GAMMA CONJECTURE VIA MIRROR SYMMETRY

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ABSTRACT. The asymptotic behaviour of solutions to the quantum differential equation of a Fano manifold F defines a characteristic class A_F of F, called the principal asymptotic class. Gamma conjecture [26] of Vasily Golyshev and the present authors claims that the principal asymptotic class A_F equals the Gamma class $\widehat{\Gamma}_F$ associated to Euler's Γ-function. We illustrate in the case of toric varieties, toric complete intersections and Grassmannians how this conjecture follows from mirror symmetry. We also prove that Gamma conjecture is compatible with taking hyperplane sections, and give a heuristic argument how the mirror oscillatory integral and the Gamma class for the projective space arise from the polynomial loop space.

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1. Introduction

1.1. **Gamma conjecture.** Gamma conjecture [26] is a conjecture which relates quantum cohomology of a Fano manifold with its topology. The small quantum cohomology of F defines a flat connection (quantum connection) over $\mathbb{C}^\times$ and its solution is given by a (multi-valued) cohomology-valued function $J_F(t)$ called the *J-function*. Under a certain condition (Property $\mathcal{O}$), the limit of the J-function:

$$A_F := \lim_{t\to+\infty} \frac{J_F(t)}{\langle [\mathrm{pt}], J_F(t)\rangle} \in H^\bullet(F)$$

exists and defines the *principal asymptotic class* A_F of F. *Gamma conjecture I* says that A_F equals the Gamma class $\widehat{\Gamma}_F = \widehat{\Gamma}(TF)$ of the tangent bundle of F (see §3.2):

$$A_F = \widehat{\Gamma}_F.$$

Suppose that the quantum cohomology of F is semisimple. In this case, we can define *higher* asymptotic classes $A_{F,i}$, $1 \leqslant i \leqslant N = \dim H^\bullet(F)$ from exponential asymptotics of flat sections of the quantum connection. *Gamma conjecture II* says that there exists a full exceptional collection $E_1, E_2, \dots, E_N$ of $D^b_{\mathrm{coh}}(F)$ such that we have

$$A_{F,i} = \widehat{\Gamma}_F \cdot \mathrm{Ch}(E_i) \qquad i = 1, \dots, N.$$

Here we write $\mathrm{Ch}(E) := (2\pi\sqrt{-1})^{\frac{\deg}{2}} \mathrm{ch}(E) = \sum_{p=0}^{\dim F} (2\pi\sqrt{-1})^p \mathrm{ch}_p(E)$ for the $(2\pi\sqrt{-1})$-modified version of the Chern character. The principal asymptotic class A_F corresponds to the exceptional object $E = \mathcal{O}_F$.

1.2. **Riemann–Roch.** One may view Gamma conjecture as a *square root* of the index theorem. Recall the Hirzebruch–Riemann–Roch formula:

$$\chi(E_1, E_2) = \int_F \mathrm{ch}(E_1^\vee) \cdot \mathrm{ch}(E_2) \cdot \mathrm{td}_F$$

for vector bundles E_1, E_2 on F, where $\chi(E_1, E_2) = \sum_{i=0}^{\dim F} (-1)^i \dim \mathrm{Ext}^i(E_1, E_2)$ is the Euler pairing and $\mathrm{td}_F = \mathrm{td}(TF)$ is the Todd class of F. The famous identity

$$\frac{x}{1-e^{-x}} = e^{x/2} \Gamma\left(1 - \frac{x}{2\pi\sqrt{-1}}\right) \Gamma\left(1 + \frac{x}{2\pi\sqrt{-1}}\right)$$

gives the factorization $(2\pi\sqrt{-1})^{\frac{\deg}{2}} \mathrm{td}_F = e^{\pi\sqrt{-1}c_1(F)} \cdot \widehat{\Gamma}_F \cdot \widehat{\Gamma}_F^*$ of the Todd class, which in turn factorizes the Hirzebruch–Riemann–Roch formula as:

$$\chi(E_1, E_2) = \left[\mathrm{Ch}(E_1) \cdot \widehat{\Gamma}_F, \mathrm{Ch}(E_2) \cdot \widehat{\Gamma}_F\right). \tag{1}$$

Here we set $\widehat{\Gamma}_F^* = (-1)^{\frac{\deg}{2}}\widehat{\Gamma}_F$ and $[\cdot,\cdot)$ is a non-symmetric pairing on $H^\bullet(F)$ given by

$$[\alpha,\beta) = \frac{1}{(2\pi)^{\dim F}}\int_F (e^{\pi\sqrt{-1}c_1(X)}e^{\pi\sqrt{-1}\mu}\alpha)\cup\beta \tag{2}$$

with $\mu \in \operatorname{End}(H^\bullet(F))$ defined by $\mu(\phi) = (p-\frac{\dim F}{2})\phi$ for $\phi \in H^{2p}(F)$. Via the factorization (1), Gamma conjecture II implies part of Dubrovin's conjecture [18]: the Euler matrix $\chi(E_i,E_j)$ of the exceptional collection equals the Stokes matrix $S_{ij} = [A_{F,i}, A_{F,j})$ of the quantum differential equation.

1.3. **Mirror symmetry.** In the B-side of mirror symmetry, solutions to Picard–Fuchs equations are often given by hypergeometric series whose coefficients are ratios of Γ-functions. Recall that a mirror of a quintic threefold $Q \subset \mathbb{P}^4$ is given by the pencil of hypersurfaces $Y_t = \{f(x) = t^{-1}\}$ in the torus $(\mathbb{C}^\times)^4$ which can be compactified to smooth Calabi–Yau threefolds $\overline{Y}_t$ [12], where f is the Laurent polynomial given by

$$f(x) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1x_2x_3x_4}.$$

For the holomorphic volume form $\Omega_t = d\log x_1 \wedge \cdots \wedge d\log x_4/df$ on $\overline{Y}_t$ and a real 3-cycle $C \subset \overline{Y}_t$, the period $\int_C \Omega_t$ satisfies the Picard–Fuchs equations

$$\left(\theta^4 - 5^5t^5(\theta+\tfrac{1}{5})(\theta+\tfrac{2}{5})(\theta+\tfrac{3}{5})(\theta+\tfrac{4}{5})\right)\int_C \Omega_t = 0$$

with $\theta = t\frac{\partial}{\partial t}$. The Frobenius method yields the following solution to this differential equation:

$$\Phi(t) = \sum_{n=0}^\infty \frac{\Gamma(1+5n+5\epsilon)}{\Gamma(1+n+\epsilon)^5} t^{5n+5\epsilon}$$

where ϵ is an infinitesimal parameter satisfying $\epsilon^4 = 0$. Regarding ϵ as a hyperplane class on the quintic Q, we may identify the leading term of the series $\Phi(t)$ with the inverse Gamma class of Q:

$$\frac{\Gamma(1+5\epsilon)}{\Gamma(1+\epsilon)^5} = \frac{1}{\widehat{\Gamma}_Q}.$$

This is how the Gamma class originally arose in the context of mirror symmetry [40, 51]. Hosono [39] conjectured (more generally for a complete intersection Calabi–Yau) that the period $\int_C \Omega_t$ of an integral 3-cycle $C \subset \overline{Y}_t$ should be written in the form

$$\int_Q \Phi(t)\cdot \operatorname{Ch}(V)\cdot \operatorname{Td}_Q \tag{3}$$

for a vector bundle $V \to Q$ which is "mirror" to C, where $\operatorname{Td}_Q = (2\pi\sqrt{-1})^{\frac{\deg}{2}}\operatorname{td}_Q$. In physics terminology, the period $\int_C \Omega_t$ (or the quantity (3)) is called the *central charge* of the D-branes C (resp. V). Hosono's conjecture has been answered affirmatively in [44] by showing that the natural integral structure $H^3(\overline{Y}_t,\mathbb{Z})$ agrees with the $\widehat{\Gamma}$-integral structure in quantum cohomology of Q, see also [38, 10, 42, 33].

A main purpose of this article is to explain a relationship between Gamma conjecture and mirror symmetry. In fact, Hosono's conjecture for a quintic Q is closely related to the truth of Gamma conjecture for the ambient Fano manifold $\mathbb{P}^4$. A mirror of $\mathbb{P}^4$ is given by the

Landau–Ginzburg model $f\colon (\mathbb{C}^\times)^4 \to \mathbb{C}$ and the quantum differential equation for $\mathbb{P}^4$ has a solution given by the oscillatory integral:

$$\int_\Gamma \exp(tf(x)) \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_4}{x_4}.$$

When the cycle Γ is a Lefschetz thimble of $f(x)$ and C is the associated vanishing cycle, this oscillatory integral can be written as a Laplace transform of the period $\int_C \Omega_t$. Correspondingly, by quantum Lefschetz principle [29, 48, 50, 14], the quantum differential equation for Q arises from a Laplace transform of the quantum differential equation for $\mathbb{P}^4$ [19, 45]. Gamma conjecture relates a Lefschetz thimble of f with an exceptional object E in $D^b_{\mathrm{coh}}(\mathbb{P}^4)$ via the exponential asymptotics of the corresponding oscillatory integral; then the vanishing cycle C associated with Γ corresponds to the spherical object $V = i^*E$ on Q under Hosono's conjecture, see [44, Theorem 6.9]. See the comparison table below for quantum differential equations (QDE) of a Fano manifold F and its anti-canonical section Q.

	Fano		Calabi–Yau
space X	F		$Q \in \lvert -K_F \rvert$
mirror	$f\colon (\mathbb{C}^\times)^n \to \mathbb{C}$		$\overline{Y}_t = \overline{f^{-1}(1/t)}$
singularities of QDE	irregular		regular
solutions (central charges)	oscillatory integral of f	$\overset{\text{Laplace}}{\leftrightarrow}$	period integral of $\overline{Y}_t$
cycles on the mirror	Lefschetz thimble	$\overset{\text{fiber}}{\to}$	vanishing cycle
objects of $D^b_{\mathrm{coh}}(X)$	exceptional object E_i	$\overset{i^*}{\to}$	spherical object V_i
monodromy data	Stokes $S_{ij} = \chi(E_i, E_j)$		reflection $S_{ij} - (-1)^n S_{ji}$

TABLE 1. We expect a mirror correspondence between objects of $D^b_{\mathrm{coh}}(X)$ and integration cycles on the mirror; when a vanishing cycle C arises as a fiber of a Lefschetz thimble Γ, the spherical object V on Q mirror to C should be the pull-back of the exceptional object E on F mirror to Γ. A Lefschetz thimble of f gives a solution to the quantum differential equation of F which has a specific exponential asymptotics as $t \to \infty$.

1.4. **Plan of the paper.** In §2, we review definitions and basic facts on quantum cohomology and quantum connection. In §3 and §4, we discuss Gamma conjecture I and II respectively. This part is a review of our previous paper [26] with Vasily Golyshev. In §5, we give a heuristic argument which gives mirror oscillatory integral and the Gamma class in terms of polynomial loop spaces. In §6 and §7, we discuss Gamma conjecture for toric varieties and toric complete intersections using Batyrev–Borisov/Givental/Hori–Vafa mirrors. In §8, we discuss compatibility of Gamma conjecture I with taking hyperplane sections (quantum Lefschetz). In §9, we discuss Gamma conjecture for Grassmannians $\mathrm{Gr}(r,n)$ using the Hori–Vafa mirror which is the rth alternate product of the mirrors of $\mathbb{P}^{n-1}$.

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2. Quantum cohomology and quantum connection

Let $H^{\bullet}(F) = H^{\bullet}(F, \mathbb{C})$ denote the even degree part of the Betti cohomology group of F. The (small) quantum product $\star_0$ on $H^{\bullet}(F)$ is defined by the formula:

$$(\alpha \star_0 \beta, \gamma)_F = \sum_{d \in H_2(F,\mathbb{Z})} \langle \alpha, \beta, \gamma \rangle_{0,3,d}$$

where $(\cdot, \cdot)_F$ is the Poincaré pairing on F and $\langle \cdots \rangle_{0,3,d}$ is the genus-zero three point Gromov–Witten invariants, which roughly speaking counts the number of rational curves in F passing through the cycles Poincaré dual to α, β and γ (see e.g. [53]). By the dimension axiom, these Gromov–Witten invariants are non-zero only if $2c_1(F) \cdot d + 2 \dim F = \deg \alpha + \deg \beta + \deg \gamma$; since F is Fano, there are finitely many such curve classes d and the above sum is finite. The product $\star_0$ is associative and commutative, and $(H^{\bullet}(F), \star_0)$ becomes a finite dimensional algebra. More generally we can define the big quantum product $\star_\tau$ for $\tau \in H^{\bullet}(F)$:

$$(\alpha \star_\tau \beta, \gamma) = \sum_{d \in H_2(F,\mathbb{Z})} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \alpha, \beta, \gamma, \tau, \dots, \tau \rangle_{0,3+n,d} \,.$$

This defines a formal[1] deformation of the small quantum cohomology as a commutative ring. In this paper, we will restrict our attention to the small quantum product $\star_0$.

The quantum connection is a meromorphic flat connection on the trivial $H^{\bullet}(F)$-bundle over $\mathbb{P}^1$. It is given by:

$$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}(c_1(F)\star_0) + \mu \tag{4}$$

where z is an inhomogeneous co-ordinate on $\mathbb{P}^1$ and $\mu \in \mathrm{End}(H^{\bullet}(F))$ is defined by $\mu(\phi) = (p - \frac{\dim F}{2})\phi$ for $\phi \in H^{2p}(F)$. This is regular singular (logarithmic) at $z = \infty$ and irregular singular at $z = 0$. We have a canonical fundamental solution around $z = \infty$ as follows:

Proposition 2.1 ([17, 26]). *There exists a unique* $\mathrm{End}(H^{\bullet}(F))$*-valued power series* $S(z) = \mathrm{id} + S_1 z^{-1} + S_2 z^{-2} + \cdots$ *which converges over the whole* z^{-1}*-plane such that*

$$\nabla(S(z) z^{-\mu} z^{c_1(F)} \phi) = 0 \qquad \forall \phi \in H^{\bullet}(F)$$
$$T(z) = z^{\mu} S(z) z^{-\mu} \text{ is regular at } z = \infty \text{ and } T(\infty) = \mathrm{id}.$$

Here we set $z^{c_1(F)} = \exp(c_1(F) \log z)$ *and* $z^{-\mu} = \exp(-\mu \log z)$.

The fundamental solution $S(z) z^{-\mu} z^{c_1(F)}$ identifies the space of flat sections with the cohomology group $H^{\bullet}(F)$. In other words, a basis of the cohomology group $H^{\bullet}(F)$ yields a basis of flat sections via the map $S(z) z^{-\mu} z^{c_1(F)}$ — this amounts to solving the quantum differential equation by the *Frobenius method* around $z = \infty$.

[1] the convergence of the big quantum product is not known in general.

Remark 2.2. The quantum connection can be extended to a meromorphic flat connection on the trivial $H^\bullet(F)$-bundle over $H^\bullet(F) \times \mathbb{P}^1$ via the big quantum product:

$$\begin{aligned} \nabla_\alpha &= \partial_\alpha + \frac{1}{z}(\alpha \star_\tau) \qquad \alpha \in H^\bullet(F), \\ \nabla_{z\partial_z} &= z\partial_z - \frac{1}{z}(E\star_\tau) + \mu \end{aligned} \tag{5}$$

where $E = c_1(F) + \sum_{i=1}^N (1 - \frac{\deg \phi_i}{2})\tau^i \phi_i$ with $\{\phi_i\}_{i=1}^N$ a homogeneous basis of $H^\bullet(F)$ and $\tau = \sum_{i=1}^N \tau^i \phi_i$.

Remark 2.3 ([26]). Via the gauge transformation z^μ and the change $z = t^{-1}$ of variables, we have

$$z^\mu \nabla_{z\partial_z} z^{-\mu} = z\frac{\partial}{\partial z} - (c_1(F)\star_{-c_1(F)\log z}) = -\left(t\frac{\partial}{\partial t} + c_1(F)\star_{c_1(F)\log t}\right)$$

This gives the quantum connection along the "anticanonical line" $\mathbb{C}c_1(F)$.

3. Gamma conjecture I (and I')

3.1. **Property $\mathcal{O}$.** We start with Property $\mathcal{O}$ for a Fano manifold. This is a supplementary condition we need to formulate Gamma conjecture I.

Definition 3.1 (Property $\mathcal{O}$ [26])**.** Let F be a Fano manifold and define a non-negative real number T as

$$T := \max\{|u| : u \in \mathbb{C} \text{ is an eigenvalue of } (c_1(F)\star_0)\} \in \overline{\mathbb{Q}}. \tag{6}$$

We say that F satisfies *Property* $\mathcal{O}$ if the following conditions are satisfied:

(1) T is an eigenvalue of $(c_1(F)\star_0)$ of multiplicity one;
(2) if u is an eigenvalue of $(c_1(F)\star_0)$ with $|u| = T$, there exists an rth root ζ of unity such that $u = \zeta T$, where r is the Fano index of F, i.e. the maximal integer $r > 0$ such that $c_1(F)/r$ is an integral class.

Remark 3.2. '$\mathcal{O}$' means the structure sheaf of F. Under mirror symmetry, it is conjectured that the set of eigenvalues of $(c_1(F)\star_0)$ agrees with the set of critical values of the mirror Landau–Ginzburg potential f. Under Property $\mathcal{O}$, number T should be a critical value of f and the Lefschetz thimble corresponding to T should be mirror to the structure sheaf $\mathcal{O}$. Conjecture $\mathcal{O}$ [26] says that every Fano manifold satisfies Property $\mathcal{O}$. Some Fano *orbifolds* with non-trivial $\pi_1^{\mathrm{orb}}(F)$ do not satisfy Property $\mathcal{O}$ [25, 26].

Remark 3.3. Perron–Frobenius theorem says that an irreducible square matrix with non-negative entries has a positive eigenvalue with the biggest norm whose multiplicity is one. It is likely that $c_1(F)\star_0$ is represented by a non-negative matrix if we have a basis of $H^\bullet(F)$ consisting of 'positive' (algebraic) cycles. This remark is due to Kaoru Ono.

Remark 3.4. Property $\mathcal{O}$ for homogeneous spaces G/P was recently proved by Cheong [13].

3.2. **Gamma class.** The Gamma class [51, 52, 42] is a characteristic class defined for an almost complex manifold F. Let $\delta_1, \dots, \delta_n$ be the Chern roots of the tangent bundle TF of F such that $c(TF) = (1 + \delta_1)(1 + \delta_2) \cdots (1 + \delta_n)$. The Gamma class $\widehat{\Gamma}_F$ is defined to be

$$\widehat{\Gamma}_F = \prod_{i=1}^n \Gamma(1 + \delta_i) \in H^\bullet(F, \mathbb{R})$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ is Euler's Γ-function. Since $\Gamma(z)$ is holomorphic at $z = 1$, via the Taylor expansion, the right-hand side makes sense as a symmetric power series in $\delta_1, \dots, \delta_n$, and therefore as a (real) cohomology class of F. This is a transcendental class and is given by

$$\widehat{\Gamma}_F = \exp\left(-\gamma c_1(F) + \sum_{k=2}^{\infty} (-1)^{k-1} (k-1)! \zeta(k) \operatorname{ch}_k(TF)\right)$$

where $\zeta(z)$ is the Riemann zeta function. As explained in the Introduction (see §1.2), the formula $(2\pi\sqrt{-1})^{\frac{\deg}{2}} \operatorname{td}_F = e^{\pi\sqrt{-1}c_1(F)} \widehat{\Gamma}_F \widehat{\Gamma}_F^*$ shows that the Gamma class $\widehat{\Gamma}_F$ can be regarded as a square root of the Todd class td_F. There is also an interpretation of the Gamma class in terms of the free loop space $\mathcal{L}F$ of F due to Lu [52] (see also [43, 26]): $\widehat{\Gamma}_F$ arises from the ζ-function regularization of the S^1-equivariant Euler class $e_{S^1}(\mathcal{N}_+)$, where $\mathcal{N}_+$ is the positive normal bundle of the locus F of constant loops in $\mathcal{L}F$.

3.3. **Principal asymptotic class and Gamma conjecture I.** Consider the space of flat sections for the quantum connection ∇ (4) over the positive real line $\mathbb{R}_{>0}$. We introduce the subspace $\mathcal{A}$ of flat sections having the smallest asymptotics $\sim e^{-T/z}$ as $z \to +0$.

$$\mathcal{A} := \left\{ s \colon \mathbb{R}_{>0} \to H^\bullet(F) : \nabla s(z) = 0,\ \|e^{T/z} s(z)\| = O(z^{-m}) \text{ as } z \to +0\ (\exists m) \right\}$$

where $T > 0$ is the number in Definition 3.1.

