Multivariate Distributions: Moment Problems

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PLAN:

Discussion on recent works on probability distributions in dimension $n \geq 2$ and their characterization as being unique (**M-determinate**) or non-unique (**M-indeterminate**) in terms of the moments.

We use standard notations and terminology, as in the 1-dimensional case.

Picture Today: .

Analytic (real and complex analysis): Petersen (1982), Berg-Thill (1991), Schmüdgen-Putinar (2008)

Not too much done/known for multivariate distributions ..:
Paper: C. Kleiber & JS (2013), J. Multivariate Analysis + a few references therein

Multivariate Distributions: Moment Problem

Random vector $X = (X_1, \dots, X_n) \in \mathbb{R}^n$, $n \geq 2$, with arbitrary d.f.
$F(x) = \mathbf{P}[X \leq x]$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Later we specify the dependence structure of X.
We assume the absolute integrability (k_j are in $\mathbb{N}_0$):

$$\mathbf{E}[|X_1^{k_1} \cdots X_n^{k_n}|] = \int_{\mathbb{R}^n} |x_1^{k_1} \cdots x_n^{k_n}| \mathrm{d}F(x) < \infty \text{ for all } k_j \geq 0,\ j = 1, \dots, n.$$

Introduce the **multi-indexed moments** (or, mixed moments)

$$m_{k_1,\dots,k_n} = \mathbf{E}[X_1^{k_1} \cdots X_n^{k_n}],\ k_j \geq 0,\ j = 1, \dots, n.$$

They are finite, the collection $\{m_{k_1,\dots,k_n}\}$ is the moment sequence of X.

If F is the only n-dim. d.f. with the moments $\{m_{k_1,\dots,k_n}\}$, F is M-det. Equivalently, we say also that the random vector X is M-det (because of one of the Kolmogorov theorems).

If the moments $\{m_{k_1,\dots,k_n}\}$ do not determine uniquely F, F is M-indet. In such a case we have to write at least one n-dim. d.f., G, such that

$F \neq G$, but $m_{k_1,\dots,k_n}(F) = m_{k_1,\dots,k_n}(G)$ for all $k_j \geq 0$, $j = 1, \dots, n$.

If all components of $X = (X_1, \dots, X_n)$ are positive, $X \in (\mathbb{R}_+)^n$, we call this **Stieltjes moment problem**.

Otherwise, it is **Hamburger moment problem**.

Question: When is F M-det, and when M-indet?

No reasonable answer. There are some uncheckable conditions.

This is why the main attention is on conditions, which are only sufficient or only necessary for either F to be M-det, or M-indet.

Cramér-Wold: For a random vector $X \sim F$ on $\mathbb{R}^n$, let the m.g.f. exist:

$$M(t) = \mathbf{E}[e^{\langle t,X\rangle}] < \infty \text{ for } t \in (-t_0, t_0) \subset \mathbb{R}^n,\ t_0 > 0 \text{ (\textbf{light tails})}.$$

Then, two statements hold:

- All multi-indexed moments $m_{k_1,\dots,k_n}$ of X are finite. (diff., $t = 0$)
- The d.f. F and the random vector X are M-det.

If no m.g.f., F has **heavy tails**, and F is either M-det, or M-indet.

Consider the components $X_1, \ldots, X_n$ of the vector $X \in \mathbb{R}^n$ and let $F_1, \ldots, F_n$ be the corresponding marginal 1-dim. d.f.s. We have

$X_1 \sim F_1, \ \ldots, \ X_n \sim F_n.$

The existence of the moments of X implies that each X_j has finite moments, say $m_{j,k_j} = \mathbf{E}[X_j^{k_j}], \ j = 1, \ldots, n, \ k_j \geq 0.$

Corollary: If $X \in \mathbb{R}^n$ has a m.g.f., then each of the r.v.s $X_1, \ldots, X_n$ also has a m.g.f. Hence each of $X_1, \ldots, X_n$ is M-det.
E.g., X_1 is the only r.v. with moments $\{m_{1,k_1}, \ k_1 \geq 0\}$.

The converse is also true: If each X_j has a m.g.f., $j = 1, \ldots, n$, then the random vector X obeys a m.g.f., and hence X is M-det, in $\mathbb{R}^n$.

Notice, both these are true for any dependence structure of X.

Carleman's condition: In dim. 1, most useful. Analog in dim. $n \geq 2$.

