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Instability, asymptotic trajectories and dimension of the phase space

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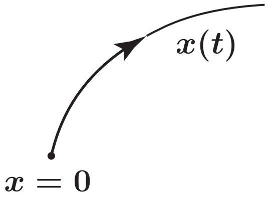
C^∞ smooth autonomous ODE system

$$\dot{x} = v(x), \qquad x \in \mathbb{R}^n,$$

with *isolated* singular point $x = 0$. Suppose that the equilibrium $x = 0$ is Lyapunov *unstable*.

It is true that there are always exist a solution $t \mapsto x(t)$, $x(t) \neq 0$, *asymptotic* to the equilibrium: $x(t) \to 0$ as $t \to -\infty$?



If $n \leqslant 2$ (and the isolated equilibrium is unstable) then there always exists an *outgoing* asymptotic trajectory.

($n = 2$: the Poincaré–Bendixson theory)

If n is *odd* and $n \geqslant 3$ then there exist divergence free systems with polynomial components of v such that

- **the equilibrium $x = 0$ is isolated and unstable,**
- **there are no *outgoing* asymptotic trajectories,**
- **there are *incoming* asymptotic trajectories.**

If n is *even* and $n \geqslant 4$ then there exist divergence free systems with polynomial components of v such that

- **the equilibrium $x = 0$ is isolated and unstable,**
- **there are no *outgoing* and *incoming* asymptotic trajectories.**

Conjecture. *If n is odd, the system is real-analytic and the singular point is isolated, then there always exist non-twisting asymtotic (incoming or outgoing) trajectories.*

$$\textit{non-twisting}: \qquad \lim_{t\to\sigma\infty} \frac{x(t)}{|x(t)|} = e, \qquad \sigma \text{ is "+" or "−"}.$$

Corollary. *If n is odd and the system admits an invariant measure with a positive continuous density then all isolated equilibriums are unstable (this conjecture was proposed by V. Ten in 1998).*

In smooth category Ten's conjecture (and our conjecture) is false (see V. Kozlov, D. Treschev. Math. Notes, 65:5 (1999), 565–570).

Examples

1. $n = 3$

$$\dot{x} = y + xz^2,$$
$$\dot{y} = -x + yz^2,$$
$$\dot{z} = -\tfrac{2}{3}z^3.$$

Reduction by the group of rotation:

$$\dot{u} = 2uz^2,$$
$$\dot{z} = -\tfrac{2}{3}z^3;$$
$$u = x^2 + y^2,$$

$\operatorname{div} v = 0$ **and**

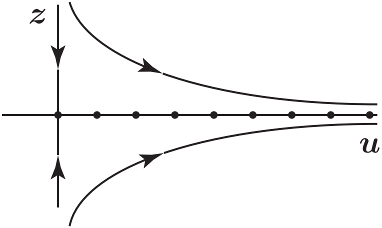
uz^3 **is the first integral.**

There are two ***incoming*** **trajectories**

$$\gamma_\pm = \{x = y = 0,\ \pm z > 0\}.$$

$$\operatorname{div} v = 0,$$

$F = (x^2 + y^2)z^3$ **is the first integral.**



Phase portrait of the reduced system

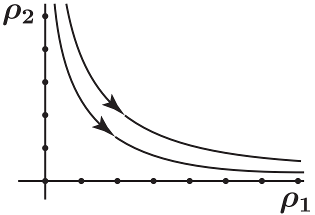
$n = 3 + 2k$: $\dot{p}_j = -q_j$, $\dot{q}_j = p_j$; $j = 1, \ldots, k$.