Proposition 3.5 ([26, Proposition 3.3.1])**.** *Suppose that a Fano manifold F satisfies Property $\mathcal{O}$. We have* $\dim_{\mathbb{C}} \mathcal{A} = 1$. *Moreover, for every element $s(z) \in \mathcal{A}$, the limit* $\lim_{z \to +0} e^{T/z} s(z)$ *exists and lies in the T-eigenspace $E(T)$ of $(c_1(F) \star_0)$.*

The principal asymptotic class of F is defined to be the class corresponding to a generator of the one-dimensional space $\mathcal{A}$.

Definition 3.6 ([26])**.** Suppose that a Fano manifold F satisfies Property $\mathcal{O}$. A cohomology class $A_F \in H^\bullet(F)$ satisfying

$$\mathcal{A} = \mathbb{C}\left[S(z) z^{-\mu} z^{c_1(F)} A_F\right]$$

is called the *principal asymptotic class*. Here $S(z) z^{-\mu} z^{c_1(F)}$ is the fundamental solution in Proposition 2.1. The principal asymptotic class is determined up to multiplication by a non-zero complex number; when $\langle [\mathrm{pt}], A_F \rangle \neq 0$, we can normalize A_F so that $\langle [\mathrm{pt}], A_F \rangle = 1$.

Since the space $\mathcal{A}$ is identified via the asymptotics near the irregular singular point $z = 0$ and the fundamental solution $S(z) z^{-\mu} z^{c_1(F)}$ is normalized at the regular singular point $z = \infty$, the definition of the class A_F involves *analytic continuation* along the positive real line $\mathbb{R}_{>0}$ on the z-plane. Note that $S(z) z^{-\mu} z^{c_1(F)}$ has a standard determination for $z \in \mathbb{R}_{>0}$ given by $z^{-\mu} z^{c_1(F)} = \exp(-\mu \log z) \exp(c_1(F) \log z)$ and $\log z \in \mathbb{R}$.

Conjecture 3.7 (Gamma Conjecture I [26])**.** *Let F be a Fano manifold satisfying Property $\mathcal{O}$. The principal asymptotic class A_F of F is given by the Gamma class $\widehat{\Gamma}_F$ of F.*

There is another description of the principal asymptotic class A_F in terms of solutions to the quantum differential equation. We introduce Givental's J-function [29] by the formula:

$$\begin{aligned} J_F(t) &= z^{\frac{\dim F}{2}} \left(S(z) z^{-\mu} z^{c_1(F)}\right)^{-1} 1 \qquad \text{with } t = z^{-1} \\ &= e^{c_1(F) \log t} \left(1 + \sum_{i=1}^{N} \sum_{d \in H_2(F,\mathbb{Z}), d \neq 0} \left\langle \frac{\phi_i}{1-\psi} \right\rangle_{0,1,d} \phi^i t^{c_1(F) \cdot d}\right) \end{aligned} \tag{7}$$

where $S(z)z^{-\mu}z^{c_1(F)}$ is the fundamental solution in Proposition 2.1 and ψ is the first Chern class of the universal cotangent line bundle over the moduli space of stable maps. This is a cohomology-valued (and multi-valued) function. Since $S(z)z^{-\mu}z^{c_1(F)}$ is a fundamental solution of the quantum connection and by Remark 2.3, we have

$$P(t,\nabla_{c_1(F)})1 = 0 \quad\Longleftrightarrow\quad P(t,t\tfrac{\partial}{\partial t})J_F(t) = 0$$

for any differential operator $P(t,t\frac{\partial}{\partial t}) \in \mathbb{C}\langle t, t\frac{\partial}{\partial t}\rangle$, where $\nabla_{c_1(F)} = t\frac{\partial}{\partial t} + c_1(F)\star_{c_1(F)\log t}$ is the quantum connection along the anticanonical line. In other words, $J_F(t)$ satisfies all the differential relations satisfied by the identity class 1 with respect to the connection $\nabla_{c_1(F)}$; in this sense $J_F(t)$ is a solution of the quantum connection. Differential operators $P(t,t\frac{\partial}{\partial t})$ annihilating $J_F(t)$ are called *quantum differential operators*. The principal asymptotic class A_F can be computed by the $t\to+\infty$ asymptotics of the J-function:

Proposition 3.8 ([26])**.** *Suppose that a Fano manifold F satisfies Property $\mathcal{O}$ and let A_F be the principal asymptotic class. Then we have an asymptotic expansion of the form:*

$$J_F(t) = Ct^{-\frac{\dim F}{2}}e^{Tt}(A_F + \alpha_1 t^{-1} + \alpha_2 t^{-2} + \cdots).$$

as $t\to+\infty$ on the positive real line, where $C\neq 0$ is a non-zero constant and $\alpha_i \in H^\bullet(F)$.

Proof. It follows from [26, Proposition 3.6.2] that $\lim_{t\to\infty} t^{\frac{\dim F}{2}}e^{-Tt}J_F(t)$ exists and is proportional to A_F. The fact that the remainder admits an asymptotic expansion of the form $\alpha_1 t^{-1}+\alpha_2 t^{-2}+\cdots$ follows from the proof there, in particular from [26, Proposition 3.2.1]. $\square$

This proposition says that $\mathbb{C}J_F(t)$ converges to $\mathbb{C}A_F$ in the projective space $\mathbb{P}(H^\bullet(F))$ as $t\to+\infty$. Since $\langle[\mathrm{pt}],\widehat{\Gamma}_F\rangle = 1$, we obtain the following corollary.

Corollary 3.9 ([26, Corollary 3.6.9])**.** *Let F be a Fano manifold satisfying Property $\mathcal{O}$. Gamma conjecture I holds for F if and only if we have*

$$\widehat{\Gamma}_F = \lim_{t\to\infty}\frac{J_F(t)}{\langle[\mathrm{pt}],J_F(t)\rangle}.$$

We can replace the continuous limit in the above corollary with a discrete limit of ratios of the Taylor coefficients. Expand the J-function as:

$$J_F(t) = e^{c_1\log t}\sum_{n=0}^{\infty} J_n t^n. \tag{8}$$

Note that $J_n = 0$ if n is not divisible by the Fano index r of F. We have the following:

Proposition 3.10 ([26, Theorem 3.7.1])**.** *Suppose that a Fano manifold F satisfies Property $\mathcal{O}$ and Gamma conjecture I. Let r be the Fano index of F. Then we have*

$$\liminf_{n\to\infty}\left|\frac{\langle\alpha,J_{rn}\rangle}{\langle[\mathrm{pt}],J_{rn}\rangle} - \langle\alpha,\widehat{\Gamma}_F\rangle\right| = 0$$

for every $\alpha\in H_\bullet(F)$ with $\alpha\cap c_1(F) = 0$.

Define the (unregularized and regularized) *quantum period* of F [15] to be

$$\begin{aligned} G_F(t) &= \langle[\mathrm{pt}],J_F(t)\rangle = \sum_{n=0}^{\infty} G_n t^n \\ \widehat{G}_F(\kappa) &= \sum_{n=0}^{\infty} n! G_n \kappa^n = \frac{1}{\kappa}\int_0^\infty G_F(t)e^{-t/\kappa}dt \end{aligned} \tag{9}$$

where $G_n = \langle [\mathrm{pt}], J_n\rangle$. It is shown in [26, Lemma 3.7.6] that if F satisfies Property $\mathcal{O}$ and if $\langle [\mathrm{pt}], A_F\rangle \neq 0$, the convergence radius of $\widehat{G}_F(\kappa)$ equals $1/T$. In particular

$$\limsup_{n\to\infty} \sqrt[rn]{(rn)!|G_{rn}|} = T. \tag{10}$$

Suppose that this limit sup (10) can be replaced with the limit, i.e. $\lim_{n\to\infty} \sqrt[rn]{(rn)!|G_{rn}|} = T$. Then the argument in the proof of [26, Theorem 3.7.1] shows, under the same assumption as in Proposition 3.10, that

$$\lim_{n\to\infty} \frac{\langle \alpha, J_{rn}\rangle}{\langle [\mathrm{pt}], J_{rn}\rangle} = \langle \alpha, \widehat{\Gamma}_F\rangle \tag{11}$$

for a class $\alpha \in H_{\bullet}(F)$ such that $\alpha \cap c_1(F) = 0$. Therefore we can consider the following variant of Gamma conjecture I:

Conjecture 3.11 (Gamma conjecture I'). *For a Fano manifold F and a class $\alpha \in H_{\bullet}(F)$ with $\alpha \cap c_1(F) = 0$, the limit formula* (11) *holds.*

Remark 3.12. By the above discussion, Gamma conjecture I' holds if Gamma conjecture I holds and one has $\lim_{n\to\infty} \sqrt[rn]{(rn)!|G_{rn}|} = T$.

The discrete limit in the left-hand side of (11) is called the *Apéry constant* (or Apéry limit) and was studied by Almkvist–van-Straten–Zudilin [1] in the context of Calabi–Yau differential equations and by Golyshev [31] and Galkin [23] for Fano manifolds. Under Gamma conjecture I', these limits are expressed in terms of the zeta values $\zeta(2), \zeta(3), \zeta(4), \dots$. For some Fano manifolds, they are precisely the limits which Apéry used to prove the irrationality of $\zeta(2)$ and $\zeta(3)$: an Apéry limit for the Grassmannian $G(2,5)$ gives a fast approximation of $\zeta(2)$ and an Apéry limit for the orthogonal Grassmannian $OG(5,10)$ gives a fast approximation of $\zeta(3)$ [23, 31]. Most of the Apéry limits of Fano manifolds are not fast enough to prove irrationality (see [23]). It would be very interesting to find a Fano manifold which gives a fast approximation of $\zeta(5)$, for example.

We give a sufficient condition that ensures that the limit sup in (10) can be replaced with the limit. When a Laurent polynomial $f(x) \in \mathbb{C}[x_1^{\pm}, \dots, x_m^{\pm}]$ is mirror to F, we expect that the quantum period $G_F(t)$ (9) for F should be given by the *constant term series* of f:

$$G_F(t) = \frac{1}{(2\pi\sqrt{-1})^m} \int_{(S^1)^m} e^{tf(x)} \frac{dx_1 \cdots dx_m}{x_1 \cdots x_m} = \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{Const}(f^n) t^n$$

where $\operatorname{Const}(f^n)$ denotes the constant term of the Laurent polynomial $f(x)^n$. When this holds, $f(x)$ is said to be a *weak Landau–Ginzburg model* of F [58].

Lemma 3.13. *Let F be a Fano manifold of index r. Suppose that F admits a weak Landau–Ginzburg model $f(x) \in \mathbb{C}[x_1^{\pm}, \dots, x_m^{\pm}]$ whose coefficients are non-negative real numbers. Suppose also that* $\operatorname{Const}(f^{rn}) \neq 0$ *for all but finitely many $n \in \mathbb{Z}_{\geqslant 0}$. Then the coefficients G_n of the quantum period* (9) *are non-negative and the limit*

$$\lim_{n\to\infty} \sqrt[rn]{(rn)!|G_{rn}|} = \lim_{n\to\infty} \sqrt[rn]{\operatorname{Const}(f^{rn})}$$

exists.

Proof. Some of the techniques here are borrowed from [24]. We set $\alpha_n = \log(\sqrt[rn]{\operatorname{Const}(f^{rn})})$. It suffices to show that $\lim_{n\to\infty} \alpha_n$ exists. By assumption, there exists $n_0 \in \mathbb{Z}_{\geqslant 0}$ such that α_n is well-defined for all $n \geqslant n_0$. Since $\operatorname{Const}(f^{r(n+m)}) \geqslant \operatorname{Const}(f^{rn}) \operatorname{Const}(f^{rm})$, we have

$$\alpha_{n+m} \geqslant \frac{n}{n+m}\alpha_n + \frac{m}{n+m}\alpha_m$$

Set $\alpha := \limsup_{n\to\infty} \alpha_n$. For any $\epsilon > 0$, there exists $n_1 \geqslant 1$ such that $\alpha_{n_1} \geqslant \alpha - \epsilon$. Then we have, for all $k \geqslant 1$ and $0 \leqslant i < n_1$,

$$\alpha_{kn_1+n_0+i} \geqslant \frac{n_0+i}{kn_1+n_0+i}\alpha_{n_0+i} + \frac{kn_1}{kn_1+n_0+i}\alpha_{n_1}$$

The right-hand side converges to α_{n_1} as $k \to \infty$. This implies that there exists $n_2 \geqslant n_0$ such that for all $n \geqslant n_2$, we have

$$\alpha_n \geqslant \alpha - 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, this implies $\liminf_{n\to\infty} \alpha_n \geqslant \alpha$ and the conclusion follows. □

Remark 3.14. When discussing the continuous limit, we only need to assume Part (1) of Property $\mathcal{O}$ (Definition 3.1). More precisely, Propositions 3.5, 3.8 and Corollary 3.9 hold for Fano manifolds satisfying only Part (1) of Property $\mathcal{O}$, and Gamma conjecture I (Conjecture 3.7) makes sense for such Fano manifolds. On the other hand, we need Part (2) of Property $\mathcal{O}$ in the proof of Proposition 3.10.

Remark 3.15. Golyshev and Zagier have announced the proof of Gamma conjecture I for Fano threefolds of Picard rank one.

4. Gamma conjecture II

The small quantum cohomology $(H^{\bullet}(F), \star_0)$ of a Fano manifold F is said to be semisimple if it is isomorphic to the direct sum of $\mathbb{C}$ as a ring. This is equivalent to the condition that $(H^{\bullet}(F), \star_0)$ has no nilpotent elements. In this section, we give a refinement of Gamma conjecture I for a Fano manifold with semisimple quantum cohomology.

4.1. **Formal fundamental solution.** Suppose that $(H^{\bullet}(F), \star_0)$ is semisimple. Then we have an idempotent basis $\psi_1, \dots, \psi_N$ of $H^{\bullet}(F)$ such that $\psi_i \star_0 \psi_j = \delta_{ij}\psi_i$. Define the normalized idempotent basis $\Psi_1, \dots, \Psi_N$ to be $\Psi_i = \psi_i/\sqrt{(\psi_i, \psi_i)}$, where (ψ_i, ψ_i) denotes the Poincaré pairing of ψ_i with itself. Note that $(\Psi_1, \dots, \Psi_N)$ is unique up to sign and ordering. We set

$$\Psi = \begin{pmatrix} | & | & & | \\ \Psi_1 & \Psi_2 & \cdots & \Psi_N \\ | & | & & | \end{pmatrix}.$$

This is a matrix with column vectors Ψ_i. We may regard it as a linear map $\mathbb{C}^N \to H^{\bullet}(F)$. Let $u_1, \dots, u_N$ be the eigenvalues of $(c_1(F)\star_0)$ such that $c_1(F) \star_0 \Psi_i = u_i \Psi_i$. Let U be the diagonal matrix with entries $u_1, \dots, u_N$:

$$U = \begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_N \end{pmatrix}$$

The following proposition is well-known in the context of Frobenius manifolds.

Proposition 4.1 ([17, Lectures 4,5], [61, Theorem 8.15])**.** *Suppose that the small quantum cohomology $(H^{\bullet}(F), \star_0)$ is semisimple. The quantum connection* (4) *near the irregular singular point $z = 0$ admits a formal fundamental matrix solution of the form:*

$$\Psi R(z) e^{-U/z}$$

where $R(z) = \mathrm{id} + R_1 z + R_2 z^2 + \cdots \in \mathrm{End}(\mathbb{C}^N)[\![z]\!]$ is a matrix-valued formal power series. The formal solution $\Psi R(z) e^{-U/z}$ is unique up to multiplication by a signed permutation matrix from the right (which corresponds to the ambiguity of $\Psi_1, \dots, \Psi_N$).

Remark 4.2. Since $R(z)$ is a formal power series in positive powers of z, the product $R(z)e^{-U/z}$ does not make sense as a power series. The meaning of the proposition is that the formal gauge transformation by $\Psi R(z)$ turns $\nabla_{z\partial_z}$ into $z\partial_z - U/z$.

Remark 4.3. We do not need to assume that $u_1, \dots, u_N$ are mutually distinct.

4.2. **Lift to an analytic solution.** By choosing an angular sector, the above formal solution can be lifted to an actual analytic solution. This is an instance of the *Hukuhara–Turrittin theorem* for irregular connections (see e.g. [64, Theorem 19.1], [60, II, 5.d]). We say that a phase $\phi \in \mathbb{R}$ (or $e^{\sqrt{-1}\phi} \in S^1$) is *admissible* for a multiset $\{u_1, u_2, \dots, u_N\} \subset \mathbb{C}$ if $\mathrm{Im}(u_i e^{-\sqrt{-1}\phi}) \neq \mathrm{Im}(u_j e^{-\sqrt{-1}\phi})$ for every pair (u_i, u_j) with $u_i \neq u_j$, i.e. $e^{\sqrt{-1}\phi}$ is not parallel to any non-zero difference $u_i - u_j$.

Proposition 4.4 ([64, Theorem 12.2], [4, Theorem A], [17, Lectures 4,5], [11, §8], [26, Proposition 2.5.1])**.** *Let $\phi \in \mathbb{R}$ be an admissible phase for the spectrum $\{u_1, u_2, \dots, u_N\}$ of $(c_1(F)\star_0)$. There exist $\epsilon > 0$ and an analytic fundamental solution $Y_\phi(z) = (y_1^\phi(z), \dots, y_N^\phi(z))$ for the quantum connection* (4) *on the angular sector $|\arg(z) - \phi| < \frac{\pi}{2} + \epsilon$ around $z = 0$ such that one has the asymptotic expansion*

$$Y_\phi(z)e^{U/z} \sim \Psi R(z) \tag{12}$$

as $z \to 0$ in the sector $|\arg(z) - \phi| < \frac{\pi}{2} + \epsilon$, where $\Psi R(z)e^{-U/z}$ is the formal fundamental solution in Proposition 4.1. Such an analytic solution $Y_\phi(z)$ is unique when we fix the sign and the ordering of $\Psi_1, \dots, \Psi_N$.

Remark 4.5. Notice that each flat section $y_i^\phi(z)$ has the exponential asymptotics $\sim e^{-u_i/z}\Psi_i$ as $z \to 0$ in the sector $|\arg(z) - \phi| < \frac{\pi}{2} + \epsilon$.

Remark 4.6. The precise meaning of the asymptotic expansion (12) is as follows. For any $0 < \epsilon' < \epsilon$ and $n \in \mathbb{Z}_{\geqslant 0}$, there exists a constant $C = C(\epsilon', n)$ such that

$$\left\| Y_\phi(z)e^{-U/z} - \sum_{k=0}^{n} \Psi R_k z^k \right\| \leqslant C|z|^{n+1}$$

for all z with $|\arg z - \phi| \leqslant \frac{\pi}{2} + \epsilon'$ and $|z| \leqslant 1$, where we write $R(z) = \sum_{k=0}^\infty R_k z^k$.

4.3. **Asymptotic basis and Gamma conjecture II.** Let $Y_\phi(z) = (y_1^\phi(z), \dots, y_N^\phi(z))$ be the analytic fundamental solution in Proposition 4.4. We regard $Y_\phi(z)$ as a function defined on the universal cover of $\mathbb{C}^\times$; initially it is defined on the angular sector $|\arg(z) - \phi| < \pi + \epsilon$ with $|z| \ll 1$, but can be analytically continued to the whole universal cover since it is a solution to a linear differential equation. For an admissible phase ϕ for $\{u_1, \dots, u_N\}$, we define the higher asymptotic classes $A_{F,i}^\phi \in H^\bullet(F)$, $i = 1, \dots, N$ by

$$y_i^\phi(z)\Big|_{\substack{\text{parallel translate}\\ \text{to } \arg(z) = 0}} = \frac{1}{(2\pi)^{\dim F/2}} S(z) z^{-\mu} z^{c_1(F)} A_{F,i}^\phi$$

where $S(z)z^{-\mu}z^{c_1(F)}$ is the fundamental solution for the quantum connection in Proposition 2.1. We call $\{A_{F,1}^\phi, A_{F,2}^\phi, \dots, A_{F,N}^\phi\}$ the *asymptotic basis* at the phase ϕ. The asymptotic basis is the same as what Dubrovin [17] called the *central connection matrix*.

Remark 4.7. Suppose that F satisfies Property $\mathcal{O}$. When we take an admissible phase ϕ from the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, the class $A_{F,i}^\phi$ corresponding to the eigenvalue $u_i = T$ is proportional to the principal asymptotic class A_F.

Remark 4.8. The asymptotic basis at a phase ϕ is unique up to sign and ordering. The sign and the ordering depend on those of the normalized idempotents $\Psi_1, \dots, \Psi_N$. Each asymptotic class $A^{\phi}_{F,i}$ is marked by the eigenvalue u_i of $(c_1(F)\star_0)$. With respect to the pairing $[\cdot,\cdot)$ in (2), these data $\{(A^{\phi}_{F,i}, u_i)\}_{i=1}^N$ form a *marked reflection system* [26], see also Remark 4.13.

