Δ-link homotopy of links in S^3 and invariants of link maps in S^4

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Two classical links (or string links) are *self C_k-equivalent* if they are connected by a sequence of C_k-moves of Gusarov and Habiro such that each of these C_k-moves involves strands only from one component. Type k invariants in the sense of Kirk–Livingston are invariant under self C_{k+1}-equivalence. These include type k invariants in the sense of Vassiliev; but Kirk and Livingston conjectured that the group of type 2 invariants in their sense has infinite rank for 2-component links with any linking number.

Self C_1-equivalence is better known as link homotopy. Self C_2-equivalence is also known as Δ*-link homotopy* (since C_2-moves are also known as Δ-moves). Yasuhara (2007) proved that a link is Δ-link homotopic to the unlink if and only if all its $\bar{\mu}$-invariants with at most two occurrences of each index vanish. However, $\bar{\mu}$-invariants generally do not suffice even to detect triviality: by a result of Fleming and Yasuhara (2005), the Whitehead double of the Whitehead link (which is a boundary link, so all its $\bar{\mu}$-invariants vanish) is not self C_3-equivalent to the unlink.

In 2003, Nakanishi and Ohyama obtained a classification of 2-component links up to Δ-link homotopy. Namely, they are classified by the linking number and the generalized Sato-Levine invariant — which are the first two coefficients of the power series $\nabla_L/(\nabla_{K_1}\nabla_{K_2})$, where ∇_L and ∇_{K_i} are the Conway polynomials of the link and of its components. These two invariants also generate all Kirk–Livingston invariants of type ≤ 1 for two-component links. Using Kirk's invariant of link maps $S^2 \sqcup S^2 \to S^4$ and its variation due to Koschorke, we obtain a simple proof of the Nakanishi–Ohyama theorem, and also its version for string links.

In the 3-component case, the strongest known result appears to be Yasuhara's Δ-link homotopy classification of those 3-component links whose $\bar{\mu}$-invariants of lengths 2, 3, 4 and 5 vanish, by $\bar{\mu}$-invariants with precisely two occurrences of each index (so, of length 6). To simplify matters, we consider *weak* Δ*-link homotopy*, which is generated by Δ-link homotopy and those C_3-moves for which two adjacent vertices of the clasper tree correspond to one of the components. This is strictly stronger than Yasuhara's $(C_2^s + C_3^a)$-equivalence, which is defined similarly, but omitting "adjacent".

Invariants of weak Δ-link homotopy of 3-component links include $\bar{\mu}$-invariants of length ≤ 4 and coefficients at the linear terms (i.e., xy, yz, xz and xyz) of the power series obtained by expanding the two- and three-variable Alexander polynomials (i.e., Conway's potential functions) in Conway's variables $z_i = x_i - x_i^{-1}$. We prove that 3-component links that are trivial up to link homotopy are classified up to weak Δ-link homotopy by $\bar{\mu}$-invariants of length ≤ 4. The proof uses a computation of the image of Koschorke's $\tilde{\beta}$-invariant of link maps $S^2 \sqcup S^2 \sqcup S^2 \to S^4$ (which is strictly stronger than Gui-Song Li's version of Kirk's invariant). This computation in is turn based on Yasuhara's results about Δ-link homotopy.

This talk is based on a joint work with Yuka Kotorii.