde{x}\Big|_{t_0}^{t_1} + 2\int_{t_0}^{t_1}\Big(-\frac{d}{dt}(A\dot h + Ch) + C\dot h + Bh\Big)\widetilde{x}\,dt =$$
$$= 2\big(A(t_1)\dot h(t_1)x_1 - A(t_0)\dot h(t_0)x_0\big) = 2G.$$

После интегрирования по частям интеграл равен нулю, поскольку выражение в скобках под знаком интеграла обращается в нуль, так как функция h удовлетворяет уравнению Эйлера–Якоби. При переходе использовали также, что $h(t_1) = h(t_0) = 0$, $\widetilde{x}(t_1) = x_1$, $\widetilde{x}(t_0) = x_0$.

Следовательно, $J(\widetilde{x} + \lambda h) = J(\widetilde{x}) + 2\lambda G$. Так как допустимая экстремаль не существует, то по теореме 1 $G \neq 0$. Поэтому $J(\widetilde{x} + \lambda h) \to -\infty$ при $\lambda \to +\infty$ или $-\infty$, т.е. $S_{\text{absmin}} = -\infty$. $\square$

Теорема 2 в квадратично-линейном случае содержится в [2].

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On Hamiltonians induced from Fuchsian system and their applications

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Consider a system of linear ordinary differential equations of the form

$$(z\mathbb{I} - B)\dot X = AX, \tag{1}$$

where X is a complex-valued (n-column) vector function of a complex variable z, dot denotes differentiation with respect to z, A and B are constant matrices of size $n \times n$, and $\mathbb{I} = I_n$ is the identity matrix of the same size.

System (1) is called a system in Okubo normal form if the matrix B is diagonal. Certain physically interesting Schrödinger equations can be rewritten as Fuchsian systems of Okubo form. It is known that every accessory parameter free system of differential equations of Okubo normal form is rigid and has an integral representation of solutions; therefore, such systems are used to describe real physical processes (see [1]).

Consider a Schrödinger equation of the form

$$i\frac{\partial\Psi(t)}{\partial t} = H(t)\Psi(t) \tag{2}$$

with time dependent Hamiltonian $H(t) = (H_{ij}(t))$, $i, j = 1, \dots, N$, where

$$H_{11} = \varepsilon(t),\ H_{12} = V_2,\ H_{13} = V_3,\ \dots,\ H_{1N} = V_N,$$

$$H_{21} = V_2,\ H_{31} = V_3,\ \dots,\ H_{N1} = V_N, \text{ and } H_{ij} = 0 \text{ otherwise.}$$

Here $\Psi(t) = (\psi_1(t), \dots, \psi_N(t))$ is a wave function, V_j, $j = 2, \dots, n$, are constants, and the time dependent part ε has the form $\varepsilon(t) = E_1 \tanh(t/T)$ with constant E_1 and T.

Theorem 1 [2]**.** *Equation* (2) *is reducible to an N-dimensional Fuchsian system of Okubo type*

$$(zI_N - B)\frac{d\Phi(z)}{dz} = A\Phi(z)$$

with $B = \operatorname{diag}(i, -i, \dots, -i)$.

Remark 1. Hamiltonians of the above type appear in the theory of non-adiabatic transition. There are also Hamiltonians of other types which permit Fuchsian reduction. Reversing the argument used in the proof, one can obtain Schrödinger realizations for certain systems of Okubo type. In this way one can construct Schrödinger equations with prescribed qualitative properties of solutions.

In the two-dimensional case (i.e., for $N = 2$) this result can be generalized by permitting Hamiltonians of more general form. Namely, consider a Schrödinger equation with two-component phase function

$$i\frac{\partial f(t)}{\partial t} = H(t)f(t), \tag{3}$$

where $f(t) = (f_1(t), f_2(t))$ and time dependent Hamiltonian $H(t)$ has the form

$$H(t) = \begin{pmatrix} \varepsilon(t) & V(t) \\ V(t) & -\varepsilon(t) \end{pmatrix}.$$

Here

$$\varepsilon(t) = \frac{E_0T + E_1Ty}{1 + y^2}\frac{dy}{dt}, \qquad V(t) = \frac{V_0T}{\sqrt{1 + y^2}}\frac{dy}{dt}$$

with some monotonically increasing differentiable function $y\colon \mathbb{R} \to \mathbb{R}$ satisfying $y(t) \to \pm\infty$ as $t \to \pm\infty$.

Theorem 2. *Equation* (3) *is reducible to a system of two hypergeometric equations of the form*

$$z(z-1)\frac{d^2g}{dz^2} + (\gamma - (1+\alpha+\beta))\frac{dg}{dz} - \alpha\beta g(z) = 0$$

with appropriate constants α, β, *and* γ.

Since a single Fuchsian equation can always be rewritten as a system of Okubo type, this theorem provides a reduction of equation (3) to a system of Okubo type. For generalized hypergeometric equations, explicit formulae for the coefficients of a resulting system of Okubo type are given in following example.

Example. Consider the Gauss' hypergeometric equation written as

$$\left[(\delta+\alpha_1)(\delta+\alpha_2) - \frac{1}{z}\delta(\delta+\beta_1-1)\right]u = 0,$$

where $\delta = z\frac{d}{dz}$. Introduce a vector function $X(z)$ by $X_1(z) = u(z)$ and $X_2(z) = (\delta+\beta_1-1)X_1(z)$. Then it can be verified that $X(z)$ satisfies a system of Okubo type with $B = \mathrm{diag}(0,1)$, $A = (a_{ij})$, where $a_{11} = 1-\beta_1$, $a_{12} = 1$, $a_{21} = -(\alpha_1-\beta_1+1)(\alpha_2-\beta_1+1)$, and $a_{22} = -(1-\beta_1)-(\alpha_1+\alpha_2)$.

Thus Schrödinger equations with potentials as above can be written in Okubo form with explicitly known coefficient matrices, which enables one to compute the monodromy and asymptotics of solutions.

Remark 2. Theorem 2 is applicable to a number of physically interesting equations. In particular, taking $y(t) = \sinh(t/T)$, we obtain $\varepsilon(t) = E_0\,\mathrm{sech}(t/T) + E_1 \tanh(t/T)$, $V(t) = V_0$, where E_0, E_1, T, and V_0 are real constants. This Hamiltonian was used for analytic calculation of nonadiabatic transition probabilities from monodromy of differential equations.

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