Conjecture 4.9 (Gamma conjecture II [26])**.** *Suppose that a Fano manifold F has a semisimple small quantum cohomology and that $D^b_{\mathrm{coh}}(F)$ has a full exceptional collection. Let ϕ be an admissible phase for the spectrum $\{u_1, \dots, u_N\}$ of $(c_1(F)\star_0)$. We number the eigenvalues $u_1, \dots, u_N$ so that $\mathrm{Im}(e^{-\sqrt{-1}\phi}u_1) \geqslant \mathrm{Im}(e^{-\sqrt{-1}\phi}u_2) \geqslant \cdots \geqslant \mathrm{Im}(e^{-\sqrt{-1}\phi}u_N)$. There exists a full exceptional collection $E^{\phi}_1, \dots, E^{\phi}_N$ such that $A^{\phi}_{F,i} = \widehat{\Gamma}_F \cdot \mathrm{Ch}(E^{\phi}_i)$.*

Recall that $\mathrm{Ch}(E) = \sum_{p=0}^{\dim F} (2\pi\sqrt{-1})^p \,\mathrm{ch}_p(E)$ is the modified Chern character.

Remark 4.10. Part (3) of Dubrovin's conjecture [18, Conjecture 4.2.2] says that the columns of the central connection matrix are given by $C'(\mathrm{Ch}(E_i))$ for some linear operator $C' \in \mathrm{End}(H^{\bullet}(F))$ commuting with $c_1(F)\cup$. Gamma conjecture II says that $C' = \widehat{\Gamma}_F\cup$. Recently Dubrovin [20] also proposed the same conjecture as Gamma conjecture II.

4.4. **Stokes matrix.** Let $Y_\phi(z)$ and $Y^-_\phi(z)$ be the analytic fundamental solutions from Proposition 4.4 associated respectively to admissible directions $e^{\sqrt{-1}\phi}$ and $-e^{\sqrt{-1}\phi}$. The domains of definitions of Y_ϕ and Y^-_ϕ are shown in Figure 1. Let $\Pi_\pm$ be the angular regions as in Figure 1 which are components of the intersection of the domains of Y_ϕ and Y^-_ϕ. The *Stokes matrices* are the constant matrices S^ϕ and S^ϕ_- satisfying

$$
\begin{aligned}
Y_\phi(z) &= Y^-_\phi(z) S^\phi && \text{for } z \in \Pi_+;\\
Y_\phi(z) &= Y^-_\phi(z) S^\phi_- && \text{for } z \in \Pi_-.
\end{aligned}
$$

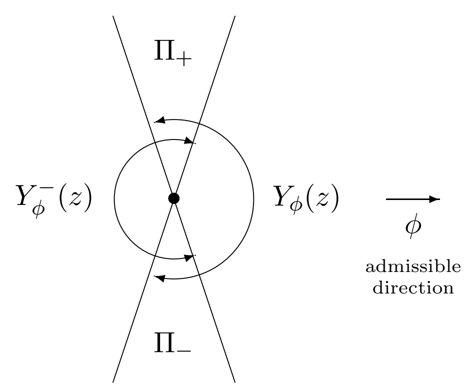

FIGURE 1.

Proposition 4.11 ([17, Theorem 4.3], [26, Proposition 2.6.4])**.** *Let $Y_\phi(z) = (y^\phi_1(z), \dots, y^\phi_N(z))$ be the fundamental solution from Proposition 4.4. The Stokes matrices at phase ϕ are given by $S^\phi_{ij} = S^\phi_{-,ji} = (y^\phi_i(e^{-\pi\sqrt{-1}}z), y^\phi_j(z))$, where $y^\phi_i(e^{-\pi\sqrt{-1}}z)$ denotes the analytic continuation* *of $y_i(z)$ along the path $[0,\pi] \ni \theta \mapsto e^{-\sqrt{-1}\theta} z$. The flat sections $y_1^\phi, \dots, y_N^\phi$ are semi-orthogonal in the following sense:*

$$S_{ij}^\phi = \begin{cases} 0 & \text{if } (i \neq j \text{ and } u_i = u_j) \text{ or } \operatorname{Im}(e^{-\sqrt{-1}\phi} u_i) < \operatorname{Im}(e^{-\sqrt{-1}\phi} u_j); \\ 1 & \text{if } i = j. \end{cases}$$

Using the non-symmetric pairing $[\cdot,\cdot)$ given in (2), we have

$$\begin{aligned} S_{ij}^\phi &= (y_i^\phi(e^{-\sqrt{-1}\pi} z), y_j^\phi(z)) \\ &= \frac{1}{(2\pi)^{\dim F}} \left(S(-z) z^{-\mu} e^{\pi\sqrt{-1}\mu} z^{c_1(F)} e^{-\pi\sqrt{-1} c_1(F)} A_{F,i}^\phi, S(z) z^{-\mu} z^{c_1(F)} A_{F,j}^\phi \right) \\ &= [A_{F,i}^\phi, A_{F,j}^\phi) \end{aligned}$$

where we used the fact that $(S(-z)\alpha, S(z)\beta) = (\alpha,\beta)$ and $(z^{-\mu}\alpha, z^{-\mu}\beta) = (\alpha,\beta)$ (see [17]). Therefore, the factorization (1) of the Hirzebruch–Riemann–Roch formula implies the following corollary.

Corollary 4.12. *Suppose that a Fano manifold F satisfies Gamma conjecture II. Then there exists a full exceptional collection $E_1, \dots, E_N$ of $D^b_{\mathrm{coh}}(F)$ such that $\chi(E_i, E_j)$ equals the Stokes matrix S_{ij}. (This conclusion is part (2) of Dubrovin's conjecture* [18, Conjecture 4.2.2]*).*

Remark 4.13. The asymptotic basis can be defined similarly for the big quantum product $\star_\tau$ with $\tau \in H^\bullet(F)$ as far as $\star_\tau$ is semisimple. We have an asymptotic basis $A_{F,i}^{\phi,\tau}$ depending on both ϕ and τ, and Gamma conjecture II makes sense at general $\tau \in H^\bullet(F)$. The truth of the Gamma conjecture II is, however, independent of the choice of (τ,ϕ). This is because the asymptotic basis changes (discontinuously) by mutation as (τ,ϕ) varies, and we can consider the corresponding mutation for exceptional collections. The right mutation of asymptotic bases takes the form

$$(A_1, A_2, \dots, \overset{i}{A_i}, \overset{i+1}{A_{i+1}}, \dots A_N) \mapsto (A_1, A_2, \dots, \overset{i}{A_{i+1}}, A_i - [A_i, \overset{i+1}{A_{i+1}}) A_{i+1}, \cdots, A_N)$$

if we order the asymptotic basis so that $\operatorname{Im}(e^{-\sqrt{-1}\phi} u_1) \geqslant \operatorname{Im}(e^{-\sqrt{-1}\phi} u_2) \geqslant \cdots \geqslant \operatorname{Im}(e^{-\sqrt{-1}\phi} u_N)$. The right mutation happens when the eigenvalue u_{i+1} crosses the ray $u_i + \mathbb{R}_{\geqslant 0} e^{\sqrt{-1}\phi}$. (The left mutation is the inverse of the right mutation). The braid group action on asymptotic bases and Stokes matrices has been studied by Dubrovin, see [17, 26] for more details.

Remark 4.14. In general, the quantum connection of a smooth projective variety (or a projective orbifold) X is underlain by the integral local system consisting of flat sections $(2\pi)^{-\frac{\dim F}{2}} S(z) z^{-\mu} z^{c_1(F)} \widehat{\Gamma}_X \operatorname{Ch}(E)$ with $E \in K^0(X)$. This is called the $\widehat{\Gamma}$*-integral structure* [42, 47]. In this language, Gamma conjecture implies that this integral structure should be compatible with the Stokes structure at the irregular singular point $z = 0$ of the quantum connection. Katzarkov-Kontsevich-Pantev [47] imposed the compatibility of rational structure with Stokes structure as part of conditions for nc-Hodge structure. Our Gamma conjecture makes their compatibility condition explicit for Fano manifolds. Note that Gamma conjecture II implies the integrality of the Stokes matrix S_{ij}.

5. Mirror heuristics

In this section we give a heuristic argument which gives the mirror oscillatory integral and the Gamma class from the polynomial loop space (quasi map space) for $\mathbb{P}^{N-1}$. This is motivated by Givental's equivariant Floer theory heuristics [27, 28]. The argument in this section can be applied more generally to toric varieties to yield their mirrors and Gamma classes.

5.1. **Polynomial loop space.** Givental [27, 28] conjectured that the quantum D-module (i.e. quantum connection) should be identified with the S^1-equivariant Floer theory for the free loop space (see also [63, 41, 2]). Following Givental, we consider an algebraic version of the loop space instead of the actual free loop space. The (Laurent) *polynomial loop space* of $\mathbb{P}^{N-1}$ is defined to be

$$L_{\mathrm{poly}}\mathbb{P}^{N-1} = \left(\mathbb{C}[\zeta,\zeta^{-1}]^N \setminus \{0\}\right)/\mathbb{C}^\times.$$

where $\mathbb{C}^\times$ acts on $\mathbb{C}[\zeta,\zeta^{-1}]^N$ by scalar multiplication. The polynomial loop space $L_{\mathrm{poly}}\mathbb{P}^{N-1}$ can be also described as a symplectic reduction of $\mathbb{C}[\zeta,\zeta^{-1}]^N \cong \mathbb{C}^\infty$ by the diagonal S^1-action. For a point $(a_1(\zeta),\dots,a_N(\zeta)) \in \mathbb{C}[\zeta,\zeta^{-1}]^N$, we write $a_i(\zeta) = \sum_{n\in\mathbb{Z}} a_{i,n}\zeta^n$ and regard $(a_{i,n} : 1 \leqslant i \leqslant N, n \in \mathbb{Z})$ as a co-ordinate system on $\mathbb{C}[\zeta,\zeta^{-1}]^N$. With respect to the standard Kähler form

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^N \sum_{n\in\mathbb{Z}} da_{i,n} \wedge d\overline{a_{i,n}} \tag{13}$$

on $\mathbb{C}[\zeta,\zeta^{-1}]^N$, the diagonal S^1-action admits a moment map[2] $\mu\colon \mathbb{C}[\zeta,\zeta^{-1}]^N \to \mathrm{Lie}(S^1)^* \cong \mathbb{R}$ given by

$$\mu(a_1(\zeta),\dots,a_N(\zeta)) = \pi \sum_{i=1}^N \sum_{n\in\mathbb{Z}} |a_{i,n}|^2.$$

This satisfies $\iota_X\omega + d\mu = 0$ for the vector field $X = 2\pi\sqrt{-1}\sum_{i=1}^N \sum_{n\in\mathbb{Z}}(a_{i,n}\frac{\partial}{\partial a_{i,n}} - \overline{a_{i,n}}\frac{\partial}{\partial \overline{a_{i,n}}})$ generating the S^1-action. Then we have

$$L_{\mathrm{poly}}\mathbb{P}^{N-1} \cong \mu^{-1}(u)/S^1$$

for every $u \in \mathbb{R}_{>0}$. Via the symplectic reduction, $L_{\mathrm{poly}}\mathbb{P}^{N-1}$ is equipped with the reduced symplectic form ω_u such that the pull-back of ω_u to $\mu^{-1}(u)$ equals the restriction $\omega|_{\mu^{-1}(u)}$. The class ω_u represents the cohomology class $uc_1(\mathcal{O}(1))$ on $L_{\mathrm{poly}}\mathbb{P}^{N-1}$.

The loop rotation defines the S^1-action on $L_{\mathrm{poly}}\mathbb{P}^{N-1}$ given by $[a_1(\zeta),\dots,a_N(\zeta)] \mapsto [a_1(\lambda\zeta),\dots,a_N(\lambda\zeta)]$ with $\lambda \in S^1$. With respect to the reduced symplectic form ω_u on $L_{\mathrm{poly}}\mathbb{P}^{N-1}$, this S^1-action admits a moment map H_u given by

$$H_u([a_1(\zeta),\dots,a_N(\zeta)]) = \pi \sum_{i=1}^N \sum_{n\in\mathbb{Z}} n|a_{i,n}|^2 \qquad \text{with } (a_1(\zeta),\dots,a_N(\zeta)) \in \mu^{-1}(u).$$

The function H_u is an analogue of the *action functional* on the free loop space.

Remark 5.1. More precisely, the polynomial loop space should be regarded as an analogue of the *universal cover* of the free loop space $\mathcal{L}\mathbb{P}^{N-1}$. In fact, we have an analogue of the deck transformation on $L_{\mathrm{poly}}\mathbb{P}^{N-1}$ given by $[a_1(\zeta),\dots,a_N(\zeta)] \mapsto [\zeta a_1(\zeta),\dots,\zeta a_N(\zeta)]$; this corresponds to a generator of $\pi_1(\mathcal{L}\mathbb{P}^{N-1}) \cong \pi_2(\mathbb{P}^{N-1}) = \mathbb{Z}$. Recall that the action functional is defined on the universal cover of $\mathcal{L}\mathbb{P}^{N-1}$.

[2]We identify $\mathrm{Lie}(S^1)^*$ with $\mathbb{R}$ so that the radian angular form $d\theta$ on $S^1 = \{e^{\sqrt{-1}\theta} : \theta \in \mathbb{R}\}$ corresponds to 2π. With this choice, the reduced symplectic form ω_u on $\mu^{-1}(u)/S^1$ represents an integral cohomology class precisely when $u \in \mathbb{Z}$.

5.2. **Solution as a path integral.** Recall that symplectic Floer theory is an infinite-dimensional analogue of the Morse theory with respect to the action functional on the loop space. We consider the Morse theory on $L_{\text{poly}}\mathbb{P}^{N-1}$ with respect to the Bott–Morse function H_u. Note that the critical set of H_u is a disjoint union of infinite copies of $\mathbb{P}^{N-1}$ given by $(\mathbb{P}^{N-1})_n = \{[a_1\zeta^n, \dots, a_N\zeta^n]\} \subset L_{\text{poly}}\mathbb{P}^{N-1}$ for each $n \in \mathbb{Z}$. The Floer fundamental cycle Δ is defined to be the closure of the stable manifold associated to the critical component $(\mathbb{P}^{N-1})_0$. This is given by

$$\Delta = \left\{[a_1(\zeta), \dots, a_N(\zeta)] \in L_{\text{poly}}\mathbb{P}^{N-1} : a_i(\zeta) \in \mathbb{C}[\zeta]\right\}.$$

Under the isomorphism between the Floer homology and the quantum cohomology [57], the Floer fundamental cycle corresponds to the identity class $1 \in H^\bullet(\mathbb{P}^{N-1})$. Consider the following equivariant 2-form on $L_{\text{poly}}\mathbb{P}^{N-1}$ (in the Cartan model)

$$\Omega_u = \omega_u - zH_u$$

where $z \in H^2_{S^1}(\text{pt}, \mathbb{Z})$ is a positive generator. Note that Ω is equivariantly closed since H_u is a Hamiltonian for the S^1-action. Givental [27, 28] proposed that the infinite-dimensional integral (which could be viewed as a Feynman path integral)

$$\int_\Delta e^{\Omega_u/z} \tag{14}$$

should give a solution to the quantum differential equation for $\mathbb{P}^{N-1}$. This may be viewed as the image of the Floer fundamental cycle Δ under the homomorphism $C \mapsto \int_C e^{\Omega_u/z}$. The integral does not have a rigorous definition in mathematics, but we can heuristically compute this quantity in two different ways — one is by a direct computation and the other is by localization. The former method yields a mirror oscillatory integral and the latter (due to Givental [27, 28]) yields the J-function of $\mathbb{P}^{N-1}$. In Givental's original calculation, however, an infinite (constant) factor corresponding to the Gamma class has been ignored. From this calculation we obtain a (mirror) integral representation of the $\widehat{\Gamma}_{\mathbb{P}^{N-1}}$-component of the J-function.

5.3. **Direct calculation.** We compute the infinite dimensional integral (14) directly. We regard z as a positive real parameter. Since $\Delta = (\mu^{-1}(u) \cap \mathbb{C}[\zeta]^N)/S^1$, we have

$$(14) = \int_{(\mu^{-1}(u)\cap\mathbb{C}[\zeta]^N)/S^1} e^{-H_u} e^{\omega_u/z} = \int_{\mu^{-1}(u)\cap\mathbb{C}[\zeta]^N} e^{-H_u}\frac{d\theta}{2\pi} \wedge e^{\omega/z}$$

where $d\theta$ is the angular form (connection form) on the principal S^1-bundle $\mu^{-1}(u) \to \mu^{-1}(u)/S^1$ given by $d\theta = \frac{\pi}{2\mu\sqrt{-1}}\sum_{i=1}^N \sum_{k\in\mathbb{Z}}(\overline{a_{i,n}}da_{i,n} - a_{i,n}d\overline{a_{i,n}})$ (satisfying $d\theta(X) = 2\pi$) and ω is the Kähler form (13). Changing co-ordinates $a_{i,n} \to \sqrt{z}a_{i,n}$, we find that this equals

$$\int_{\mu^{-1}(u/z)\cap\mathbb{C}[\zeta]^N} e^{-zH_u}\frac{d\theta}{2\pi} \wedge e^{\omega}.$$

If $\mathbb{C}[\zeta]^N$ were a finite dimensional vector space, the top-degree component of the differential form $d\mu \wedge \frac{d\theta}{2\pi} \wedge e^\omega$ would equal the Liouville volume form on $\mathbb{C}[\zeta]^N$ associated to ω. Therefore we may write this as

$$\int_{\mathbb{C}[\zeta]^N} \delta(\mu - u/z)e^{-zH_u}\, d\,\text{vol}$$

where $d\,\mathrm{vol} = \bigwedge_{i=1}^N \bigwedge_{n=0}^\infty \left(\frac{\sqrt{-1}}{2} da_{i,n} \wedge d\overline{a_{i,n}}\right)$ and $\delta(x)$ is the Dirac delta-function. Using $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{\sqrt{-1}x\xi} d\xi$, we can compute this as:

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{C}[\zeta]^N} d\,\mathrm{vol} \int_{-\infty}^\infty d\xi e^{\sqrt{-1}\xi(\mu - u/z)} e^{-zH_u} \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty d\xi e^{-\sqrt{-1}\xi u/z} \int_{\mathbb{C}[\zeta]^N} d\,\mathrm{vol} \prod_{i=1}^N \prod_{n=0}^\infty e^{-\pi|a_{i,n}|^2(nz - \sqrt{-1}\xi)} \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty d\xi e^{-\sqrt{-1}\xi u/z} \prod_{i=1}^N \prod_{n=0}^\infty \frac{1}{nz - \sqrt{-1}\xi}
\end{aligned} \tag{15}$$

By the ζ-function regularization, we can regularize the infinite product to get:

$$\frac{1}{\prod_{n=0}^\infty (nz - \sqrt{-1}\xi)} \sim (2\pi z)^{-1/2} z^{-\sqrt{-1}\xi/z}\Gamma(-\sqrt{-1}\xi/z).$$

Therefore (15) should equal, after the change $\xi \to z\xi$ of co-ordinates,

$$\frac{z}{2\pi(2\pi z)^{N/2}} \int_{-\infty}^\infty d\xi e^{-\sqrt{-1}(u + N\log z)\xi}\Gamma(-\sqrt{-1}\xi)^N. \tag{16}$$

This integral makes sense if we perturb the integration contour so that it avoids the singularity at $\xi = 0$. We will consider the perturbed contour from $-\infty + \sqrt{-1}\epsilon$ to $\infty + \sqrt{-1}\epsilon$ with $\epsilon > 0$. Then the integral (16) is just a (finite-dimensional) Fourier transform and the following discussion can be made completely rigorous. Using the integral representation $\Gamma(z) = \int_0^\infty e^{-x} z^{x-1} dx$ of the Γ-function, we find that (16) equals

$$\begin{aligned}
\frac{z}{2\pi(2\pi z)^{N/2}} \int_{-\infty}^\infty d\xi \int_{[0,\infty)^N} \frac{dx_1}{x_1}\cdots\frac{dx_N}{x_N} e^{-\sqrt{-1}\xi(u + N\log z + \sum_{i=1}^N \log x_i)} e^{-(x_1+\cdots+x_N)} \\
&= \frac{z}{(2\pi z)^{N/2}} \int_{[0,\infty)^N} \frac{dx_1}{x_1}\cdots\frac{dx_N}{x_N} \delta(u + N\log z + \textstyle\sum_{i=1}^N \log x_i) e^{-(x_1+\cdots+x_N)} \\
&= \frac{z}{(2\pi z)^{N/2}} \int_{[0,\infty)^{N-1}} \frac{dx_1}{x_1}\cdots\frac{dx_{N-1}}{x_{N-1}} e^{-\left(x_1+\cdots+x_{N-1}+\frac{e^{-u}}{x_1\cdots x_{N-1}}\right)/z}
\end{aligned} \tag{17}$$

where in the last line we considered the co-ordinate change $x_i \to x_i/z$. The Landau–Ginzburg mirror of $\mathbb{P}^{N-1}$ is given by the Laurent polynomial function $f(x_1,\dots,x_{N-1}) = x_1+\cdots+x_{N-1}+\frac{e^{-u}}{x_1\cdots x_{N-1}}$ and this is the associated oscillatory integral [27, 37].

