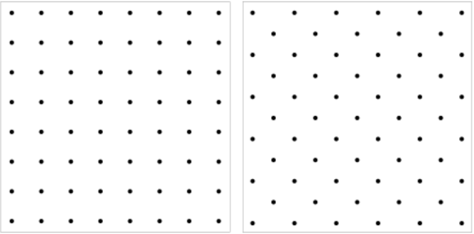

An improvement of Liouville theorem for discrete harmonic functions on $\mathbb{Z}^2$

Eugenia Malinnikova, NTNU
joint work with L. Buhovsky, A. Logunov, M. Sodin (Tel Aviv University)

Moscow, October 9th, 2017

A historical remark

Discrete Laplace operator on $(h\mathbb{Z})^n$ and discrete harmonic (preharmonic) functions

$$\Delta_h u(x) = h^{-2}\left(\sum_{j=1}^{n}(u(x+he_j) + u(x-he_j) - 2nu(x)\right).$$

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A statement about discrete harmonic functions (on bounded domains) implies the corresponding statement about continuous ones. But not vice versa.

Some usual tools still work

— Maximum Principle (simple)

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— Dirichlet Problem (linear algebra, or energy minimizer)

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Rotated coordinate system in $\mathbf{Z}^2$

$$u(x,y) = \frac{1}{4}\big(u(x+1,y+1) + u(x+1,y-1) + u(x-1,y+1) + u(x-1,y-1)\big)$$

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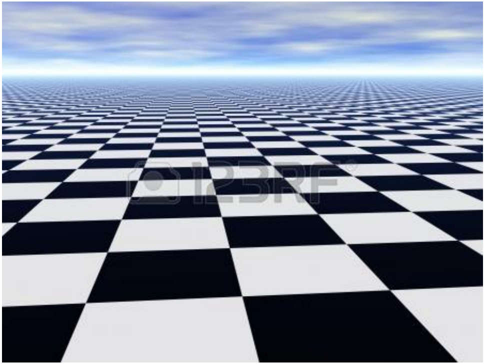

From now on we leave on white cells of an infinite chessboard.
Convention: cells of the chessboard are indexed by $\mathbb{Z}^2$ and $(0,0)$ is white.

Liouville theorem

Any bounded harmonic function is a constant.

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— Nice books on problem solving.

Main result

Let $\varepsilon \in (0,1)$ be a positive number.
We say that $|u|$ is bounded by 1 on $(1-\varepsilon)$ portion of the chessboard if

$$|\mathcal{Q}_N \cap \{|u| \leq 1\}| \geq (1-\varepsilon)|\mathcal{Q}_N|$$

for all $N \geq N_0$, where $\mathcal{Q}_N = \{-N \leq x, y \leq N\}$.

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Theorem

There exists $\varepsilon > 0$ such that if u is a discrete harmonic function and $|u|$ is bounded by 1 on $(1-\varepsilon)$ portion of $\mathbb{Z}^2$, then u is identically constant.

L. Buhovsky, A. Logunov, E.M., M.Sodin, 2017

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— On a diagonal half-plane any Cauchy boundary value problem has a unique solution which extends uniquely to a harmonic function on the whole plane.

— There is a non-constant harmonic function which is bounded on 3/4 of the chessboard ($\varepsilon < 0.25$).

Two competing results

Suppose that u is discrete harmonic and

$$(*) \qquad |\mathcal{Q}_K \cap \{|u| \le 1\}| \ge (1-\varepsilon)|\mathcal{Q}_K|$$

Theorem (A)

For all sufficiently small $\varepsilon > 0$*, there exist* $a = a(\varepsilon) > 0$*,* $a(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, and* $N_0 = N_0(\varepsilon)$ *such that if* $(*)$ *holds for* $K = N > N_0(\varepsilon)$ *then*

$$\max_{\mathcal{Q}_{N/2}} |u| \le e^{aN}.$$

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Theorem (**B**)

If $\varepsilon > 0$ *is sufficiently small,* N *is sufficiently large,* $\max_{\mathcal{Q}_{[\sqrt{N}]}} |u| \ge 2$*, and* $(*)$ *holds for every* $K \in [\sqrt{N}, N]$*, then*