Start with 1-dim. r.v. $\xi \sim G$, $m_k = \mathbf{E}[\xi^k]$, $k = 1, 2, \ldots$. Depending on the support, $\mathbb{R}^1$ or $\mathbb{R}_+$, we define the following series:

$$\mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(m_{2k})^{1/2k}}, \qquad \mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(m_k)^{1/2k}}.$$

Theorem: $\mathrm{C} = \infty \Rightarrow G$ is M-det. (Only sufficient.)

Proofs by Carleman (1926), Koosis (1979) use quasi-analytic functions.

Remark: There is a result using the converse to Carleman's condition, i.e., assuming that $\mathrm{C} < \infty$. Under an additional condition, G will be M-indet. (Due to P. Koosis, A. Pakes, G.D. Lin.) This is only in dim. 1.

Question: Can we find an analog of Carleman's condition in dim. $n \geq 2$? Answer is "yes", however first we comment on this condition in dim. 1.

How does Carleman's condition imply M-uniqueness?

Idea: Method of metric distances (V. Zolotarev, S. Rachev).

$\mathcal{D} =$ all d.f.s (on the real line), metric, $d(F,G)$ between $F, G \in \mathcal{D}$. Then:

(a) $d(F,G)$ is symmetric; (b) $d(F,G)$ satisfies the triangle inequality;

(c) $d(F,G) = 0 \iff F = G$.

Work by Lev Klebanov et al. ~ 1982. Assume $d(F,G)$ is the Lévy metric, or the Kolmogorov (uniform) metric.

Result: Let F and G have finite all moments and the first $2n$ coincide: $m_k(F) = m_k(G) = m_k$, $k = 1, 2, \ldots, 2n$. Denote $C_n = \sum_{k=1}^{n} (m_{2k})^{-1/2k}$.

$$d(F,G) \leq K_2 \frac{\log(1 + C_{n-1})}{(C_{n-1})^{1/4}} \quad (\text{here} \quad K_2 = K_2(m_1, m_2)).$$

Corollary: If $C = \sum_{k=1}^{\infty} (m_{2k})^{-1/2k} = \infty$ (Carleman's condition), then $C_n \to \infty$, as $n \to \infty$, and, see (c), $d(F,G) \to 0 \Rightarrow F = G$.

Result coming soon: (K. Lykov, TPA, no. 4 (2017)):

Theorem: Suppose the r.v. ξ has all moments finite. Then:

- ξ is a sum of two r.v.s, ξ_1 and ξ_2, whose supports are disjoint;
- the moments of each of ξ_1 and ξ_2 satisfy Carleman's condition (!), hence, each of ξ_1 and ξ_2 is M-det.

Notice, if ξ itself is M-det, there is nothing to prove.

Interesting is that ξ with finite moments being M-indet (we do not specify how), has the above decomposition as a sum of two r.v.s each being M-det (here we do specify, M-det, by Carleman's condition).

Warning: Writing $\xi = \xi_1 + \xi_2$, the Carleman's condition holds for ξ_1 and ξ_2, but not for ξ. Or: a linear combination of two Carleman sequences is not always a Carleman sequence. Similarly, for a product.

Carleman's Condition in Dimension n:

We need the numbers M_{2k} and M_k, for F on $\mathbb{R}^n$ and $\mathbb{R}^n_+$:

$$M_{2k} = m_{2k,0,\ldots,0} + m_{0,2k,0\ldots,0} + \ldots + m_{0,0,\ldots,0,2k} \quad \text{(Hamburger)},$$

$$M_k = m_{k,0,\ldots,0} + m_{0,k,0\ldots,0} + \ldots + m_{0,0,\ldots,0,k} \quad \text{(Stieltjes)}.$$

Now the n-Carleman quantity is defined, respectively, as follows:

$$\mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(M_{2k})^{1/2k}} \quad \text{and} \quad \mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(M_k)^{1/2k}}.$$

Theorem: $\mathrm{C} = \infty \Rightarrow$ the n-dimensional d.f. F is M-det.

If n-Carleman holds for X, then 1-Carleman holds for each X_j. The converse is not in general true, as mentioned above.

General Result (Petersen 1982):

Given is $X \sim F$ in $\mathbb{R}^n$ with marginals $X_1 \sim F_1, \ldots, X_n \sim F_n$.

(a) If each of $F_1, \ldots, F_n$ is M-det, then the n-dim. d.f. F is M-det.