5.4. **Calculation by localization.** Next we calculate the quantity (14) using the localization formula of equivariant cohomology (or Duistermaat–Heckman formula) [21, 8, 3]. The S^1-fixed set in Δ is the disjoint union of $(\mathbb{P}^{N-1})_n$ with $n \geqslant 0$ and we have $[\Omega_u]|_{(\mathbb{P}^{N-1})_n} = uh - znu$, where $h := c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^{N-1})$ is the hyperplane class. Therefore

$$\int_\Delta e^{\Omega_u/z} = \sum_{n=0}^\infty \int_{(\mathbb{P}^{N-1})_n} \frac{e^{uh/z - nu}}{e_{S^1}(\mathcal{N}_n)}$$

where $\mathcal{N}_n$ is the (infinite-rank) normal bundle of $(\mathbb{P}^{N-1})_n$ in Δ and

$$e_{S^1}(\mathcal{N}_n) = \prod_{k\geqslant -n, k\neq 0} (h + kz)^N = \prod_{k=1}^\infty (h+kz)^N \prod_{k=1}^n (h - kz)^N.$$

The infinite factor $\prod_{k=1}^{\infty}(h+kz)^N$ was discarded in the original calculation of Givental [28]. Using again the ζ-function regularization, we find that this factor yields the Gamma class:

$$\frac{1}{\prod_{k=1}^{\infty}(h+kz)} \sim \left(\frac{z}{2\pi}\right)^{1/2} z^{h/z}\Gamma(1+h/z).$$

Thus we should have

$$\begin{aligned}\int_{\Delta} e^{\Omega_u/z} &= \sum_{n=0}^{\infty}\int_{\mathbb{P}^{N-1}} \left(\frac{z}{2\pi}\right)^{N/2} z^{Nh/z}\Gamma(1+h/z)^N \frac{e^{uh/z-nu}}{\prod_{k=1}^{n}(h-kz)^N} \\ &= \frac{z}{(2\pi z)^{N/2}}\int_{\mathbb{P}^{N-1}} \Gamma(1+h)^N \sum_{n=0}^{\infty}\frac{(e^u z^N)^{h-n}}{\prod_{k=1}^{n}(h-k)^N}.\end{aligned}$$

Recall that the J-function (7) of $\mathbb{P}^{N-1}$ is given by [29]

$$J_{\mathbb{P}^{N-1}}(t) = \sum_{n=0}^{\infty}\frac{t^{N(h+n)}}{\prod_{k=1}^{n}(h+k)^N}. \tag{18}$$

Thus, using $\widehat{\Gamma}_{\mathbb{P}^{N-1}} = \Gamma(1+h)^N$, we obtain

$$\int_{\Delta} e^{\Omega_u/z} = \frac{z}{(2\pi z)^{N/2}}(2\pi\sqrt{-1})^{N-1}\left[J_{\mathbb{P}^{N-1}}(e^{\pi\sqrt{-1}}t), \widehat{\Gamma}_{\mathbb{P}^{N-1}}\right) \tag{19}$$

under the identification $t = e^{-u/N}z^{-1}$, where $[\cdot,\cdot)$ is the non-symmetric pairing defined in (2).

Remark 5.2. The quantity (19) coincides, up to a factor, with the *quantum cohomology central charge* of $\mathcal{O}_{\mathbb{P}^{N-1}}$ [42, 26]. For a vector bundle E on a Fano manifold F, the quantum cohomology central charge $Z(E)$ is defined to be:

$$\begin{aligned}Z(E) &= (2\pi\sqrt{-1})^{\dim F}\left[J_F(e^{\pi\sqrt{-1}}t), \widehat{\Gamma}_F \operatorname{Ch}(E)\right) \\ &= z^{\frac{\dim F}{2}}\left(1, S(z)z^{-\mu}z^{c_1(F)}\widehat{\Gamma}_F \operatorname{Ch}(E)\right).\end{aligned}$$

where $t = z^{-1}$.

5.5. **Comparison.** We computed the infinite dimensional integral (14) in two ways. Comparing (17) and (19), we should have the equality

$$\int_{[0,\infty)^{N-1}} e^{-\left(x_1+\cdots+x_{N-1}+\frac{e^{-u}}{x_1\cdots x_{N-1}}\right)/z}\frac{dx_1}{x_1}\cdots\frac{dx_{N-1}}{x_{N-1}} = (2\pi\sqrt{-1})^{N-1}\left[J_{\mathbb{P}^{N-1}}(e^{\pi\sqrt{-1}}t), \widehat{\Gamma}_{\mathbb{P}^{N-1}}\right) \tag{20}$$

with $t = e^{-u/N}z^{-1}$. This oscillatory integral representation yields the asymptotic expansion (as $t \to +\infty$)

$$Z(\mathcal{O}_{\mathbb{P}^{N-1}}) = (2\pi\sqrt{-1})^{N-1}\left[J_{\mathbb{P}^{N-1}}(e^{\pi\sqrt{-1}}t), \widehat{\Gamma}_{\mathbb{P}^{N-1}}\right) \sim \text{const} \times t^{-\frac{N-1}{2}}e^{-Nt}$$

which can be used to prove the Gamma conjecture for $\mathbb{P}^{N-1}$. See §6 and [26, §3.8].

Remark 5.3. We have a rigorous independent proof of the equality (20) (see [42, 47]). Recall that we can write the left-hand side as the Fourier transform (16) of $\Gamma(-\sqrt{-1}\xi)^N$; by closing the integration contour in the lower half ξ-plane and writing the integral as the sum of residues at $\xi = 0, -\sqrt{-1}, -2\sqrt{-1}, -3\sqrt{-1}, \ldots$, we arrive at the expression in the right-hand side.

Remark 5.4. A similar regularization of an infinite dimensional integral appears in the computation of (sphere or hemisphere) partition functions of (2,2) supersymmetric gauge theories, see Benini-Cremonesi [7], Doroud–Gomis–Le-Floch–Lee [16] and Hori-Romo [36]. It appears that the computations in §5.3 and §5.4 correspond, in the terminology of [7, 16], to the localization on the *Coulomb branch* and on the *Higgs branch* respectively.

6. Toric Manifold

In this section, we discuss Gamma conjecture for Fano toric manifolds. We prove Gamma conjecture I by assuming a certain condition for the mirror Laurent polynomial f which is analogous to Property $\mathcal{O}$.

Let X be an n-dimensional Fano toric manifold. A mirror of X is given by the Laurent polynomial [27, 37, 30]:

$$f(x) = x^{b_1} + x^{b_2} + \cdots + x^{b_m}$$

where $x = (x_1, \dots, x_n) \in (\mathbb{C}^\times)^n$, $b_1, \dots, b_m \in \mathbb{Z}^n$ are primitive generators of the 1-dimensional cones of the fan of X and $x^{b_i} = x_1^{b_{i1}} \cdots x_n^{b_{in}}$ for $b_i = (b_{i1}, \dots, b_{in})$. By mirror symmetry [30, 42], the spectrum of $(c_1(X)\star_0)$ is the set of critical values of f. The restriction of f to the real locus $(\mathbb{R}_{>0})^n$ is strictly convex since the logarithmic Hessian

$$\frac{\partial^2 f}{\partial \log x_i \partial \log x_j}(x) = \sum_{k=1}^{m} b_{kj} b_{ki} x^{b_k}$$

is positive definite for any $x \in (\mathbb{R}_{>0})^n$. One can also show that $f|_{(\mathbb{R}_{>0})^n}$ is proper and bounded from below since the convex hull of $b_1, \dots, b_m$ contains the origin in its interior (cf. Remark 7.4). Therefore $f|_{(\mathbb{R}_{>0})^n}$ admits a global minimum at a unique point $x_{\mathrm{con}} \in (\mathbb{R}_{>0})^n$. We call x_{con} the *conifold point* [25, 26]. Consider the following condition:

Condition 6.1 (analogue of Property $\mathcal{O}$ for toric manifolds)**.** Let X be a Fano toric manifold and f be its mirror Laurent polynomial. Let $T_{\mathrm{con}} = f(x_{\mathrm{con}})$ be the value of f at the conifold point x_{con}. One has

(a) every critical value u of f satisfies $|u| \leqslant T_{\mathrm{con}}$;
(b) the conifold point is the unique critical point of f contained in $f^{-1}(T_{\mathrm{con}})$.

Note that Condition 6.1 implies Part (1) of Property $\mathcal{O}$ (Definition 3.1) which is sufficient to make sense of Gamma Conjecture I (see Remark 3.14).

Remark 6.2. One can define the conifold point for every Laurent polynomial such that the Newton polytope contains the origin in its interior and that all the coefficients are positive, but Condition 6.1 does not always hold. For instance, the one-dimensional Laurent polynomial $f(x) = x^{-1} + x + tx^2$ with a sufficiently small $t > 0$ does not satisfy Part (a) of the condition. Also, if the lattice generated by $b_1, \dots, b_m$ is not equal to $\mathbb{Z}^n$, then $f(x) = x^{b_1} + x^{b_2} + \cdots + x^{b_m}$ has non-trivial diagonal symmetry $\{\zeta \in (\mathbb{C}^\times)^n : f(x) = f(\zeta x)\} \neq \{1\}$ and Part (b) of the condition fails.

Theorem 6.3. *Suppose that a Fano toric manifold X satisfies Condition 6.1. Then X satisfies Gamma Conjecture I.*

Proof. It follows from the argument in [42, §4.3.1] that

$$z^{n/2}\left(\phi, S(z) z^{-\mu} z^{c_1(X)} \widehat{\Gamma}_X\right)_X = \int_{(\mathbb{R}_{>0})^n} e^{-f(x)/z} \varphi(x,z) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

for $z > 0$, where $\phi \in H^\bullet(X)$ and $\varphi(x,z) \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm, z]$ is such that the class $[\varphi(x,z)e^{-f(x)/z}dx_1 \cdots dx_n/(x_1 \cdots x_n)]$ corresponds to ϕ under the mirror isomorphism in [42, Proposition 4.8] and $n = \dim X$. When $\varphi = 1$, one has $\phi = 1$ and this gives the integral representation

$$Z(\mathcal{O}_X) = \int_{(\mathbb{R}_{>0})^n} e^{-f(x)/z} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

of the quantum cohomology central charge $Z(\mathcal{O}_X)$ from Remark 5.2. We already saw this in (20) for $X = \mathbb{P}^{N-1}$. This integral representation implies that the flat section $S(z)z^{-\mu}z^{c_1(X)}\widehat{\Gamma}_X$ has the smallest asymptotics as $z \to +0$, i.e.

$$\left\| e^{T_{\mathrm{con}}/z} S(z) z^{-\mu} z^{c_1(X)} \widehat{\Gamma}_X \right\| = O(1)$$

and the conclusion follows. □

A toric manifold has a generically semisimple quantum cohomology. We note that the following weaker version of Gamma Conjecture II (Conjecture 4.9) can be shown for a toric manifold.

Theorem 6.4 ([42]). *Let X be a Fano toric manifold. We choose a semisimple point $\tau \in H^\bullet(X)$ and an admissible phase ϕ. There exists a $\mathbb{Z}$-basis $\{[E_1], \dots, [E_N]\}$ of the K-group such that the matrix $(\chi(E_i, E_j))_{1 \leqslant i,j \leqslant N}$ is uni-uppertriangular and that the asymptotic basis (see §4.3) of X at τ with respect to ϕ is given by $\widehat{\Gamma}_X \operatorname{Ch}(E_i)$, $i = 1, \dots, N$.*

Proof. Recall from Remark 4.13 that the asymptotic basis makes sense also for the big quantum product $\star_\tau$ and that it changes by mutation as τ varies. By the theory of mutation, it suffices to prove the theorem at a single semisimple point τ. If τ is in the image of the mirror map, we can take $[E_i]$ to be the mirror images of the Lefschetz thimbles (with phase ϕ) under the isomorphism between the integral structures of the A-model and of the B-model, given in [42, Theorem 4.11]. □

Remark 6.5 ([42]). Results similar to Theorems 6.3, 6.4 hold for a weak-Fano toric orbifold. In the weak-Fano case, however, we need to take into consideration the effect of the mirror map.

Remark 6.6. Kontsevich's homological mirror symmetry suggests that the basis $[E_1], \dots, [E_N]$ in the K-group should be lifted to an exceptional collection in $D^b_{\mathrm{coh}}(X)$ because the corresponding Lefschetz thimbles form an exceptional collection in the Fukaya–Seidel category of the mirror.

7. Toric complete intersections

In this section we discuss Gamma conjecture I for a Fano complete intersection Y in a toric manifold X. Again the problem comes down to the truth of a mirror analogue of Property $\mathcal{O}$.

We begin with the remark that the number T can be evaluated on the ambient part. Let

$$H^\bullet_{\mathrm{amb}}(Y) := \operatorname{Im}(i^* \colon H^\bullet(X) \to H^\bullet(Y))$$

denote the *ambient part* of the cohomology group, where $i \colon Y \to X$ is the inclusion. It is shown in [56, Proposition 4], [44, Corollary 2.5] that the ambient part $H^\bullet_{\mathrm{amb}}(Y)$ is closed under quantum multiplication $\star_\tau$ when $\tau \in H^\bullet_{\mathrm{amb}}(Y)$. Note that Givental's mirror theorem [30] determines this ambient quantum cohomology $(H^\bullet_{\mathrm{amb}}(Y), \star_\tau)$.

Proposition 7.1. *The spectrum of $(c_1(Y)\star_0)$ on $H^\bullet(Y)$ is the same as the spectrum of $(c_1(Y)\star_0)$ on $H^\bullet_{\mathrm{amb}}(Y)$ as a set (when we ignore the multiplicities). In particular the number T for Y (see* (6)*) can be evaluated on the ambient part.*

Proof. It follows from the fact that $c_1(Y)$ belongs to $H^\bullet_{\mathrm{amb}}(Y)$ and the fact that $H^\bullet(Y)$ is a module over the algebra $(H^\bullet_{\mathrm{amb}}(Y), \star_0)$. □

We assume that Y is a complete intersection in X obtained from the following *nef partition* [5, 30]. Let $D_1, \dots, D_m$ denote prime toric divisors of X. Each prime toric divisor D_i corresponds to a primitive generator $b_i \in \mathbb{Z}^n$ of a 1-dimensional cone of the fan of X. We assume that there exists a partition $\{1, \dots, m\} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_l$ such that $\sum_{i \in I_k} D_i$ is nef for $k = 1, \dots, l$ and $\sum_{i \in I_0} D_i$ is ample. Let $\mathcal{L}_k := \mathcal{O}(\sum_{i \in I_k} D_i)$ be the corresponding nef line bundle. We assume that Y is the zero-locus of a transverse section of $\bigoplus_{k=1}^l \mathcal{L}_k$ over X. Then $c_1(Y) = c_1(X) - \sum_{k=1}^l c_1(\mathcal{L}_k) = \sum_{i \in I_0} [D_i]$ is ample and Y is a Fano manifold. Define the Laurent polynomial functions $f_0, f_1, \dots, f_l \colon (\mathbb{C}^\times)^n \to \mathbb{C}$ as follows:

$$f_k(x) = -c_0 \delta_{0,k} + \sum_{i \in I_k} x^{b_i}$$

where c_0 is the constant arising from Givental's mirror map [30]:

$$c_0 := \sum_{\substack{d \in \mathrm{Eff}(X) : c_1(Y)\cdot d = 1 \\ [D_i]\cdot d \geqslant 0\,(\forall i)}} \frac{\prod_{k=1}^l (c_1(\mathcal{L}_k)\cdot d)!}{\prod_{i=1}^m ([D_i]\cdot d)!}.$$

A mirror of Y [37, 30] is the affine variety $Z := \{x \in (\mathbb{C}^\times)^n : f_1(x) = \cdots = f_l(x) = 1\}$ equipped with a function $f_0 \colon Z \to \mathbb{C}$ and a holomorphic volume form

$$\omega_Z := \frac{\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}}{df_1 \wedge \cdots \wedge df_l}.$$

Consider the following condition, which implies Property $\mathcal{O}$ for Y.

Condition 7.2 (analogue of Property $\mathcal{O}$ for Y)**.** Let Y, Z and $f_0 \colon Z \to \mathbb{C}$ be as above. Let $Z_{\mathrm{real}} := \{(x_1, \dots, x_n) \in Z : x_i \in \mathbb{R}_{>0}\ (\forall i)\}$ be the positive real locus of Z. We have:

(a) Z is a smooth complete intersection of dimension $n - l$;
(b) $f_0|_{Z_{\mathrm{real}}} \colon Z_{\mathrm{real}} \to \mathbb{R}$ attains global minimum at a unique critical point $x_{\mathrm{con}} \in Z_{\mathrm{real}}$; we call it *the conifold point*;
(c) the conifold point x_{con} is a non-degenerate critical point of f_0;
(d) the number T (6) for Y coincides with $T_{\mathrm{con}} := f_0(x_{\mathrm{con}})$;
(e) T_{con} is an eigenvalue of $(c_1(Y)\star_0)$ with multiplicity one.

Variant. We can consider the weaker version of Condition 7.2 where part (e) is replaced with

(e′) T_{con} is an eigenvalue of $(c_1(Y)\star_0)|_{H_{\mathrm{amb}}(Y)}$ with multiplicity one.

When (a), (b), (c), (d), (e′) hold, we say that Y satisfies Condition 7.2 *on the ambient part.*

Example 7.3. Let Y be a degree d hypersurface in $\mathbb{P}^n$ with $d < n + 1$. The mirror is given by the function

$$f_0(x) = \frac{1}{x_1 x_2 \cdots x_n} + x_1 + \cdots + x_{n-d} - \delta_{d,n} d!$$

on the affine variety $Z = \{x \in (\mathbb{C}^\times)^n : f_1(x) = x_{n-d+1} + x_{n-d+2} + \cdots + x_n = 1\}$. Following Przyjalkowski [59], introduce homogeneous co-ordinates $[y_{n-d+1} : \cdots : y_n]$ of $\mathbb{P}^{n-d-1}$ and consider the change of variables:

$$x_i = \frac{y_i}{y_{n-d+1} + \cdots + y_n} \quad \text{for} \quad n - d + 1 \leqslant i \leqslant n.$$

Then we have

$$f_0(x) = \frac{(y_{n-d+1} + \cdots + y_n)^d}{\prod_{i=1}^{n-d} x_i \cdot \prod_{j=n-d+1}^{n} y_j} + x_1 + \cdots + x_{n-d} - \delta_{n,d} d!$$

Setting $y_n = 1$ we obtain a Laurent polynomial expression (with positive coefficients) for f_0 in the variables $x_1, \dots, x_{n-d}, y_{n-d+1}, \dots, y_{n-1}$. Note that the change of variables maps Z_{real} isomorphically onto the real locus $\{x_i > 0, y_j > 0 : 1 \leqslant i \leqslant n-d, n-d+1 \leqslant j \leqslant n-1\}$. As in the case of mirrors of toric manifolds (see §6), the Laurent polynomial expression of f_0 shows the existence of a conifold point in Condition 7.2. (Similarly, the mirrors of complete intersections in weighted projective spaces have Laurent polynomial expressions [59].)

We have $T_{\text{con}} = (n+1-d)d^{d/(n+1-d)} - \delta_{n,d} d!$. It is easy to check that T_{con} coincides with T for Y and gives a simple eigenvalue of $(c_1(Y)\star_0)|_{H^\bullet_{\text{amb}}(Y)}$ restricted to the ambient part (using Givental's mirror theorem [30]). Therefore Y satisfies Condition 7.2 on the ambient part.

Furthermore, if the Fano index $n+1-d$ is greater than one, one has $c_1(Y)\star_0 \theta = 0$ for every primitive class $\theta \in H^{n-1}(Y)$: this follows from the fact that $(c_1(Y)\star_0)$ preserves the primitive part in $H^{n-1}(Y)$ and for degree reasons. Therefore Y satisfies Condition 7.2 if $d < n$.