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Quantitative real analyticity

Lemma

There exist positive constants $\gamma, q < 1$ *and* C *such that for any discrete harmonic function* u *on* $\mathcal{Q}_N$ *and any positive integer* d *there is a polynomial* P_d *of degree* $\leq d$ *such that*

$$|u(x,y) - P_d(x,y)| \leq C \max_{\mathcal{Q}_N} |u| q^d, \quad \textit{when } -\gamma N \leq x, y \leq \gamma N.$$

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The statement is trivial when d is small and when d is large. The point is that the constants do not depend on N.
The reason is that $u(x,y)$ admits a bounded holomorphic continuation to some domain in $\mathbb{C}^2$.

A useful tool

Theorem (Remez' inequality, 1936)

$P \in \mathbf{R}[x]$ is a polynomial of degree d, and $E \subset I = [a, b]$,

$$\sup_I |P| \leq \left(\frac{4|I|}{|E|}\right)^d \sup_E |P|$$

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Corollary (Discrete version)

Suppose that a polynomial P of degree less than or equal to d is bounded by M at $2d$ integer points on an interval I, then

$$\max_I |P_d| \leq \left(\frac{4|I|}{d}\right)^d M.$$

Three square inequality

Lemma

Let u be a discrete harmonic in $\mathcal{Q}_N$,
$|u| \leq M$ on $\mathcal{Q}_N$ and $|u| \leq \varepsilon$ on a fixed portion of $\mathcal{Q}_{N/4}$ then

$$\max_{\mathcal{Q}_{N/2}} |u| \leq C(M^{\alpha}\varepsilon^{1-\alpha} + \delta^N M).$$

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- When ε is not too small we obtain the same estimate as in the continuous case, but there is no information from polynomials of degree $> cN$ and if ε is small we get the second term.

Theorem B

Theorem (**B**)

If $\varepsilon > 0$ *is sufficiently small, N is sufficiently large,* $\max_{\mathcal{Q}_{[\sqrt{N}]}} |u| \geq 2$,

$$\textit{and} \quad (*) \quad |\mathcal{Q}_K \cap \{|u| \leq 1\}| \geq (1-\varepsilon)|\mathcal{Q}_K|$$

holds for every $K \in [\sqrt{N}, N]$, *then*

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Aim: $M(N) \geq e^{bN}$.

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holds for $K = N > N_0(\varepsilon)$ *then*

$$\max_{\mathcal{Q}_{N/2}} |u| \leq e^{aN}.$$

Toy question: assume that u is zero on a large portion of $\mathcal{Q}_N$, does it follow that u is zero on $\mathcal{Q}_{N/2}$?

An observation

Suppose that u is harmonic in a rectangle

$$\mathcal{R} = \{a_1 \leq x \leq a_2,\ b_1 \leq y \leq b_2\}$$

$$\text{and} \quad u(a_1, y) = u(a_1 + 1, y) = 0$$

then $(-1)^y u(a_1 + k, y)$ is a polynomial of degree $< k$.

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Proof by induction.

$$u(x+1, y+1) + u(x+1, y-1) = 4u(x, y) - u(x-1, y+1) - u(x-1, y-1)$$

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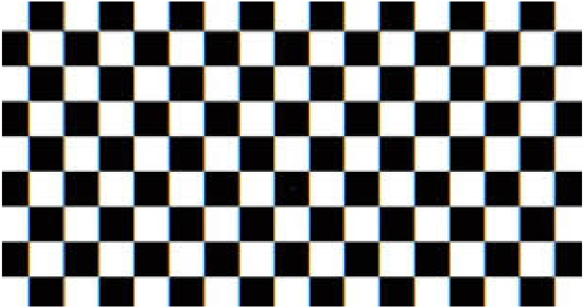

Zeros

The observation from the last slide implies:
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More precisely, if $u(x, b_1) = u(x, b_1 + 1) = 0$ when $a_1 \leq x \leq a_2$ and some $u(x, b_1 + 2) \neq 0$. Then $u(x, b_1 + k)$ has at most $k - 2$ zeros. Thus

$$|\{u = 0\} \cap \mathcal{R}| \leq 0.5|\mathcal{R}|$$

when $b = b_2 - b_1 < a_2 - a_1 = a$.