(b) If the d.f. F in $\mathbb{R}^n$ is M-det, and the r.v.s $X_1, \ldots, X_n$ are independent, then each of $F_1, \ldots, F_n$ is M-det.

Comments:

- In (a) we do not specify in which way F_j are M-det.
- For (b), we use essentially that $F(x) = F_1(x_1) \cdots F_n(x_n)$.
- Compare claims (a) and (b) with the result involving the m.g.f. of X.
- Compare (a) and (b) with, e.g., normality property of X and of $X_1, \ldots, X_n$.

Warning: There are M-det n-dim. d.f.s with M-indet marginals.

This looks strange and counter-intuitive, but it is true. Quite analytic. We do not give details here. [Two illustrations: ODEs, real life case.]

Multivariate M-indet distribution:

Start with $X \sim F$ in $\mathbb{R}^n$ with all multi-indexed moments finite.

Take one component, e.g., $X_1 \sim F_1$; the rest $(X_2, \ldots, X_n)$ is of dim. $n-1$.

Statement: Let the r.v. X_1 be independent of $(X_2, \ldots, X_n)$. Then, if X_1 is M-indet, the random vector $X = (X_1, X_2, \ldots, X_n)$ is M-indet.
This is so for any dependence structure of $(X_2, \ldots, X_n)$ in both cases, when $(X_2, \ldots, X_n)$ is M-det and M-indet.

Hint: In general, since F_1 is M-indet, there are infinitely many other distributions, continuous and discrete, all with the same moments as F_1. Eventually, we can construct Stieltjes class with center F_1.

Use of Krein's Condition:

Given a r.v. $Y \sim G$ with positive density g. Depending on the support, $\mathbb{R}^1$ or $\mathbb{R}_+$, define Krein quantity:

$$\mathrm{K}[g] = \int_{-\infty}^{\infty} \frac{-\ln g(y)}{1+y^2} \mathrm{d}y, \qquad \mathrm{K}[g] = \int_{a}^{\infty} \frac{-\ln g(y^2)}{1+y^2} \mathrm{d}y, \ a \ge 0.$$

Krein Theorem: $\mathrm{K}[g] < \infty \ \Rightarrow \ G$ is M-indet.

Statement: For $X = (X_1, \dots, X_n)$, if there is an index j such that $\mathrm{K}[f_j] < \infty$ and X_j is independent of the rest, then X is M-indet.

We need more, in dim. 1. If G, 1-dim. d.f., has density g which is positive and smooth, define Lin's condition:

$$\frac{-y\, g'(y)}{g(y)} \nearrow +\infty, \ 0 \le y_0 \le y \to \infty.$$

Applications of the Petersen's Result:

Statement: The random vector $X \sim F$ in $\mathbb{R}^n$ has marginals $X_1 \sim F_1, \ldots, X_n \sim F_n$ which are absolutely continuous and moreover, their densities $f_1, \ldots, f_n$ are strictly positive and smooth. Assume that for $j = 1, \ldots, n$, the converse to Krein's condition holds, i.e., $\mathrm{K}[f_j] = \infty$ and that Lin's condition holds, i.e. $-x_j f_j'(x_j)/f_j(x_j) \nearrow +\infty, \ x_j \to \infty$. Then for any dependence structure of X, its n-dim. d.f. F is M-det.

Hint: According to Lin's theorem (1997), each of $X_1, \ldots, X_n$ is M-det, so the claim follows from the above Petersen's general result.

Remark: Lin's theorem can be extended to absolutely continuous distributions without requiring smoothness of the densities. Joint work with P. Kopanov (Plovdiv University, BG) is in good progress.

Use Hardy's Condition: SL, TPA (2012).

Random vector $X \sim F$, arbitrary d.f. F in $\mathbb{R}^n$, finite mixed moments $m_{k_1,\dots,k_n} = \mathbf{E}[X_1^{k_1} \cdots X_n^{k_n}]$. Define: $\|X\| = \sqrt{\|X\|^2} = \sqrt{X_1^2 + \dots + X_n^2}$.

Statement: Let the 1-dim. r.v. $\|X\|$ satisfy Cramér's condition:

$$\mathbf{E}[e^{c\|X\|}] < \infty, \quad c > 0.$$

Then the n-dim. Hamburger moment problem for F has a unique solution: F is the only n-dim. d.f. with the set of moments $\{m_{k_1,\dots,k_n}\}$.