Remark 7.4 ([44]). Using an inequality of the form $\beta_1 u_1 + \cdots + \beta_m u_m \geqslant u_1^{\beta_1} \cdots u_m^{\beta_m}$ for $u_i > 0$, $\beta_i > 0$, $\sum_i \beta_i = 1$, we find that $f_0|_{Z_{\text{real}}}$ is bounded (from below) by a convex function:

$$f_0(x) + l = f_0(x) + f_1(x) + \cdots + f_l(x) \geqslant -c_0 + \epsilon \sum_{i=1}^{n} (x_i^{1/k} + x_i^{-1/k}) \qquad \forall x \in Z_{\text{real}}$$

where $\epsilon > 0$ and $k > 0$ are constants. In particular $f_0|_{Z_{\text{real}}} \colon Z_{\text{real}} \to \mathbb{R}$ is proper and attains global minimum at some point. It also follows that the oscillatory integral (21) below converges.

Theorem 7.5. *Let Y be a Fano complete intersection in a toric manifold constructed from a nef partition as above. If Y satisfies Condition 7.2, Y satisfies Gamma Conjecture I. If Y satisfies Condition 7.2 on the ambient part, the ambient quantum cohomology of Y satisfies Gamma conjecture I.*

Proof. In [44, Theorem 5.7], it is proved that the quantum cohomology central charge (see Remark 5.2) of $\mathcal{O}_Y$ has an integral representation:

$$Z(\mathcal{O}_Y) = z^{\frac{\dim Y}{2}} \left(1, S(z) z^{-\mu} z^{c_1(Y)} \widehat{\Gamma}_Y\right)_X = \int_{Z_{\text{real}}} e^{-f_0/z} \omega_Z. \tag{21}$$

We note that the constant term c_0 in f_0 comes from the value of the mirror map at $\tau = 0$. In [44], a slightly more general statement was proved: we have mirrors $(Z_\tau, f_{0,\tau}, \omega_{Z,\tau})$ parametrized by $\tau \in H^2_{\text{amb}}(Y)$ and

$$z^{\frac{\dim Y}{2}} \left(1, S(\tau, z) z^{-\mu} z^{c_1(Y)} \widehat{\Gamma}_Y\right)_Y = \int_{Z_{\tau,\text{real}}} e^{f_{0,\tau}/z} \omega_{Z,\tau}$$

where $S(\tau,z)z^{-\mu}z^{c_1(Y)}$ is the fundamental solution for the big quantum connection (5) which restricts to the one in Proposition 2.1 at $\tau=0$. By differentiating this in the τ-direction, we obtain

$$z^{\frac{\dim Y}{2}}\left(z\nabla_{\alpha_1}\cdots z\nabla_{\alpha_k}1, S(\tau,z)z^{-\mu}z^{c_1(Y)}\widehat{\Gamma}_Y\right)_Y = z\partial_{\alpha_1}\cdots z\partial_{\alpha_k}\int_{Z_{\tau,\mathrm{real}}} e^{-f_{0,\tau}/z}\omega_{Z,\tau}$$

for $\alpha_1,\dots,\alpha_k\in H^2(Y)$. Since $H^{\bullet}_{\mathrm{amb}}(Y)$ is generated by $H^2_{\mathrm{amb}}(Y)$, we obtain an integral representation of the form:

$$z^{\frac{\dim Y}{2}}\left(\phi, S(z)z^{-\mu}z^{c_1(Y)}\widehat{\Gamma}_Y\right)_Y = \int_{Z_{\mathrm{real}}}\varphi(x,z)e^{-f_0/z}\omega_Z$$

for every $\phi\in H^{\bullet}_{\mathrm{amb}}(Y)$ (for some function $\varphi(x,z)$). Since the flat section $S(z)z^{-\mu}z^{c_1(Y)}\widehat{\Gamma}_Y$ takes values in the ambient part $H^{\bullet}_{\mathrm{amb}}(Y)$, we obtain integral representations for all components of $S(z)z^{-\mu}z^{c_1(Y)}\widehat{\Gamma}_Y$. These integral representations and Condition 7.2 show that $\|e^{T/z}S(z)z^{-\mu}z^{c_1(Y)}\widehat{\Gamma}_Y\|$ grows at most polynomially as $z\to+0$. $\square$

8. Quantum Lefschetz

In this section we show that Gamma conjecture I is compatible with the quantum Lefschetz principle.

Let X be a Fano manifold of index $r\geqslant 2$. We write $-K_X=rh$ for an ample class h. Let $Y\subset X$ be a degree-a Fano hypersurface in the linear system $|ah|$ with $0<a<r$. Assuming the truth of Gamma Conjecture I for X, we study Gamma Conjecture I for Y.

We write the J-function of X (7) as

$$J_X(t)=e^{rh\log t}\sum_{n=0}^{\infty}J_{rn}t^{rn}$$

with $J_d\in H^{\bullet}(X)$. By the quantum Lefschetz theorem [50, 14], the J-function of Y is

$$J_Y(t)=e^{(r-a)h\log t-c_0t}\sum_{n=0}^{\infty}(ah+1)\cdots(ah+an)(i^*J_{rn})t^{(r-a)n}, \tag{22}$$

where $i\colon Y\to X$ is the natural inclusion and c_0 is the constant:

$$c_0=\begin{cases} a!\,\langle[\mathrm{pt}],J_r\rangle = a!\sum_{h\cdot d=1}\langle[\mathrm{pt}]\psi^{r-2}\rangle^X_{0,1,d} & \text{if } r-a=1;\\ 0 & \text{if } r-a>1.\end{cases} \tag{23}$$

The quantum Lefschetz principle can be rephrased in terms of the Laplace transformation:

Lemma 8.1. *We have*

$$J_Y(t=u^{a/(r-a)})=\frac{e^{-c_0t}}{\Gamma(1+ah)u}\int_0^{\infty}i^*J_X(q^{a/r})e^{-q/u}dq.$$

Remark 8.2. The Laplace transformation in the above lemma converges for sufficiently small $u>0$ because of the exponential asymptotics as $t\to+\infty$ in Proposition 3.8 and the growth estimate $\|J_X(t)\|\leqslant C|\log t|^{\dim X}$ as $t\to+0$.

Suppose that X satisfies Gamma Conjecture *I*. Recall from Proposition 3.8 that we have the following limit formula for J_X:

$$\widehat{\Gamma}_X\propto\lim_{t\to+\infty}t^{\frac{\dim X}{2}}e^{-T_Xt}J_X(t). \tag{24}$$

Here T_X is the number T in (6) for X. Using the stationary phase approximation in Lemma 8.1, we obtain the following result.

Theorem 8.3. *Suppose that the J-function of X satisfies the limit formula (24) and let Y be a Fano hypersurface in the linear system $|-(a/r)K_X|$, where r is the Fano index of X and $0 < a < r$. Then the J-function of Y satisfies the limit formula:*

$$\widehat{\Gamma}_Y \propto \lim_{t\to+\infty} t^{\frac{\dim Y}{2}} e^{-(T_0-c_0)t} J_Y(t)$$

where c_0 is given in (23) and the positive number $T_0 > 0$ is determined by the relation:

$$\left(\frac{T_0}{r-a}\right)^{r-a} = a^a \left(\frac{T_X}{r}\right)^r.$$

In particular, if Y satisfies Property $\mathcal{O}$ (Definition 3.1), then Y satisfies Gamma conjecture I by Corollary 3.9.

Proof. We write $n = \dim X$ and $T = T_X$ for simplicity. We set $\widetilde{J}(t) := t^{\frac{n}{2}} e^{-Tt} i^* J_X(t)$. The limit formula (24) gives $\lim_{t\to+\infty} \widetilde{J}(t) = C_1 i^*\widehat{\Gamma}_X$ for some $C_1 \neq 0$. By Lemma 8.1, we have

$$\begin{aligned} t^{\frac{n-1}{2}} e^{-(T_0-c_0)t} J_Y(t = u^{a/(r-a)}) &= \frac{t^{\frac{n-1}{2}} e^{-T_0 t}}{\Gamma(1+ah)u} \int_0^\infty q^{-an/(2r)} e^{Tq^{a/r} - q/u} \widetilde{J}(q^{a/r}) dq \\ &= \frac{\sqrt{t}}{\Gamma(1+ah)} \int_0^\infty q^{-an/(2r)} e^{-(q - Tq^{a/r} + T_0)t} \widetilde{J}(tq^{a/r}) dq \end{aligned}$$

where in the second line we performed the change of variables $q \to u^{r/(r-a)} q$ and used $t = u^{a/(r-a)}$. Consider the function $\theta(q) = q - Tq^{a/r} + T_0$ on $[0,\infty)$. This function has a unique critical point at $q_0 := (\frac{a}{r} T)^{\frac{r}{r-a}}$ and attains a global minimum at $q = q_0$; we have $\theta(q_0) = 0$ by the definition of T_0. By the stationary phase approximation, the $t \to +\infty$ asymptotics of this integral is determined by the behaviour of the integrand near $q = q_0$. To establish the asymptotics rigorously, we divide the interval $[0,\infty)$ of integration into $[0, q_0/2]$ and $[q_0/2, \infty)$. We first estimate the integral over $[0, q_0/2]$.

$$\begin{aligned} (25) \quad \left\| \sqrt{t} \int_0^{q_0/2} q^{-an/(2r)} e^{-\theta(q)t} \widetilde{J}(tq^{a/r}) dq \right\| &\leqslant e^{-t\theta(q_0/2)} \sqrt{t} \int_0^{q_0/2} q^{-an/(2r)} \|\widetilde{J}(tq^{a/r})\| dq \\ &\leqslant e^{-t\theta(q_0/2)} t^{\frac{n+1}{2} - \frac{r}{a}} \frac{r}{a} \int_0^{t(q_0/2)^{a/r}} y^{-n/2} \|\widetilde{J}(y)\| y^{(r/a)-1} dy. \end{aligned}$$

where we set $y = tq^{a/r}$. Note that $y^{-n/2}\|\widetilde{J}(y)\|y^{(r/a)-1}$ is integrable near $y = 0$ (by the definition of $\widetilde{J}$) and is of polynomial growth as $y \to +\infty$. Therefore the integral

$$\int_0^{t(q_0/2)^{a/r}} y^{-n/2} \|\widetilde{J}(y)\| y^{(r/a)-1} dy.$$

is of polynomial growth as $t \to +\infty$. Since $\theta(q_0/2) > 0$, the integral (25) over $[0, q_0/2]$ goes to zero (exponentially) as $t \to +\infty$. Next we consider the integral over $[q_0/2, \infty)$. By the change of variables $q = q_0 e^{x/\sqrt{t}}$, the integral over $[q_2/2, \infty)$ can be written as

$$(26) \qquad \frac{q_0^{1-\frac{an}{2r}}}{\Gamma(1+ah)} \int_{-\sqrt{t}\log 2}^\infty e^{(1-\frac{an}{2r})x/\sqrt{t}} e^{-t\theta(q_0 e^{x/\sqrt{t}})} \widetilde{J}\left(tq_0^{a/r} e^{\frac{a}{r}x/\sqrt{t}}\right) dx$$

Since we have an expansion of the form $\theta(q_0e^y) = a_2y^2 + \sum_{n=3}^\infty a_n y^n$, we have for a fixed $x \in \mathbb{R}$,

$$e^{(1-\frac{an}{2r})x/\sqrt{t}} \to 1, \qquad e^{-t\theta(q_0e^{x/\sqrt{t}})} \to e^{-a_2x^2}, \qquad \widetilde{J}\left(tq_0^{a/r}e^{\frac{a}{r}x/\sqrt{t}}\right) \to C_1 i^*\widehat{\Gamma}_X$$

as $t \to +\infty$. On the other hand, since $\theta(q_0e^y)$ grows exponentially as $y \to +\infty$, we have an estimate of the form

$$\theta(q_0e^y) \geqslant C_2y^2$$

on $y \in [-\log 2, \infty)$ for some $C_2 > 0$. Therefore, when $t \geqslant 1$, the integrand of (26) can be estimated as

$$\left\| e^{(1-\frac{an}{2r})x/\sqrt{t}} e^{-t\theta(q_0e^{x/\sqrt{t}})} \widetilde{J}\left(tq_0^{a/r}e^{\frac{a}{r}x/\sqrt{t}}\right)\right\| \leqslant C_3 e^{|1-\frac{an}{2r}||x|}e^{-C_2x^2}$$

for all $x \in [-\sqrt{t}\log 2, \infty)$ for some $C_3 > 0$. Thus the integrand is uniformly bounded by an integrable function and we can apply Lebesgue's dominated convergence theorem to (26) to see that the limit of (26) as $t \to +\infty$ is proportional to $i^*\widehat{\Gamma}_X/\Gamma(1+ah)$. The conclusion follows by noting that

$$\widehat{\Gamma}_Y = \frac{i^*\widehat{\Gamma}_X}{\Gamma(1+ah)}.$$

□

It is natural to ask the following questions:

Problem 8.4. Check that $T_0 - c_0$ is the number T (6) for the hypersurface Y. Also prove that Y satisfies Property $\mathcal{O}$ (assuming that X satisfies Property $\mathcal{O}$).

Remark 8.5 ([23]). It is easy to see that Apéry limits (11) (or Gamma conjecture I') are compatible with quantum Lefschetz. Suppose that X is a Fano manifold of index r satisfying Gamma conjecture I' and let $Y \in |-(a/r)K_X|$ be a Fano hypersurface of index $r-a > 1$. Then for any $\alpha \in H_\bullet(Y)$ with $c_1(Y) \cap \alpha = 0$, we have the limit formula:

$$\lim_{n\to\infty} \frac{\langle i_*\alpha, J_{X,rn}\rangle}{\langle [\mathrm{pt}], J_{X,rn}\rangle} = \langle i_*\alpha, \widehat{\Gamma}_X\rangle = \langle \alpha, \widehat{\Gamma}_Y\rangle$$

where $J_{X,n}$ is the Taylor coefficients of the J-function of X (as in (8)) and we used $h \cap i_*\alpha = 0$ in the second equality. Since the index $r-a$ is greater than one, the quantum Lefschetz (22) gives:

$$\left\langle \alpha, J_{Y,(r-a)n}\right\rangle = (an)!\,\langle i_*\alpha, J_{X,rn}\rangle, \quad \left\langle [\mathrm{pt}], J_{Y,(r-a)n}\right\rangle = (an)!\,\langle [\mathrm{pt}], J_{X,rn}\rangle$$

and thus Y also satisfies the Gamma conjecture I'.

9. Grassmannians

In this section, we prove Gamma conjecture I' (Conjecture 3.11) for Grassmannians $\mathrm{Gr}(r,n)$ using the Hori–Vafa mirror [37] and abelian/non-abelian correspondence [9]. The discussion in this section extends the proof of Dubrovin conjecture for Grassmannians by Ueda [62]. In the course of the proof, we obtain a formula for quantum cohomology central charges of $\mathrm{Gr}(r,n)$ in terms of mirror oscillatory integrals. We note that Gamma conjecture I' for $\mathrm{Gr}(2,n)$ was proved by Golyshev [31]. Gamma conjectures I and II were proved for general $\mathrm{Gr}(r,n)$ in [26] by a different (but closely related) method based on the quantum Satake principle [32].

9.1. **Abelian quotient and non-abelian quotient.** Let $\mathbb{G} = \mathrm{Gr}(r,n)$ denote the Grassmann variety of r-dimensional subspaces in $\mathbb{C}^n$ and let $\mathbb{P} = \mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}$ (r times) denote the product of r copies of the projective space $\mathbb{P}^{n-1}$. We relate these two spaces in the framework of Martin [55]: $\mathbb{G}$ and $\mathbb{P}$ arise as non-abelian and abelian quotients of the same vector space $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$:

$$\mathbb{G} = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)/\!/GL(r), \quad \mathbb{P} = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)/\!/(\mathbb{C}^\times)^r.$$

We have a rational map

$$\mathbb{P} \dashrightarrow \mathbb{G}$$

sending a collection of lines $l_1, \dots, l_r$ to the subspace spanned by $l_1, \dots, l_r$. We can also relate $\mathbb{P}$ and $\mathbb{G}$ by the following diagram [55]:

$$\begin{tikzcd} \mathbb{F} \arrow[r, "p"] \arrow[d, "\iota"'] & \mathbb{G} \\ \mathbb{P} & \end{tikzcd} \tag{27}$$

where $\mathbb{F} := \mathrm{Fl}(1, 2, \dots, r, n)$ is the partial flag variety, $p \colon \mathbb{F} \to \mathbb{G}$ is the natural projection and ι is a real-analytic embedding which sends a flag $0 \subset V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{C}^n$ to the lines $L_1, \dots, L_r$ such that L_i is the orthogonal complement of V_{i-1} in V_i. Here we need to choose a Hermitian metric on $\mathbb{C}^n$ for the definition of ι. The diagram (27) naturally comes from the following description of $\mathbb{G}$ and $\mathbb{P}$ as symplectic reductions of $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$:

$$\mathbb{G} \cong \mu_G^{-1}(\eta)/G, \quad \mathbb{F} \cong \mu_G^{-1}(\eta)/T, \quad \mathbb{P} \cong \mu_T^{-1}(\eta_0)/T. \tag{28}$$

Here μ_G and μ_T are the moment maps of $G = U(r)$ and $T = (S^1)^r$-actions on $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ respectively and $\eta \in \mathrm{Lie}(G)^\star$ and $\eta_0 \in \mathrm{Lie}(T)^\star$ are non-zero central elements (i.e. scalar multiples of the identify matrix) such that $\pi(\eta) = \eta_0$ (where $\pi \colon \mathrm{Lie}(G)^\star \to \mathrm{Lie}(T)^\star$ is the natural projection):

$$\mu_G(A) = A^\dagger A, \quad \mu_T(A) = \left((A^\dagger A)_{i,i}\right)_{i=1}^r, \qquad \begin{tikzcd} \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n) \arrow[r, "\mu_G"] \arrow[dr, "\mu_T"'] & \mathrm{Lie}(G)^\star \arrow[d, "\pi"] \\ & \mathrm{Lie}(T)^\star \end{tikzcd}$$

The map ι in (27) is then induced from the inclusion $\mu_G^{-1}(\eta) \subset \mu_T^{-1}(\eta_0)$.

We describe the cohomology groups of $\mathbb{G}$ and $\mathbb{P}$. Let $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_r$ denote the tautological bundles (with $\operatorname{rank} \mathcal{V}_i = i$) over $\mathbb{F}$ and set

$$\mathcal{L}_i := (\mathcal{V}_i/\mathcal{V}_{i-1})^\vee.$$

The line bundle $\mathcal{L}_i$ is the pull-back of the line bundle $\mathcal{O}(1)$ over the ith factor $\cong \mathbb{P}^{n-1}$ of $\mathbb{P}$ under the non-algebraic map ι. We denote the corresponding line bundle over $\mathbb{P}$ also by $\mathcal{L}_i$, i.e. $\iota^\star \mathcal{L}_i = \mathcal{L}_i$. We set

$$x_i := c_1(\mathcal{L}_i) \in H^\bullet(\mathbb{P}) \text{ or } H^\bullet(\mathbb{F}).$$

A symmetric polynomial in $x_1, \dots, x_r$ can be written in terms of Chern classes of $\mathcal{V}_r$ and thus makes sense as a cohomology class on $\mathbb{G}$. The cohomology rings of $\mathbb{P}$, $\mathbb{G}$ and $\mathbb{F}$ are described in terms of x_i as follows:

$$\begin{aligned} H^\bullet(\mathbb{P}) &\cong \mathbb{C}[x_1,\dots,x_r]\Big/\langle x_1^n,\dots,x_r^n\rangle, \\ H^\bullet(\mathbb{G}) &\cong \mathbb{C}[x_1,\dots,x_r]^{\mathfrak{S}_r}\Big/\langle h_{n-r+1},\dots,h_n\rangle, \\ H^\bullet(\mathbb{F}) &\cong \mathbb{C}[x_1,\dots,x_r]\Big/\langle h_{n-r+1},\dots,h_n\rangle. \end{aligned}$$

Here $h_i = c_i(\mathbb{C}^n/\mathcal{V}_r)$ is the ith complete symmetric function of $x_1,\dots,x_r$. An additive basis of $H^\bullet(\mathbb{G})$ is given by the Schubert classes:

$$\sigma_\mu := \frac{\det\left(x_i^{\mu_j+r-j}\right)_{1\leqslant i,j\leqslant r}}{\det\left(x_i^{r-j}\right)_{1\leqslant i,j\leqslant r}} \tag{29}$$

where μ ranges over partitions such that $n-r \geqslant \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r \geqslant 0$. This is the Poincaré dual of the Schubert cycle

$$\Omega_\mu = \{V \in \mathrm{Gr}(r,n) : \dim(V\cap\mathbb{C}^{n-r+i-\mu_i}) \geqslant i,\ 1\leqslant i\leqslant r\}$$

and $\deg\sigma_\lambda = 2|\lambda| = 2\sum_{i=1}^r \lambda_i$ (degree as a cohomology class), see [34, §6, Chapter 1].