Maximal null squares

Definition

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Simple covering lemma

Assume that $\mathcal{Q}_{N/2}$ is not a null square and cover all zero cells in $\mathcal{Q}_{N/2}$ by maximal null squares $\mathcal{K}_j$. All of them are inside $\mathcal{Q}_N$ otherwise we find many non-zero cells.

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Consider squares $3\mathcal{K}_j$, they cover some area $|S| > 1/2|\mathcal{Q}_{N/2}|$. Then take a disjoint subfamily of squares $3\mathcal{K}_j$ (simple version of the Vitali covering lemma). This disjoint squares cover some set S_1 whose area is at least $1/9|S|$. We get

$$|\{|u| \neq 0\} \cap \mathcal{Q}_N| \geq 0.0005|\mathcal{Q}_{N/2}|$$

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For ε small enough we get a contradiction.

Tools: an observation

Let $\mathcal{R} = \{a_1 \le x \le a_2,\ b_1 \le y \le b_2\}$, $a = a_2 - a_1,\ b = b_2 - b_1$.

Lemma

If u is defined on

$$S = (\{(x,y) : x \in (a_1, a_1 + 1)\} \cup \{(x,y) : y \in (b_1, b_1 + 1)\}) \cap \mathcal{R}$$

then u can be uniquely extended to a discrete harmonic function on $\mathcal{R}$ and

$$\max_{\mathcal{R}} |u| \le 3^{a+b} \max_{S} |u|.$$

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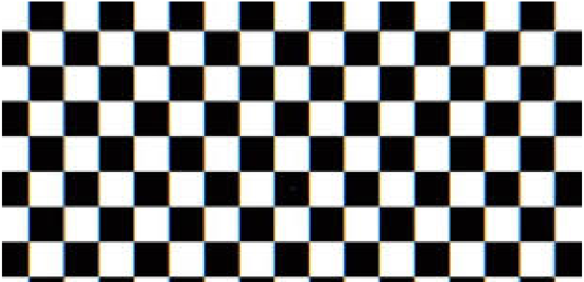


Tools: three line lemma

Lemma

Suppose that u is discrete harmonic in $\mathcal{R}$, $a > 5b$ and

$$|u(x, b_1)|, |u(x, b_1 + 1)|, |u(x, b_2)| \leq M.$$

Then

$$\max_{\mathcal{R}} |u| \leq 10^{a+b} M.$$

Tools: three line lemma

Lemma

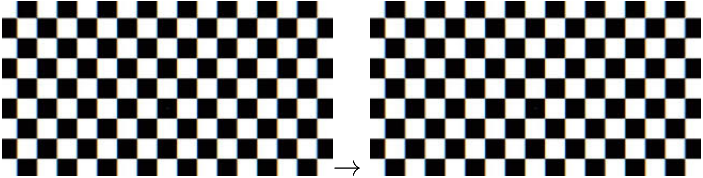
Suppose that u is discrete harmonic in $\mathcal{R}$, $a > 5b$ and

$$|u(x, b_1)|, |u(x, b_1 + 1)|, |u(x, b_2)| \leq M.$$

Then

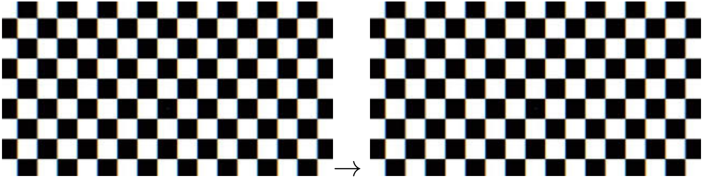
$$\max_{\mathcal{R}} |u| \leq 10^{a+b} M.$$

Step 1: Induction on $b = b_2 - b_1$, $|u(x, y)| \leq 3^{a+b} M$ if $a_1 - 2b < |x| < a_2 - 2b$



$$u(x, y) \to v(x, y) = u(x, y) + u(x + 2, y)$$

Three line lemma: proof



$$u(x, y) \to v(x, y) = u(x, y) + u(x + 2, y)$$

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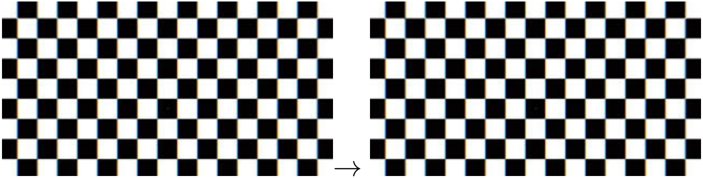

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By induction hypothesis $|v(x, y)| \leq 3^{a+b-2} 2M$, $a_1 - 2b \leq x \leq a_2 - 2b,\ b_1 \leq y \leq b_2$.