Proof: We follow two steps.

Step 1: Cramér's condition for $\|X\|$ $\Rightarrow$ Hardy's condition (Stieltjes case) holds for $\|X\|^2$: $\mathbf{E}[e^{\tilde{c}\sqrt{\|X\|^2}}] < \infty, \ \tilde{c} > 0$. Hence $\|X\|^2$ is M-det. .

Step 2: Amazing result by Putinar-Schmüdgen (2008):

If $X \sim F$ in $\mathbb{R}^n$ is such that $\|X\|^2$ is M-det (1-dim. Stieltjes case), then F is M-det (n-dim. Hamburger case).

Rate of growth of the moments:

First, dim. 1: r.v. $Y \sim G$ with unbounded support and finite $m_k = \mathbf{E}[Y^k], k = 1, 2, \ldots$ Then, as $k \to \infty$, $m_k \nearrow \infty$ for $Y > 0$, while $m_{2k} \nearrow \infty$ for $Y \in \mathbb{R}$. Define

$$\Delta_k = \frac{m_{k+1}}{m_k} \text{ (Stieltjes case)}, \ \Delta_k = \frac{m_{2k+2}}{m_{2k}} \text{ (Hamburger case)}.$$

$\{\Delta_k\}$ is strictly $\nearrow$, unique $\lim_{k\to\infty} \Delta_k = \infty$. Suppose for some $\delta \geq 0$

$$\Delta_k \approx \tilde{c}\, k^{\delta} \ \text{ for large } \ k.$$

$0 \leq \delta \leq \infty$ is called the **rate of growth of the moments** of Y.

Statement 1: If $\delta \leq 2$, then Y is M-det; $\delta = 2$ is the best possible rate for which Y is M-det.

Statement 2: If $\delta > 2$ **and** Lin's condition holds, then Y is M-indet.

Use the rate δ in dimension n

Given $X = (X_1, \ldots, X_n)$ with all multi-indexed moments finite. Consider the marginals $X_1 \sim F_1, \ldots, X_n \sim F_n$ and their rates $\delta_1, \ldots, \delta_n$.

Statement: Suppose that

$$\tilde{\delta} = \max\{\delta_1, \ldots, \delta_n\} \leq 2.$$

Then the random vector X is M-det.

Idea: The condition allows to apply Carleman's condition in dim. n.

Question: What to do if $\delta_j > 2$, or $\min\{\delta_1, \ldots, \delta_n\} > 2$?
It is not known. The reason is that there is no analog of Lin's condition in dim. $n \geq 2$. Attempts continue!

Brief Comments on Involving the Cumulants (Semi-Invariants):

Take (1-dim.) r.v. $\xi \sim G$, finite all moments $m_k = \mathbf{E}[\xi^k]$, $k = 1, 2, \ldots$.

Use the ch.f. $\psi(t) = \mathbf{E}[e^{it\xi}]$, real t and $\ln \psi(t)$. Find $\frac{d^k}{t^k} \ln \psi(t)$ at $t = 0$ to define numbers $s_k, k = 1, 2, \ldots$, called **cumulants** or **semi-invariants**.

Known: $\{m_k, \ k = 1, 2, \ldots\}$ and $\{s_k, \ k = 1, 2, \ldots\}$ are one-to-one.

Question: Can we characterize G uniquely by $\{s_k\}$? **Answer:** No!

Example: $\xi \sim Log\,\mathcal{N}(0, 1)$, $g(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left[-\frac{1}{2}(\ln x)^2\right], x > 0$; $f(x) = 0, x \leq 0$. For ξ: no m.g.f., finite moments, $m_k = e^{k^2/2}$, $k = 1, 2, \ldots$

Write the sets of r.v.s called **Stieltjes classes**

ξ_ε, $\varepsilon \in [-1, 1]$: density $g_\varepsilon(x) = g(x)\,[1 + \varepsilon \sin(2\pi \ln x)]$, $x > 0$;

All r.v.s, ξ_ε are as ξ, have the same moments $\{e^{k^2/2}\}$, hence the same semi-invariants $\{s_k\}$. Notice, $Log\,\mathcal{N}$ is M-indet.

Conjecture: If G is M-det, then G is also S-det, and vise-versa.

Open Questions:

How to write Krein's condition in dim. $n \geq 2$?

How to write Lin's condition in dim. $n \geq 2$?

Topic: Characterization of probability distributions in terms of the semi-invariants.

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