Let $J_{\mathbb{P}^{n-1}}(t)$ be the J-function (see (7), (18)) of $\mathbb{P}^{n-1}$. We define the multivariable J-function of $\mathbb{P}$ by

$$J_{\mathbb{P}}(t_1,\dots,t_r) = J_{\mathbb{P}^{n-1}}(t_1)\otimes J_{\mathbb{P}^{n-1}}(t_2)\otimes\cdots\otimes J_{\mathbb{P}^{n-1}}(t_r).$$

This takes values in $H^\bullet(\mathbb{P}) \cong H^\bullet(\mathbb{P}^{n-1})^{\otimes r}$. Bertram–Ciocan-Fontanine–Kim [9] proved the following *abelian/non-abelian correspondence* between the J-functions of $\mathbb{G}$ and $\mathbb{P}$.

Theorem 9.1 ([9]). *The J-function $J_{\mathbb{G}}(t)$ of $\mathbb{G}$ is given by*

$$p^\star J_{\mathbb{G}}(t) = e^{-\sigma_1\pi\sqrt{-1}(r-1)}\left[\frac{\left(\prod_{1\leqslant i<j\leqslant r}(\theta_i-\theta_j)\right)\iota^\star J_{\mathbb{P}}(t_1,\dots,t_r)}{n^{\binom{n}{r}}\Delta}\right]_{t_1=\cdots=t_r=\xi t}$$

where $\Delta := \prod_{i<j}(x_i-x_j)$, $\theta_i := t_i\frac{\partial}{\partial t_i}$, $\xi := e^{\pi\sqrt{-1}(r-1)/n}$ and $\sigma_1 := x_1+\cdots+x_r$.

9.2. **Preliminary lemmas.** We discuss elementary topological properties of the maps in (27). Martin [55] has shown similar results for general abelian/non-abelian quotients.

Lemma 9.2. *Let $\mathcal{N}_\iota \to \mathbb{F}$ denote the normal bundle of ι and let $\mathcal{T}_p = \mathrm{Ker}(dp) \subset T\mathbb{F}$ denote the relative tangent bundle of p. Then the complex structure I on $\mathbb{P}$ induces an isomorphism:*

$$I\colon \mathcal{T}_p \cong \mathcal{N}_\iota.$$

In particular, we have an isomorphism

$$\iota^\star T\mathbb{P} \cong p^\star T\mathbb{G} \oplus \left(\mathcal{T}_p\otimes_{\mathbb{R}}\mathbb{C},\sqrt{-1}\right)$$

of topological complex vector bundles, where the complex structure $\sqrt{-1}$ on $\mathcal{T}_p\otimes_{\mathbb{R}}\mathbb{C}$ is induced from that on the $\mathbb{C}$-factor.

Proof. Recall that $\mathbb{P}$, $\mathbb{G}$, $\mathbb{F}$ are described as symplectic reductions (28). Note that $\mathrm{rank}\,\mathcal{N}_\iota = \mathrm{rank}\,\mathcal{T}_p = \dim G - \dim T$. Let $\underline{X}$ denote the fundamental vector field associated with $X \in \mathrm{Lie}(G)$. For $x \in \mu_G^{-1}(\eta)$, it follows from the property of the moment map that $I\underline{X}_x$ is perpendicular to $T_x\mu_G^{-1}(\eta)$ for $X\in\mathrm{Lie}(G)$. Thus we have an orthogonal decomposition:

$$T_x\,\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n) = T_x\mu_G^{-1}(\eta)\oplus IT_x(G\cdot x).$$

Similarly we have an orthogonal decomposition

$$T_x \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^n) = T_x\mu_T^{-1}(\eta_0) \oplus IT_x(T \cdot x).$$

Note that the tangent space $T_{[x]}\mathbb{P}$ at $[x] \in \mathbb{P} = \mu_T^{-1}(\mu_0)/T$ is identified with the orthogonal complement of $T_x(T\cdot x)$ in $T_x\mu_T^{-1}(\eta_0)$, which is preserved by the complex structure I. Under this identification, the orthogonal complement of $T_x(T\cdot x)$ in $T_x(G\cdot x)$ is identified with the fiber $(\mathcal{T}_p)_{[x]}$; by the above decompositions, the complex structure I sends this to the orthogonal complement of $T_x\mu_G^{-1}(\eta)$ in $T_x\mu_T^{-1}(\eta_0)$. This shows $I((\mathcal{T}_p)_{[x]}) = (\mathcal{N}_\iota)_{[x]}$ and the conclusion follows. □

Remark 9.3. Note that as complex vector bundles

$$(\mathcal{T}_p \otimes_{\mathbb{R}} \mathbb{C}, \sqrt{-1}) \cong \mathcal{T}_p \oplus \overline{\mathcal{T}}_p$$

where $\overline{\mathcal{T}}_p$ denotes the conjugate of the complex vector bundle $\mathcal{T}_p$.

Because $\mathbb{F}, \mathbb{P}$ are compact oriented manifolds, the push-forward map $\iota_*\colon H^\bullet(\mathbb{F}) \to H^\bullet(\mathbb{P})$ is well-defined (even though ι is not algebraic). Here we need to be a little careful about the orientation of the normal bundle of ι.

Lemma 9.4. *The push-forward map $\iota_\star\colon H^\bullet(\mathbb{F}) \to H^\bullet(\mathbb{P})$ is given by*

$$\iota_\star(f(x)) = f(x) \cup \overline{\Delta}$$

for any polynomial $f(x) = f(x_1, \dots, x_r)$. Here $\overline{\Delta} = \prod_{i<j}(x_j - x_i) = (-1)^{\binom{r}{2}}\Delta$.

Proof. The normal bundle $\mathcal{N}_\iota$ of ι is not a complex vector bundle, but can be written as the formal difference of complex vector bundles:

$$\mathcal{N}_\iota = \iota^\star T\mathbb{P} \ominus T\mathbb{F}$$

and thus defines an element of the K-group $K^0(\mathbb{P})$ of topological complex vector bundles. Using Lemma 9.2, Remark 9.3 and a topological isomorphism $T\mathbb{F} \cong \mathcal{T}_p \oplus p^\star T\mathbb{G}$, we have:

$$\mathcal{N}_\iota = \mathcal{T}_p \otimes \mathbb{C} \ominus \mathcal{T}_p = \overline{\mathcal{T}}_p.$$

On the other hand, we have a topological isomorphism:

$$\mathcal{T}_p \cong \bigoplus_{i<j} \operatorname{Hom}(\mathcal{L}_i^\vee, \mathcal{L}_j^\vee).$$

Thus $\overline{\mathcal{T}}_p \to \mathbb{F}$ is isomorphic to the restriction of the following complex vector bundle over $\mathbb{P}$:

$$\bigoplus_{i<j} \overline{\mathcal{L}_i} \otimes \mathcal{L}_j \longrightarrow \mathbb{P}$$

and $\mathbb{F}$ is the zero locus of the section of this bundle defined by the given Hermitian metric on $\mathbb{C}^n$. Therefore, $\iota_\star(f(x)) = f(x) \cup \overline{\Delta}$. □

Lemma 9.5. *For a polynomial $f(x) = f(x_1, \dots, x_r)$, we have*

$$p_\star(f(x)) = \frac{1}{\Delta}\left(\sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(r)})\right).$$

Note that the right-hand side is a symmetric polynomial in $x_1, \dots, x_r$.

Proof. The $\mathfrak{S}_r$-action on $\mathbb{P}$ induces a non-algebraic $\mathfrak{S}_r$ action on $\mathbb{F}$ which preserves the fibration $p\colon \mathbb{F} \to \mathbb{G}$. The action of $\sigma \in \mathfrak{S}_r$ on $\mathbb{F}$ changes the orientation of each fiber of p by $\operatorname{sgn}(\sigma)$. Therefore $p_\star(f(x)) = \operatorname{sgn}(\sigma) p_\star(f(x_{\sigma(1)}, \dots, x_{\sigma(r)}))$. Thus we have

$$\begin{aligned} p_\star(f(x)) &= \frac{1}{r!} p_\star \left(\sum_{\sigma\in\mathfrak{S}_r} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(r)}) \right) \\ &= \frac{1}{r!} \frac{\sum_{\sigma\in\mathfrak{S}_r} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(r)})}{\Delta} p_\star(\Delta). \end{aligned}$$

The conclusion follows from $p_\star(\Delta) = p_\star(\operatorname{Euler}(\mathcal{T}_p)) = r!$. □

9.3. **Comparison of cohomology and K-groups.** Using the diagram (27), we identify the cohomology (or the K-group) of $\mathbb{G}$ with the anti-symmetric part of the cohomology (resp. the K-group) of $\mathbb{P}$. This identification was proved by Martin [55] for cohomology groups of general abelian/non-abelian quotients and has been generalized to K-groups by Harada–Landweber [35]. We compare the cohomological and K-theoretic identifications via the map $\mathcal{E} \mapsto \widehat{\Gamma}_F \operatorname{Ch}(\mathcal{E})$.

We identify the rth wedge product $\wedge^r H^\bullet(\mathbb{P}^{n-1})$ with the anti-symmetric part of $H^\bullet(\mathbb{P})$ (with respect to the $\mathfrak{S}_r$-action) via the map:

$$\wedge^r H^\bullet(\mathbb{P}^{n-1}) \ni \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r \longmapsto \sum_{\sigma\in\mathfrak{S}_r} \operatorname{sgn}(\sigma) \alpha_{\sigma(1)} \otimes \alpha_{\sigma(2)} \otimes \cdots \otimes \alpha_{\sigma(r)} \in H^\bullet(\mathbb{P}).$$

Similarly we identify $\wedge^r K^0(\mathbb{P})$ with the anti-symmetric part of $K^0(\mathbb{P})$, where $K^0(\cdot)$ denotes the topological K-group. For a non-increasing sequence $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r$ of integers, we set

$$\mathcal{E}_\mu := p_\star \left(\mathcal{L}_1^{\mu_1} \otimes \mathcal{L}_2^{\mu_2} \otimes \cdots \otimes \mathcal{L}_r^{\mu_r} \right).$$

By the Borel–Weil theory, $\mathcal{E}_\mu$ is the vector bundle on $\mathbb{G}$ associated to the irreducible $GL(r)$-representation of highest weight μ. Shifting μ_i by some number $\mu_i \mapsto \mu_i + \ell$ simultaneously corresponds to twisting the $GL(r)$-representation by $\det^{\otimes \ell}$, where $\det\colon GL(r) \to \mathbb{C}^\times$ corresponds to $\mathcal{O}(1)$ on $\mathbb{G}$. These vector bundles span the K-group of $\mathbb{G}$. We define a line bundle $\mathcal{R}$ on $\mathbb{F}$ by

$$\mathcal{R} = \mathcal{L}_1^{-(r-1)} \otimes \mathcal{L}_2^{-(r-2)} \otimes \cdots \otimes \mathcal{L}_{r-1}^{-1}.$$

This restricts to a half-canonical bundle[3] on each fiber of p. One can easily check that:

$$\det(\mathcal{T}_p^\vee) = p^\star(\mathcal{O}(r-1)) \otimes \mathcal{R}^{\otimes 2} \tag{30}$$

where $p^\star \mathcal{O}(1) = \mathcal{L}_1 \mathcal{L}_2 \cdots \mathcal{L}_r$. We have the following result:

Proposition 9.6. *Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r$ be a non-increasing sequence and let $k_1 > k_2 > \cdots > k_r$ be the strictly decreasing sequence given by $k_i = \mu_i + r - i$. Then we have:*

1. *The map $(r!)^{-1} p_\star \iota^\star\colon \wedge^r H^\bullet(\mathbb{P}) \to H^\bullet(\mathbb{G})$ sends the class $x_1^{k_1} \wedge x_2^{k_2} \wedge \cdots \wedge x_r^{k_r}$ to the Schubert class σ_μ (29) and is an isomorphism.*
2. *The map $(r!)^{-1} p_\star(\mathcal{R} \otimes \iota^\star(\cdot))\colon \wedge^r K^0(\mathbb{P}^{n-1}) \to K^0(\mathbb{G})$ sends the class $\mathcal{L}_1^{k_1} \wedge \mathcal{L}_2^{k_2} \wedge \cdots \wedge \mathcal{L}_r^{k_r}$ to the class $\mathcal{E}_\mu$ and is an isomorphism (over $\mathbb{Z}$).*

[3] corresponding to the half-sum ρ of positive roots

(3) *We have the commutative diagram:*

$$
\begin{tikzcd}
\wedge^r K^0(\mathbb{P}^{n-1}) \arrow[rr, "\frac{1}{r!}p_\star(\mathcal{R}\otimes\iota^\star(\cdot))"] \arrow[d, "\widehat{\Gamma}_{\mathbb{P}}\operatorname{Ch}(\cdot)"] && K^0(\mathbb{G}) \arrow[d, "\widehat{\Gamma}_{\mathbb{G}}\operatorname{Ch}(\cdot)"] \\
\wedge^r H^\bullet(\mathbb{P}^{n-1}) \arrow[rr, "\frac{\epsilon_r}{r!}e^{-\pi\sqrt{-1}(r-1)\sigma_1}p_\star\iota^\star"] && H^\bullet(\mathbb{G})
\end{tikzcd}
\tag{31}
$$

where $\epsilon_r = (2\pi\sqrt{-1})^{-\binom{r}{2}}$ *and* $\operatorname{Ch}(E) = \sum_{p\geqslant 0}(2\pi\sqrt{-1})^p \operatorname{ch}_p(E)$ *is the modified Chern character.*

Proof. (1): This follows from Lemma 9.5 and the definition of the Schubert class σ_μ (29).

(2): Using the Grothendieck–Riemann–Roch theorem, we have

$$
\begin{aligned}
\operatorname{ch}\left(p_\star(\mathcal{R}\otimes\iota^\star(\mathcal{L}_1^{k_1}\otimes\cdots\otimes\mathcal{L}_r^{k_r}))\right) &= p_\star(\operatorname{ch}(\mathcal{R}\otimes\mathcal{L}_1^{k_1}\otimes\cdots\otimes\mathcal{L}_r^{k_r}\otimes\operatorname{ch}(\mathcal{T}_p)) \\
&= p_\star\left(e^{\mu_1x_1+\cdots+\mu_rx_r}\prod_{i<j}\frac{x_i-x_j}{1-e^{x_i-x_j}}\right) \\
&= p_\star\left(e^{k_1x_1+\cdots+k_rx_r}\prod_{i<j}\frac{x_i-x_j}{e^{x_j}-e^{x_i}}\right).
\end{aligned}
$$

By Lemma 9.5, it follows that this is anti-symmetric in $k_1,\dots,k_r$. Therefore

$$
\begin{aligned}
\frac{1}{r!}\operatorname{ch}\left(p_\star(\mathcal{R}\otimes\iota^\star(\mathcal{L}_1^{k_1}\wedge\cdots\wedge\mathcal{L}_r^{k_r}))\right) &= \operatorname{ch}\left(p_\star(\mathcal{R}\otimes\iota^\star(\mathcal{L}_1^{k_1}\otimes\cdots\otimes\mathcal{L}_r^{k_r}))\right) \\
&= \operatorname{ch}\left(p_\star(\mathcal{L}_1^{\mu_1}\otimes\cdots\otimes\mathcal{L}_1^{\mu_r})\right) = \operatorname{ch}(\mathcal{E}_\mu).
\end{aligned}
$$

This shows that $(r!)^{-1}p_\star(\mathcal{R}\otimes\iota^\star(\mathcal{L}_1^{k_1}\wedge\cdots\wedge\mathcal{L}_r^{k_r})) = \mathcal{E}_\mu$.

(3): It suffices to prove the commutativity of the following two diagrams:

$$
\begin{tikzcd}
\wedge^r K^0(\mathbb{P}^{n-1}) \arrow[r, "\iota^\star"] \arrow[d, "\widehat{\Gamma}_{\mathbb{P}}\operatorname{Ch}(\cdot)"] & K^0(\mathbb{F}) \arrow[d, "(\iota^\star\widehat{\Gamma}_{\mathbb{P}})\operatorname{Ch}(\cdot)"] \\
\wedge^r H^0(\mathbb{P}^{r-1}) \arrow[r, "\iota^\star"] & H^0(\mathbb{G})
\end{tikzcd}
\qquad
\begin{tikzcd}
K^0(\mathbb{F}) \arrow[rr, "p_\star(\mathcal{R}\otimes(\cdot))"] \arrow[d, "(\iota^\star\widehat{\Gamma}_{\mathbb{P}})\operatorname{Ch}(\cdot)"] && K^0(\mathbb{G}) \arrow[d, "\widehat{\Gamma}_{\mathbb{G}}\operatorname{Ch}(\cdot)"] \\
H^\bullet(\mathbb{F}) \arrow[rr, "\epsilon_r e^{-\pi\sqrt{-1}(r-1)\sigma_1}p_\star"] && H^\bullet(\mathbb{G})
\end{tikzcd}
$$

The commutativity of the left diagram is obvious. The commutativity of the right diagram follows from Grothendieck–Riemann–Roch. For $\alpha\in K^0(\mathbb{F})$, we have

$$\operatorname{Ch}(p_\star(\mathcal{R}\otimes\alpha)) = \epsilon_r p_\star(\operatorname{Ch}(\mathcal{R}\otimes\alpha)\operatorname{Td}(\mathcal{T}_p)).$$

From $\iota^\star T\mathbb{P}\cong p^\star T\mathbb{G}\oplus\mathcal{T}_p\oplus\overline{\mathcal{T}_p}$ (Lemma 9.2 and Remark 9.3), we have

$$\iota^\star\widehat{\Gamma}_{\mathbb{P}} = (p^\star\widehat{\Gamma}_{\mathbb{P}})\widehat{\Gamma}(\mathcal{T}_p)\widehat{\Gamma}(\mathcal{T}_p^\vee) = (p^\star\widehat{\Gamma}_{\mathbb{P}})\operatorname{Td}(\mathcal{T}_p)e^{-\pi\sqrt{-1}c_1(\mathcal{T}_p)}$$

where we used the fact that the Gamma class is a square root of the Todd class (see §1.2). Therefore

$$
\begin{aligned}
\widehat{\Gamma}_{\mathbb{G}}\operatorname{Ch}(p_\star(\mathcal{R}\otimes\alpha)) &= \epsilon_r p_\star\left((p^\star\widehat{\Gamma}_{\mathbb{G}})\operatorname{Ch}(\mathcal{R}\otimes\alpha)\operatorname{Td}(\mathcal{T}_p)\right) \\
&= \epsilon_r p_\star\left((\iota^\star\widehat{\Gamma}_{\mathbb{P}})\operatorname{Ch}(\alpha)\operatorname{Ch}(\mathcal{R})e^{-\pi\sqrt{-1}c_1(\mathcal{T}_p)}\right).
\end{aligned}
$$

The relationship (30) gives $\operatorname{Ch}(\mathcal{R})e^{-\pi\sqrt{-1}c_1(\mathcal{T}_p)} = e^{-\pi\sqrt{-1}(r-1)\sigma_1}$ and the commutativity follows. $\square$

We define the Euler pairing on $\wedge^r K^0(\mathbb{P}^{n-1})$ as

$$\chi(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r, \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_r) := \det(\chi_{\mathbb{P}^{n-1}}(\alpha_i, \beta_j))_{1 \leqslant i,j \leqslant r}.$$

This is $1/r!$ of the Euler pairing induced from $K^0(\mathbb{P})$. Recall the non-symmetric pairing $[\cdot,\cdot)$ defined in (2). Similarly we define the pairing $[\cdot,\cdot)$ on $\wedge^r H^\bullet(\mathbb{P}^{n-1})$ by

$$[\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r, \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_r) := \det([\alpha_i, \beta_j))_{1 \leqslant i,j \leqslant r}. \tag{32}$$

This is again $1/r!$ of the pairing $[\cdot,\cdot)$ on $H^\bullet(\mathbb{P})$.

Proposition 9.7. *The horizontal maps in the diagram* (31) *preserves the pairings. More precisely, the map*

$$(r!)^{-1} p_\star(\mathcal{R} \otimes \iota^\star(\cdot))\colon\ \wedge^r K^0(\mathbb{P}^{n-1}) \to K^0(\mathbb{G})$$

preserves the Euler pairing χ and the map

$$(r!)^{-1} \epsilon_r e^{-\pi\sqrt{-1}(r-1)\sigma_1} p_\star \iota^\star\colon\ \wedge^r H^\bullet(\mathbb{P}^{n-1}) \to H^\bullet(\mathbb{G})$$

preserves the pairing $[\cdot,\cdot)$.