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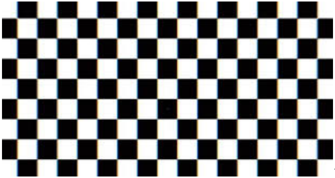
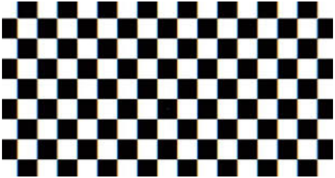
 $\to$ 

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$$u(x,y) = \frac{1}{4}(v(x-1,y-1)+v(x-1,y+1)) \le 3^{a+b-2}M, \ , \ b_1 < y < b_2$$

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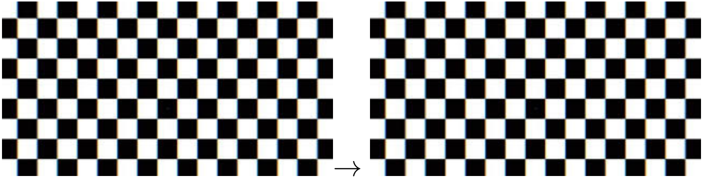

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Step II: from an earlier observation $|u(x, y)| \leq 3^{a+b}3^{a+b}M$.

Tools: Remez inequality

Consider rectangles with $a > 5b$.

Lemma

Suppose that u is discrete harmonic in $\mathcal{R}$.
If $u(x, b_1) = u(x, b_1 + 1) = 0$, $|u(x, b_2)| \leq M$ for at least half of x, $a_1 \leq x \leq a_2$. Then

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It follows immediately from the discrete Remez inequality.

Final lemma

Lemma

Suppose that u is discrete harmonic in $\mathcal{R}$.
If $|u(x, b_1)|, |u(x, b_1 + 1)| \le M$ *and* $|u(x, b_2)| \le M$ *for at least half of* $x,\ a_1 \le x \le a_2$. *Then*

$$|u(x, b_2)| \le C_2^{a+b} M.$$

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Suppose that u is discrete harmonic in $\mathcal{R}$. If $|u(x, b_1)|, |u(x, b_1 + 1)| \leq M$ and $|u(x, b_2)| \leq M$ for at least half of $x,\ a_1 \leq x \leq a_2$. Then

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Proof.

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where $u_1(x, b_1) = u(x, b_1 + 1) = 0$ and $u_2(a_1, y) = u(a_1 + 1, y) = 0$ when $y > b_1 + 1$.

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Then $|u_2| \leq M3^{a+b}$ and $|u_1(x,b_2)| \leq M3^{a+b}$ for a half of the points. Then by the previous Lemma $|u_1(x,b_2)| \leq (3C_1)^{a+b}$ and by the three line lemma $|u_2(x,y)| \leq (3C_1C)^{a+b}$. □

Maximal good squares

Definition

We say that a square $\mathcal{K}$ with side length l is good if $|u| \leq B^l$ on $\mathcal{K}$ (B depends on constants above).

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$$\max_{\mathcal{Q}_{N/2}} |u| \leq e^{aN}$$

The last step: divide and conquer

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$$|\{|u| < 1\} \cap \mathcal{Q}_N| \leq (1-\varepsilon_1)|\mathcal{Q}_N| \;\Rightarrow\; |\{|u| < 1\} \cap 2q| \leq (1-\varepsilon)|q|$$

Then $a(\varepsilon_1) \leq a(\varepsilon)/L$.

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— The result fails in $\mathbb{Z}^3$, there is a harmonic function vanishing everywhere except for a plane $x = y$.