Proof. Since the vertical maps in (31) intertwines the pairings χ and $[\cdot,\cdot)$ (see (1)), it suffices to show that the map $(r!)^{-1}\epsilon_r e^{-\pi\sqrt{-1}(r-1)\sigma_1} p_\star\iota^\star$ on cohomology preserves the pairing $[\cdot,\cdot)$. For $\alpha, \beta \in \wedge^r H^\bullet(\mathbb{P}^{n-1})$, we have

$$\begin{aligned}
&\Big[(r!)^{-1}\epsilon_r e^{-\pi\sqrt{-1}(r-1)\sigma_1} p_\star\iota^\star(\alpha), (r!)^{-1}\epsilon_r e^{-\pi\sqrt{-1}(r-1)\sigma_1} p_\star\iota^\star(\beta)\Big) \\
&= \left(\frac{\epsilon_r}{r!}\right)^2 \frac{1}{(2\pi\sqrt{-1})^{\dim \mathbb{G}}} \int_{\mathbb{G}} e^{\pi\sqrt{-1}c_1(\mathbb{G})} (e^{\pi\sqrt{-1}\deg/2} p_\star\iota^\star\alpha) \cup p_\star\iota^\star(\beta) \\
&= \frac{1}{(r!)^2} \frac{(-1)^{\binom{r}{2}}}{(2\pi\sqrt{-1})^{\dim \mathbb{P}}} \int_{\mathbb{G}} (p_\star\iota^\star e^{\pi\sqrt{-1}c_1(\mathbb{P})} e^{\pi\sqrt{-1}\deg/2}\alpha) \cup p_\star\iota^\star(\beta) \\
&= \frac{1}{(r!)^2} \frac{(-1)^{\binom{r}{2}}}{(2\pi\sqrt{-1})^{\dim \mathbb{P}}} \int_{\mathbb{P}} (e^{\pi\sqrt{-1}c_1(\mathbb{P})} e^{\pi\sqrt{-1}\deg/2}\alpha) \cup \iota_\star p^\star p_\star\iota^\star(\beta).
\end{aligned}$$

By the formulas in Lemma 9.4 and Lemma 9.5, it follows easily that $\iota_\star p^\star p_\star \iota^\star\beta = r!(-1)^{\binom{r}{2}}\beta$ for the antisymmetric element β. Therefore this gives $1/r!$ of the pairing $[\cdot,\cdot)$ on $\mathbb{P}$ and the conclusion follows. ∎

Corollary 9.8 ([62])**.** *For $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r$ and $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_r$*

$$\chi(\mathcal{E}_\mu, \mathcal{E}_\nu) = \det\left(\chi(\mathcal{O}_{\mathbb{P}^{n-1}}(l_i), \mathcal{O}_{\mathbb{P}^{n-1}}(k_i))_{1 \leqslant i,j \leqslant r}\right)$$

where $l_i = \mu_i + r - i$ and $k_i = \nu_i + r - i$.

Remark 9.9. The map $(r!)^{-1} p_\star\iota^\star\colon \wedge^r H^\bullet(\mathbb{P}^{n-1}) \to H^\bullet(\mathbb{G})$ sending $x_1^{\mu_1+r-1} \wedge x_2^{\mu_2+r-2} \wedge \cdots \wedge x_r^{\mu_r}$ to σ_μ is called the *Satake identification* in [26].

9.4. **Quantum cohomology central charge.** Here we restate the abelian/non-abelian correspondence of the J-functions in terms of quantum cohomology central charges. Recall from Remark 5.2 that the central charge $Z^{\mathbb{G}}(E)$ of a vector bundle $E \to \mathbb{G}$ is given by

$$Z^{\mathbb{G}}(E)(t) = (2\pi\sqrt{-1})^{\dim \mathbb{G}} \Big[J_{\mathbb{G}}(e^{\pi\sqrt{-1}}t), \widehat{\Gamma}_{\mathbb{G}} \operatorname{Ch}(E)\Big), \tag{33}$$

where $[\cdot,\cdot)$ is the pairing defined in (2). This is a function on the universal cover of the punctured t-plane $\mathbb{C}^\times$. A branch of this function is determined when we specify $\arg t \in \mathbb{R}$. When $t \in \mathbb{R}_{>0}$, we regard $\arg t = 0$ unless otherwise specified. For $\alpha \in K^0(\mathbb{P})$, we define

$$Z^{\mathbb{P}}(\alpha) := (2\pi\sqrt{-1})^{\dim \mathbb{P}} \Big[J_{\mathbb{P}}(e^{\pi\sqrt{-1}}t_1, \ldots, e^{\pi\sqrt{-1}}t_r), \widehat{\Gamma}_{\mathbb{P}} \operatorname{Ch}(\alpha)\Big)$$

where $[\cdot,\cdot)$ is the pairing on $H^\bullet(\mathbb{P})$. When $\alpha = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r \in \wedge^r K^0(\mathbb{P}^{n-1})$, we have

$$Z^{\mathbb{P}}(\alpha) = \det \left(Z^{\mathbb{P}^{n-1}}(\alpha_i)(t_j) \right)_{1 \leqslant i,j \leqslant r}$$

where $Z^{\mathbb{P}^{n-1}}$ is the quantum cohomology central charge for $\mathbb{P}^{n-1}$.

Proposition 9.10. *Let $Z^{\mathbb{G}}$ and $Z^{\mathbb{P}}$ denote the quantum cohomology central charges of $\mathbb{G}$ and $\mathbb{P}$ respectively. Then we have the equality:*

$$Z^{\mathbb{G}}(p_\star(\mathcal{R} \otimes \iota^\star \alpha)) = \frac{1}{(2\pi\sqrt{-1}n)^{\binom{r}{2}}} \left[\prod_{i<j} (\theta_i - \theta_j) \cdot Z^{\mathbb{P}}(\alpha) \right]_{t_1 = \cdots = t_r = \xi t},$$

where $\alpha \in \wedge^r K^0(\mathbb{P}^{n-1})$ and $\xi := e^{\pi\sqrt{-1}(r-1)/n}$.

Proof. By Lemma 9.5, the abelian/non-abelian correspondence (Theorem 9.1) can be written in the form:

$$J_{\mathbb{G}}(t) = \frac{1}{r!} e^{-\pi\sqrt{-1}(r-1)\sigma_1} p_\star \iota^\star \widetilde{J}(t)$$

with

$$\widetilde{J}(t) = \left[n^{-\binom{r}{2}} \prod_{i<j} (\theta_i - \theta_j) J_{\mathbb{P}}(t_1, \dots, t_r) \right]_{t_1 = \cdots = t_r = \xi t}.$$

By Proposition 9.6 (3), we have

$$\widehat{\Gamma}_{\mathbb{G}} \operatorname{Ch}(p_\star(\mathcal{R} \otimes \iota^\star \alpha)) = \epsilon_r e^{-\pi\sqrt{-1}(r-1)\sigma_1} p_\star \iota^\star \left(\widehat{\Gamma}_{\mathbb{P}} \operatorname{Ch}(\alpha) \right).$$

By the above equations and the fact that $\frac{\epsilon_r}{r!} e^{-\pi\sqrt{-1}(r-1)\sigma_1} p_\star \iota^\star$ preserves the pairing $[\cdot,\cdot)$ (Proposition 9.7), we obtain

$$Z^{\mathbb{G}}(p_\star(\mathcal{R} \otimes \iota^\star \alpha)) = (2\pi\sqrt{-1})^{\dim \mathbb{G} + \binom{r}{2}} r! \left[\widetilde{J}(t), \widehat{\Gamma}_{\mathbb{P}} \operatorname{Ch}(\alpha) \right)$$

where the pairing $[\cdot,\cdot)$ on the right-hand side denotes the pairing (32) on $\wedge^r H^\bullet(\mathbb{P}^{n-1})$ (which is $1/r!$ of the pairing $[\cdot,\cdot)$ on $H^\bullet(\mathbb{P})$). Using $\dim G + \binom{r}{2} = \dim \mathbb{P} - \binom{r}{2}$, we obtain the formula in the proposition. □

9.5. **Integral representation of quantum cohomology central charges.** In this section we give integral representations of quantum cohomology central charges in terms of the Hori–Vafa mirrors. Let $f \colon (\mathbb{C}^\times)^{n-1} \to \mathbb{C}$ be the mirror Laurent polynomial of $\mathbb{P}^{n-1}$ (see §5):

$$f(y) = y_1 + \cdots + y_{n-1} + \frac{1}{y_1 y_2 \cdots y_{n-1}}.$$

The critical values of f are given as

$$v_k := n e^{-2\pi\sqrt{-1}k/n}, \quad k \in \mathbb{Z}/n\mathbb{Z}.$$

Let $\phi \in \mathbb{R}$ be an admissible phase for $\{v_0, v_1, \dots, v_{n-1}\}$ (see §4.2). Let $\Gamma_k(\phi) \subset (\mathbb{C}^\times)^{n-1}$ denote the Lefschetz thimble for f associated with the critical value v_k and the vanishing path $v_k + \mathbb{R}_{\geqslant 0} e^{\sqrt{-1}\phi}$ (see Figure 2). The Lefschetz thimble $\Gamma_k(\phi)$ is homeomorphic to $\mathbb{R}^{n-1}$ and fibers over the straight half-line $f(\Gamma_k) = v_k + \mathbb{R}_{\geqslant 0} e^{\sqrt{-1}\phi}$; the fiber is a vanishing cycle homeomorphic to S^{n-2}. We write $\Gamma_k^\vee(\phi)$ for the "opposite" Lefschetz thimble associated to the critical value v_k and the path $v_k - \mathbb{R}_{\geqslant 0} e^{\sqrt{-1}\phi}$. We choose orientation of Lefschetz thimbles such that $\sharp(\Gamma_k(\phi) \cap \Gamma_k^\vee(\phi)) = (-1)^{(n-1)(n-2)/2}$.

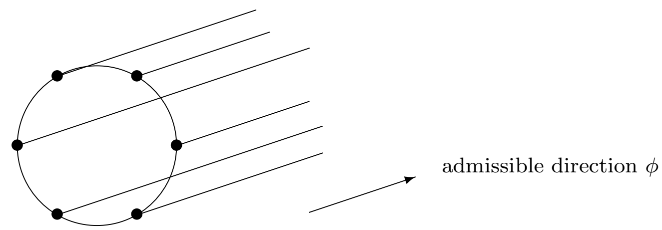

FIGURE 2. Vanishing paths in the admissible direction ϕ

Proposition 9.11. *Let $\phi \in \mathbb{R}$ be an admissible phase for $\{v_0, v_1, \dots, v_{n-1}\}$. There exist K-classes $\mathcal{F}_k(\phi), \mathcal{G}_k(\phi) \in K^0(\mathbb{P}^{n-1})$ for $k \in \mathbb{Z}/n\mathbb{Z}$ such that*

$$
\begin{aligned}
Z^{\mathbb{P}^{n-1}}(\mathcal{F}_k(\phi))(t) &= \int_{\Gamma_k(\phi)} e^{-tf(y)} \frac{dy}{y}, \\
Z^{\mathbb{P}^{n-1}}(\mathcal{G}_k(\phi))(e^{-\pi\sqrt{-1}}t) &= \int_{\Gamma_k^\vee(\phi)} e^{tf(y)} \frac{dy}{y},
\end{aligned}
\tag{34}
$$

hold when $|\arg t + \phi| < \frac{\pi}{2}$, where $\frac{dy}{y} = \bigwedge_{i=1}^{n-1} \frac{dy_i}{y_i}$. Moreover, we have:

(1) *$\chi(\mathcal{F}_k(\phi), \mathcal{G}_l(\phi)) = \delta_{kl}$;*
(2) *when $|k\frac{2\pi}{n} + \phi| < \frac{\pi}{2} + \frac{\pi}{n}$, we have $\mathcal{F}_k(\phi) = \mathcal{O}_{\mathbb{P}^{n-1}}(k)$.*

Proof. This follows from the result in [42, 47]. (See §6 and [26, §5] for a closely related discussion). By [42, Theorems 4.11, 4.14], we have an isomorphism

$$K^0(\mathbb{P}^{n-1}) \cong H_{n-1}((\mathbb{C}^\times)^{n-1}, \{y : \mathrm{Re}(tf(y)) \geqslant M\}), \quad \alpha \mapsto \Gamma(\alpha, \arg t)$$

depending on $\arg t \in \mathbb{R}$, such that $\Gamma(\alpha, \arg t)$ is Gauss–Manin flat with respect to the variation of $\arg t$ and that

$$Z^{\mathbb{P}^{n-1}}(\alpha)(t) = \int_{\Gamma(\alpha, \arg t)} e^{-tf(y)} \frac{dy}{y}.$$

Here M is a sufficiently big positive number (it is sufficient that $M \geqslant 2n|t|$). Moreover, this isomorphism intertwines the Euler pairing with the intersection pairing:

$$\chi(\alpha, \beta) = (-1)^{\frac{(n-1)(n-2)}{2}} \sharp \left(\Gamma(\alpha, \arg t + \pi) \cap \Gamma(\beta, \arg t) \right).$$

Therefore equation (34) and part (1) of the proposition hold for $\mathcal{F}_k, \mathcal{G}_k$ such that

$$\Gamma(\mathcal{F}_k(\phi), -\phi) = \Gamma_k(\phi), \quad \Gamma(\mathcal{G}_k(\phi), -\phi - \pi) = \Gamma_k^\vee(\phi).$$

(We only need to check (34) at $\arg t = -\phi$; it then follows by analytic continuation for other $t \in (-\phi - \frac{\pi}{2}, -\phi + \frac{\pi}{2})$.) Recall from (20) that we have

$$Z^{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}})(t) = \int_{\Gamma_0(0)} e^{-tf(y)} \frac{dy}{y}$$

when $\arg t = 0$. Therefore, when $\arg t = 2\pi k/n$, we have by setting $t' = e^{-2\pi\sqrt{-1}k/n} t$,

$$
\begin{aligned}
Z^{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(k))(t) &= Z^{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}})(t') \\
&= \int_{\Gamma_0(0)} e^{-t'f(y)} \frac{dy}{y} = \int_{\Gamma_k(-2\pi k/n)} e^{-tf(y)} \frac{dy}{y}
\end{aligned}
$$

where in the first line we used (18) and the definition of the central charge, and in the second line we performed the change $y_i \to e^{2\pi\sqrt{-1}k/n} y_i$ of variables and used the fact that $e^{-2\pi\sqrt{-1}k/n}\Gamma_0(0) = \Gamma_k(-2\pi k/n)$. From this it follows that

$$Z^{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(k))(t) = \int_{\Gamma_k(\phi)} e^{-tf(y)} \frac{dy}{y}$$

when $|\phi + k\frac{2\pi}{n}| < \frac{\pi}{2} + \frac{\pi}{n}$ and $\arg t = -\phi$. This implies $\mathcal{F}_k(\phi) = \mathcal{O}_{\mathbb{P}^{n-1}}(k)$ for such ϕ and k and part (2) follows. □

Remark 9.12. The sign $(-1)^{(n-1)(n-2)/2}$ in the intersection pairing was missing in [42], and this has been corrected in [44, footnote 16].

Remark 9.13. Part (2) of the proposition determines roughly half of $\mathcal{F}_k(\phi)$'s. The whole collection $\{\mathcal{F}_k(\phi) : k \in \mathbb{Z}/n\mathbb{Z}\}$ is obtained from the exceptional collection $\{\mathcal{O}(k) : -\phi - \pi < k\frac{2\pi}{n} < -\phi + \pi\}$ by a sequence of mutations (see [26, §5]), and thus is an exceptional collection. Because $\mathcal{F}_k(\phi)$ is mirror to the Lefschetz thimble $\Gamma_k(\phi)$, the set $\{\widehat{\Gamma}_{\mathbb{P}^{n-1}} \operatorname{Ch}(\mathcal{F}_k(\phi)) : k \in \mathbb{Z}/n\mathbb{Z}\}$ gives the asymptotic basis of $\mathbb{P}^{n-1}$ at phase ϕ (see the proof of Theorem 6.4) and the Gamma conjecture II (§4.3) holds for $\mathbb{P}^{n-1}$.

The Hori–Vafa mirror $\tilde{g}$ of $\mathbb{P} = (\mathbb{P}^{n-1})^r$ is given by:

$$\begin{aligned}\tilde{g}(y) &:= f(\vec{y}_1) + f(\vec{y}_2) + \cdots + f(\vec{y}_r) \\ &= \sum_{i=1}^{r} \left(y_{i,1} + y_{i,2} + \cdots + y_{i,n-1} + \frac{1}{y_{i,1}y_{i,2}\cdots y_{i,n-1}} \right)\end{aligned}$$

where $y = (\vec{y}_1, \dots, \vec{y}_r) \in (\mathbb{C}^\times)^{r(n-1)}$ and $\vec{y}_i = (y_{i,1}, \dots, y_{i,n-1}) \in (\mathbb{C}^\times)^{n-1}$. Critical values of $\tilde{g}$ are given by

$$\tilde{v}_K := v_{k_1} + \cdots + v_{k_r} \quad \text{for } K = (k_1, \dots, k_r) \in \mathbb{Z}^r.$$

The product $\Gamma_{k_1}(\phi) \times \cdots \times \Gamma_{k_r}(\phi)$ of Lefschetz thimbles for f gives a Lefschetz thimble for $\tilde{g}$ associated to the critical value $\tilde{v}_K$ and the path $\tilde{v}_K + \mathbb{R}_{\geqslant 0} e^{\sqrt{-1}\phi}$.

The Hori–Vafa mirror of $\mathbb{G}$ is obtained from the mirror of $\mathbb{P}$ by shifting the phase by $(r-1)\pi/n$ and restricting to "anti-symmetrized" Lefschetz thimbles (see [37, 62, 49]). We set

$$g(y) := \xi \tilde{g}(y)$$

with $\xi = e^{\pi\sqrt{-1}(r-1)/n}$. Then critical values of $g(y)$ are

$$v_K := \xi \tilde{v}_K \quad \text{for } K = (k_1, \dots, k_r) \in \mathbb{Z}^r.$$

For a tuple $K = (k_1, \dots, k_r) \in \mathbb{Z}^r$ and $\phi \in \mathbb{R}$, we define the anti-symmetrized Lefschetz thimble for $g(y)$ as:

$$\begin{aligned}\Gamma_K(\phi) &:= \Gamma_{k_1}(\phi') \wedge \cdots \wedge \Gamma_{k_r}(\phi') = \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) \Gamma_{k_{\sigma(1)}}(\phi') \times \cdots \times \Gamma_{k_{\sigma(r)}}(\phi') \\ \Gamma_K^\vee(\phi) &:= \Gamma_{k_1}^\vee(\phi') \wedge \cdots \wedge \Gamma_{k_r}^\vee(\phi') = \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) \Gamma^\vee_{k_{\sigma(1)}}(\phi') \times \cdots \times \Gamma^\vee_{k_{\sigma(r)}}(\phi')\end{aligned}$$

with $\phi' = \phi - \frac{(r-1)\pi}{n}$. They are elements of the relative homology groups

$$H_{r(n-1)}((\mathbb{C}^\times)^{r(n-1)}, \{y : \pm \operatorname{Re}(e^{-\sqrt{-1}\phi} g(y)) \geqslant M\})$$

with M sufficiently large. We also define K-classes $\mathcal{F}_K(\phi), \mathcal{G}_K(\phi) \in K^0(\mathbb{G})$ as:

$$\begin{aligned} \mathcal{F}_K(\phi) &:= \frac{1}{r!} p_\star \left(\mathcal{R} \otimes \iota^\star(\mathcal{F}_{k_1}(\phi') \wedge \cdots \wedge \mathcal{F}_{k_r}(\phi'))\right) \\ \mathcal{G}_K(\phi) &:= \frac{1}{r!} p_\star \left(\mathcal{R} \otimes \iota^\star(\mathcal{G}_{k_1}(\phi') \wedge \cdots \wedge \mathcal{G}_{k_r}(\phi'))\right) \end{aligned} \tag{35}$$

with $\phi' = \phi - \frac{(r-1)\pi}{n}$, where $\mathcal{F}_k(\phi)$ and $\mathcal{G}_k(\phi)$ are as in Proposition 9.11. Note that $\{\mathcal{F}_K(\phi)\}_K$ or $\{\mathcal{G}_K(\phi)\}_K$ gives an integral basis of $K^0(\mathbb{G})$ by Proposition 9.6.

Remark 9.14. By the change of variables $y_{i,j} = \xi y'_{i,j}$, we can also write:

$$g(y) = \sum_{i=1}^{r} \left(y'_{i,1} + y'_{i,2} + \cdots + y'_{i,n-1} + \frac{(-1)^{r-1}}{y'_{i,1} y'_{i,2} \cdots y'_{i,n-1}} \right)$$

Theorem 9.15. *Let $\phi \in \mathbb{R}$ be such that $\phi' = \phi - \frac{(r-1)\pi}{n}$ is admissible for $\{v_0, v_1, \dots, v_{n-1}\}$. For a mutually distinct r-tuple $K = (k_1, \dots, k_r) \in (\mathbb{Z}/n\mathbb{Z})^r$, we have*

$$Z^{\mathbb{G}}(\mathcal{F}_K(\phi))(t) = \frac{1}{r!} \left(\frac{-\xi t}{2\pi\sqrt{-1}n} \right)^{\binom{r}{2}} \int_{\Gamma_K(\phi)} \frac{dy}{y} e^{-tg(y)} \cdot \prod_{i<j} (f(\vec{y}_i) - f(\vec{y}_j))$$

$$Z^{\mathbb{G}}(\mathcal{G}_K(\phi))(e^{-\pi\sqrt{-1}} t) = \frac{1}{r!} \left(\frac{\xi t}{2\pi\sqrt{-1}n} \right)^{\binom{r}{2}} \int_{\Gamma^\vee_K(\phi)} \frac{dy}{y} e^{tg(y)} \cdot \prod_{i<j} (f(\vec{y}_i) - f(\vec{y}_j))$$

for $|\arg t + \phi| < \frac{\pi}{2}$, where $\xi := e^{\pi\sqrt{-1}(r-1)/n}$ and $\frac{dy}{y} := \bigwedge_{i=1}^r \bigwedge_{j=1}^{n-1} \frac{dy_{i,j}}{y_{i,j}}$. Moreover, we have

$$\chi(\mathcal{F}_K(\phi), \mathcal{G}_L(\phi)) = \delta_{K,L}$$

when the tuples K, L are ordered with respect to a fixed choice of a total order of $\mathbb{Z}/n\mathbb{Z}$.

Proof. Combining Proposition 9.10 and Proposition 9.11, we have that

$$\begin{aligned} Z^{\mathbb{G}}(\mathcal{F}_K(\phi))(t) &= \frac{1}{r!(2\pi\sqrt{-1}n)^{\binom{r}{2}}} \left[\prod_{i<j} (\theta_i - \theta_j) \cdot Z^{\mathbb{P}}(\mathcal{F}_{k_1}(\phi') \wedge \cdots \wedge \mathcal{F}_{k_r}(\phi')) \right]_{t_1 = \cdots = t_r = \xi t} \\ &= \frac{1}{r!(2\pi\sqrt{-1}n)^{\binom{r}{2}}} \left[\prod_{i<j} (\theta_i - \theta_j) \cdot \int_{\Gamma_K(\phi)} e^{-(t_1 f(\vec{y}_1) + \cdots + t_r f(\vec{y}_r))} \frac{dy}{y} \right]_{t_1 = \cdots = t_r = \xi t} \\ &= \frac{1}{r!} \left(\frac{-\xi t}{2\pi\sqrt{-1}n} \right)^{\binom{r}{2}} \int_{\Gamma_K(\phi)} \frac{dy}{y} e^{-tg(y)} \prod_{i<j} (f(\vec{y}_i) - f(\vec{y}_j)), \end{aligned}$$

where in the last line, we used Lemma 9.16 below. The formula for $Z^{\mathbb{G}}(\mathcal{G}_K(\phi))(e^{-\pi\sqrt{-1}} t)$ follows similarly. The orthogonality $\chi(\mathcal{F}_K(\phi), \mathcal{G}_L(\phi)) = \delta_{K,L}$ follows easily from Propositions 9.7 and 9.11. ∎

Lemma 9.16. *We have*

$$\left(\prod_{i<j} (\theta_i - \theta_j) \right) e^{\sum_{i=1}^r \alpha_i t_i} = e^{\sum_{i=1}^r \alpha_i t_i} \prod_{i<j} (\alpha_i t_i - \alpha_j t_j)$$

where $\theta_i = t_i \frac{\partial}{\partial t_i}$.

Proof. The left-hand side can be written in the form $\varphi(\alpha_1 t_1, \dots, \alpha_r t_r) e^{\sum_{i=1}^r \alpha_i t_i}$ for some polynomial φ and the highest order term of φ is $\prod_{i<j}(\alpha_i t_i - \alpha_j t_j)$. On the other hand, φ is anti-symmetric in the arguments, and thus should be divisible by $\prod_{i<j}(\alpha_i t_i - \alpha_j t_j)$. This implies the lemma. □

Remark 9.17. The collection $\{\mathcal{F}_K(\phi)\}_K$, where K ranges over distinct r elements of $\mathbb{Z}/n\mathbb{Z}$, yields the asymptotic basis $\{\widehat{\Gamma}_{\mathbb{G}} \operatorname{Ch}(\mathcal{F}_K(\phi))\}_K$ of $\mathbb{G}$ at phase ϕ. This follows either by studying mirror oscillatory integrals in more details or by combining [26, Proposition 6.5.1], Remark 9.13 and Proposition 9.6. It follows from the deformation argument in [26, §6] that this is mutation equivalent to Kapranov's exceptional collection $\{\mathcal{E}_\mu : n-r \geqslant \mu_1 \geqslant \cdots \geqslant \mu_r \geqslant 0\}$ [46].

9.6. **Eguchi–Hori–Xiong mirror and quantum period.** There is another mirror description for Grassmannians due to Eguchi–Hori–Xiong [22] (see also [6]). This is a Laurent polynomial mirror of dimension $\dim \mathbb{G} = r(n-r)$. We use this description to show that the limit sup in (10) can be replaced with the limit for Grassmannians.

We introduce $r(n-r)$ independent variables $X_{i,j}$ with $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n-r$. Define the Laurent polynomial $W(X)$ in these variables by

$$W(X) = \sum_{i=1}^{r} \sum_{j=1}^{n-r-1} \frac{X_{i,j+1}}{X_{i,j}} + \sum_{j=1}^{n-r} \sum_{i=1}^{r-1} \frac{X_{i+1,j}}{X_{i,j}} + \frac{1}{X_{1,n-r}} + \frac{1}{X_{r,1}}.$$

Batyrev–Ciocan-Fontanine–Kim–van-Straten [6] showed that the Newton polytope of W equals the fan polytope of a toric degeneration of $\mathbb{G} = \operatorname{Gr}(r,n)$ and conjectured that $W(X)$ is a weak Landau–Ginzburg model of $\mathbb{G}$. Recently, Marsh and Rietsch [54] proved this conjecture by constructing a compactification of the Eguchi–Hori–Xiong mirror and showing an isomorphism between the quantum connection and the Gauss–Manin connection.

Theorem 9.18 ([54])**.** *The quantum period* (9) $G_{\mathbb{G}}(t) = \langle [\mathrm{pt}], J_{\mathbb{G}}(t)\rangle$ *of the Grassmannian* $\mathbb{G}$ *is given by the constant term series of the Eguchi–Hori–Xiong mirror* $W(X)$*, i.e.* $G_{\mathbb{G}}(t) = \sum_{i=0}^\infty \frac{1}{i!} \operatorname{Const}(W^i) t^i$.

It is easy to check that the constant term series of $W(X)$ is of the form $\sum_{k=0}^\infty a_k t^{kn}$ with $a_k \neq 0$ for all $k \in \mathbb{Z}_{\geqslant 0}$ (see [6, §5.2] for the explicit form). Therefore, by applying Lemma 3.13, we obtain the following. (It seems difficult to deduce this from Hori–Vafa mirrors).

Proposition 9.19. *Let* $G_{\mathbb{G}}(t) = \langle [\mathrm{pt}], J_{\mathbb{G}}(t)\rangle = \sum_{k=0}^\infty G_{kn} t^{kn}$ *be the quantum period of* $\mathbb{G}$*. Then* $G_{kn} > 0$ *for all* $k \in \mathbb{Z}_{\geqslant 0}$ *and* $\lim_{k\to\infty} \sqrt[kn]{(kn)! G_{kn}}$ *exists.*

9.7. **Apéry constants.** In this section we prove Gamma conjecture I' (Conjecture 3.11) for $\mathbb{G} = \operatorname{Gr}(r,n)$.

Theorem 9.20. *The Grassmannian* $\operatorname{Gr}(r,n)$ *satisfies Gamma conjecture I'.*

The rest of the paper is devoted to the proof of Theorem 9.20. Since $\operatorname{Gr}(r,n) \cong \operatorname{Gr}(n-r,n)$, we may assume that $r \leqslant n/2$. We fix a sufficiently small phase $\phi > 0$ such that $\phi' = \phi - \frac{(r-1)\pi}{n}$ is admissible for $\{v_0, v_1, \dots, v_{n-1}\}$. Let Λ denote the index set of mutually distinct r-tuples K of elements of $\mathbb{Z}/n\mathbb{Z}$:

$$\Lambda = \{K = (k_1, \dots, k_r) \in (\mathbb{Z}/n\mathbb{Z})^r : n-1 \geqslant k_1 > k_2 > \cdots > k_r \geqslant 0\}.$$

For $K \in \Lambda$, we write $\mathcal{F}_K = \mathcal{F}_K(\phi)$ and $\mathcal{G}_K = \mathcal{G}_K(\phi)$ for the elements in (35).

Lemma 9.21. *Set* $T := \max\{|v_K| : K \in \Lambda\}$.

(1) *We have* $T = n\frac{\sin(\pi r/n)}{\sin(\pi/n)}$.
(2) *If* $T = |v_K|$ *for* $K \in \Lambda$, *then* K *is given by a consecutive* r*-tuple of elements in* $\mathbb{Z}/n\mathbb{Z}$ *and* $v_K = Te^{-2\pi\sqrt{-1}k/n}$ *for some* $k \in \mathbb{Z}$.
(3) *We have* $T = v_{K_0}$ *for* $K_0 := (r-1, r-2, \dots, 1, 0)$. *Moreover,* $\mathcal{F}_{K_0} = \mathcal{O}_{\mathbb{G}}$.

Proof. Parts (1) and (2) follow easily from the definition of v_K. To see Part (3), note that Proposition 9.11 (2) gives $\mathcal{F}_k(\phi') = \mathcal{O}_{\mathbb{P}^{n-1}}(k)$ for $0 \leqslant k \leqslant r-1$ and $\phi' = \phi - \frac{(r-1)\pi}{n}$ (since $|\phi|$ is small and $r \leqslant n/2$). Then Proposition 9.6 (2) implies the conclusion. $\square$

Let $\alpha \in H_{2|\alpha|}(\mathbb{G})$ be a homology class such that $c_1(\mathbb{G}) \cap \alpha = 0$. We want to show that the limit formula (see (11))

$$\lim_{n\to\infty} \frac{\langle \alpha, J_{kn}\rangle}{\langle [\mathrm{pt}], J_{kn}\rangle} = \langle \alpha, \widehat{\Gamma}_{\mathbb{G}}\rangle$$

holds where we set $J_{\mathbb{G}}(t) = e^{c_1(\mathbb{G})\log t}\sum_{k=0}^\infty J_{kn}t^{kn}$. We start with noting that it suffices to show this formula when $\langle \alpha, \widehat{\Gamma}_{\mathbb{G}}\rangle = 0$. Indeed, we obtain the general case by applying the formula to $\alpha' = \alpha - \langle \alpha, \widehat{\Gamma}_{\mathbb{G}}\rangle[\mathrm{pt}]$ (which satisfies $\langle \alpha', \widehat{\Gamma}_{\mathbb{G}}\rangle = 0$).

Let $\widehat{\alpha} \in K^0(\mathbb{G}) \otimes \mathbb{C}$ be a complexified K-class such that

$$\mathrm{PD}(\alpha) = \widehat{\Gamma}_{\mathbb{G}} \operatorname{Ch}(\widehat{\alpha}).$$

Then we have, using the definition of $Z^{\mathbb{G}}$ (see (33) and (2)),

$$\begin{aligned}
\langle \alpha, J_{\mathbb{G}}(t)\rangle &= \int_{\mathbb{G}} J_{\mathbb{G}}(t) \cup \mathrm{PD}(\alpha) \\
&= (\sqrt{-1})^{\dim \mathbb{G} - 2|\alpha|}\int_{\mathbb{G}} J_{\mathbb{G}}(t) \cup e^{-\pi\sqrt{-1}\mu}e^{\pi\sqrt{-1}c_1(\mathbb{G})}\,\mathrm{PD}(\alpha) \\
&= (-1)^{|\alpha|}(2\pi\sqrt{-1})^{\dim\mathbb{G}}\left[J_{\mathbb{G}}(t), \widehat{\Gamma}_{\mathbb{G}}\operatorname{Ch}(\widehat{\alpha})\right) = (-1)^{|\alpha|}Z^{\mathbb{G}}(\widehat{\alpha})(e^{-\pi\sqrt{-1}}t).
\end{aligned}$$

Using the dual bases $\{\mathcal{F}_K\}_{K\in\Lambda}$ and $\{\mathcal{G}_K\}_{K\in\Lambda}$ of $K^0(\mathbb{G})$, we can expand

$$\widehat{\alpha} = \sum_{K\in\Lambda}\chi(\mathcal{F}_K, \widehat{\alpha})\mathcal{G}_K.$$

Thus by Theorem 9.15, we obtain the integral representation for $\langle \alpha, J_{\mathbb{G}}(t)\rangle$:

$$\langle \alpha, J_{\mathbb{G}}(t)\rangle = (-1)^{|\alpha|}c_{n,r}\sum_{K\in\Lambda}\chi(\mathcal{F}_K,\widehat{\alpha})\cdot t^{\binom{r}{2}}\int_{\Gamma_K^\vee(\phi)}\frac{dy}{y}e^{tg(y)}\prod_{i<j}(f(\vec{y}_i) - f(\vec{y}_j)) \tag{36}$$

for $|\arg t + \phi| < \pi/2$, where we set $c_{n,r} = \frac{1}{r!}(\xi/(2\pi\sqrt{-1}n))^{\binom{r}{2}}$. Let $C_K(\lambda)$ denote the "anti-symmetrized" vanishing cycle in the fiber $g^{-1}(\lambda)$:

$$C_K(\lambda) = \Gamma_K^\vee(\phi) \cap g^{-1}(\lambda) = \sum_{\sigma\in\mathfrak{S}_r}\operatorname{sgn}(\sigma)C_{k_{\sigma(1)}}(\lambda)\times\cdots\times C_{k_{\sigma(r)}}(\lambda)$$

where $\lambda \in v_K - \mathbb{R}_{\geqslant 0}e^{\sqrt{-1}\phi}$ and $C_{k_i}(\lambda) = \Gamma_{k_i}(\phi') \cap f^{-1}(\xi^{-1}\lambda)$ is the vanishing cycle for f. We define the period integral $P_K(\lambda)$ as:

$$P_K(\lambda) := \int_{C_K(\lambda)\subset g^{-1}(\lambda)}\prod_{i<j}(f(\vec{y}_i) - f(\vec{y}_j))\left.\frac{\bigwedge_{i=1}^r\bigwedge_{j=1}^{n-1}\frac{dy_{i,j}}{y_{i,j}}}{dg}\right|_{g^{-1}(\lambda)}.$$

Then we may rewrite (36) as the Laplace transform of the period:

$$\langle \alpha, J_{\mathbb{G}}(t)\rangle = (-1)^{|\alpha|} c_{n,r} \sum_{K\in\Lambda} \chi(\mathcal{F}_K, \widehat{\alpha}) \int_{v_K - e^{\sqrt{-1}\phi}\mathbb{R}_{\geqslant 0}} d\lambda \cdot t^{\binom{r}{2}} e^{t\lambda} P_K(\lambda). \tag{37}$$

Applying the Laplace transformation

$$\varphi(t) \mapsto \int_0^\infty \varphi(t) e^{-ut} dt$$

to both sides of (37), we obtain

$$\sum_{k=0}^{\infty} (kn)!\, \langle \alpha, J_{kn}\rangle\, u^{-kn-1} = (-1)^{|\alpha|} c'_{n,r} \sum_{K\in\Lambda} \chi(\mathcal{F}_K, \widehat{\alpha}) \int_{v_K - e^{\sqrt{-1}\phi}\mathbb{R}_{\geqslant 0}} d\lambda \frac{P_K(\lambda)}{(u-\lambda)^{\binom{r}{2}+1}} \tag{38}$$

when $\mathrm{Re}(u - v_K) > 0$ for all $K \in \Lambda$, where $c'_{n,r} = \binom{r}{2}! c_{n,r}$. Note that the right-hand side can be analytically continued to a holomorphic function outside the branch cut:

$$\bigcup_{K\in\Lambda} v_K - e^{\sqrt{-1}\phi}\mathbb{R}_{\geqslant 0}.$$

See Figure 3. Moreover, this can be analytically continued to the universal cover of $\mathbb{C} \setminus \{v_K : K \in \Lambda\}$. Since the left-hand side is regular at $u = \infty$, it follows from Lemma 9.21 that the

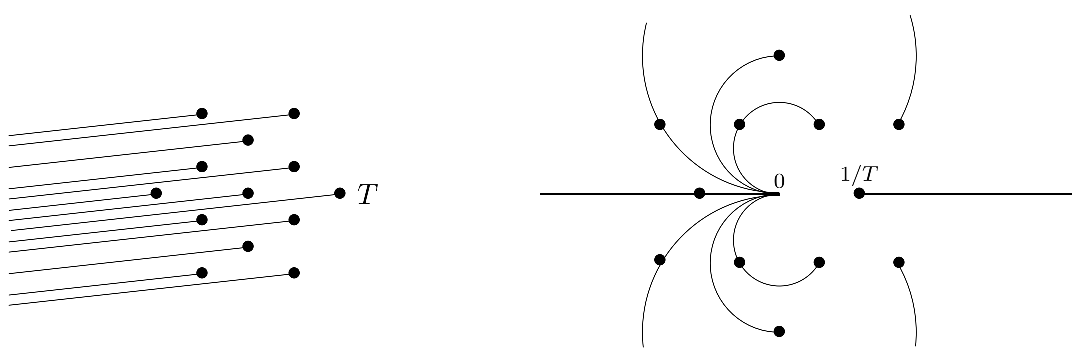

FIGURE 3. Branch cut in the u-plane (left) and the u^{-1}-plane (right) ($n = 6, r = 2$). In the right picture, we set $\phi = 0$ for simplicity.

convergence radius of the left-hand side of (38) (as a power series in u^{-1}) is bigger than or equal to

$$\frac{1}{T} = \frac{1}{n} \frac{\sin(\pi/n)}{\sin(\pi r/n)}.$$

By Cauchy's integral formula, it follows that the jump of the function (38) across the cut $v_K - e^{\sqrt{-1}\phi}\mathbb{R}_{\geqslant 0}$ is proportional to $\chi(\mathcal{F}_K, \widehat{\alpha}) \cdot (\partial_u)^{\binom{r}{2}} P_K(u)$. From this it follows that:

(a) if $\alpha = [\mathrm{pt}]$, we have $\chi(\mathcal{F}_{K_0}, \widehat{\alpha}) = (2\pi\sqrt{-1})^{-\dim \mathbb{G}} \chi(\mathcal{O}_{\mathbb{G}}, \mathcal{O}_{\mathrm{pt}}) \neq 0$ and thus the convergence radius of $\sum_{k=0}^\infty (kn)!\, \langle [\mathrm{pt}], J_{kn}\rangle\, x^{kn}$ is exactly $1/T$;

(b) if $\langle \alpha, \widehat{\Gamma}_{\mathbb{G}}\rangle = 0$, then we have $\chi(\mathcal{F}_{K_0}, \widehat{\alpha}) = \chi(\mathcal{O}_{\mathbb{G}}, \widehat{\alpha}) = [\widehat{\Gamma}_{\mathbb{G}}, \alpha) = 0$ (by (1)); this implies that (38) is holomorphic at $u = v_{K_0} = T$; since the left-hand side of (38) is a power series in u^{-n} (multiplied by u^{-1}), it is holomorphic at any other v_K with $|v_K| = T$ and the series $\sum_{k=0}^\infty (kn)!\, \langle \alpha, J_{kn}\rangle\, x^{kn}$ has a convergence radius $R(\alpha) > 1/T$.

Here $K_0 = (r-1, r-2, \ldots, 1, 0)$ and recall from Lemma 9.21 that $\mathcal{F}_{K_0} = \mathcal{O}_{\mathbb{G}}$. From part (a) and Proposition 9.19, it follows that

$$\lim_{k\to\infty} \sqrt[kn]{(kn)!\,\langle [\mathrm{pt}], J_{kn}\rangle} = T.$$

From part (b), it follows that, when $\langle \alpha, \widehat{\Gamma}_{\mathbb{G}}\rangle = 0$,

$$\lim_{k\to\infty} \frac{\langle \alpha, J_{kn}\rangle}{\langle [\mathrm{pt}], J_{kn}\rangle} = \lim_{k\to\infty} \frac{(kn)!\,\langle \alpha, J_{kn}\rangle}{(kn)!\,\langle [\mathrm{pt}], J_{kn}\rangle} = 0.$$

This completes the proof of Theorem 9.20.

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