2. $n = 4$

$$\dot{x}_1 = y_1 + x_1\rho_1\rho_2, \quad \dot{y}_1 = -x_1 + y_1\rho_1\rho_2,$$
$$\dot{x}_2 = y_2 - x_2\rho_1\rho_2, \quad \dot{y}_2 = -x_2 - y_2\rho_1\rho_2,$$
$$\rho_k = x_k^2 + y_k^2, \quad k = 1, 2,$$

$\operatorname{div} v = 0$ and $F = \rho_1\rho_2$ is the first integral.

$$\dot{\rho}_1 = 2\rho_1^2\rho_2,$$
$$\rho_2 = -2\rho_1\rho_2^2.$$



Phase portrait of the reduced system

$$\Phi = \frac{x_1y_2 - x_2y_1}{x_1x_2 + y_1y_2}$$ is the rational integral.

The case of an odd dimension

$$v(x) = Ax + O(|x|^2), \qquad \det A \neq 0$$

Theorem 1. *Suppose* $\det A \neq 0$, *and* n *is odd. Then the system has two nontwisting asymptotic trajectories.*

The characteristic polynomial $f(\lambda) = \det(A - \lambda E)$ has a nonzero real root $\lambda = a$ ($f(0) \neq 0$ and $f(\lambda) \to \mp\infty$ as $\lambda \to \pm\infty$).
Then (by Lyapunov), system has solutions

$$x(t) = \xi e^{at} + o(e^{at}) \quad \text{as } t \to +\infty \text{ or } t \to -\infty,$$

$$\frac{x(t)}{|x(t)|} \to \frac{\xi}{|\xi|} \quad \text{as } t \to +\infty \text{ or } t \to -\infty.$$

Definition 1. $v(x)$ is quasihomogeneous vector field of degree $m \in \mathbb{N}$, $m > 1$, with mutually prime integer quasihomogeneity exponents $g_1, \ldots, g_n > 0$ if

$$v_i(\lambda^{g_1} x_1, \ldots, \lambda^{g_n} x_n) = \lambda^{g_i+m-1} v_i(x_1, \ldots, x_n)$$

for all $\lambda \in \mathbb{R}$.

Definition 2. The smooth vector field v is semiquasihomogeneous if

$$v = v^{(m)} + \sum_{\alpha > m} v^{(\alpha)},$$

where $v^{(k)}$ are quasihomogeneous fields of degree k with the same exponents $g_1, \ldots, g_n$.

Example. $\dot{x}_1 = x_2^2$, $\dot{x}_2 = x_1^3$. If $g_1 = g_2 = 1$ then $m = 2$ and quasihomogeneous truncation is $\dot{x}_1 = x_2^2$, $\dot{x}_2 = 0$. If $g_1 = 3$ and $g_2 = 4$ then $m = 6$ and the system is quasihomogeneous.

Theorem 2. *Suppose the v is smooth semiquasihomogeneous vector field. If $x = 0$ is on isolated singular point of $v^{(m)}$ and n is odd then there exists a nontwisting asymptotic trajectory.*

The case of a zero root

Let $n = 2p + 1$, and $\mathbb{R}^{2p+1} = \{x_1, \ldots, x_{2p}, z\}$.

$$\dot{x} = Bx + az^2 + \cdots, \qquad \dot{z} = \langle b, x\rangle + \alpha z^2 + \cdots, \tag{$*$}$$

$$C = \begin{pmatrix} B & a \\ b^\top & \alpha \end{pmatrix}, \qquad \det B \neq 0.$$

Proposition. *The singular point $x = 0$, $z = 0$ of $(*)$ is isolated if the following equivalent conditions hold:*

1) $\det C \neq 0$,

2) $c = \alpha - \langle b, B^{-1}a\rangle \neq 0$.

The quasihomogeneous truncation of $(*)$:

$$Bx + az^2 = 0, \quad \dot{z} = \langle b, x\rangle + \alpha z^2 \quad (g_1 = \cdots = g_{2p} = 2, \quad g_{2p+1} = 1).$$

This system has two asymptotic solutions:

$$x = -\frac{1}{c^2t^2}B^{-1}a, \qquad z = -\frac{1}{ct} \qquad (t \to +\infty \text{ and } t \to -\infty).$$

Theorem 3. *Suppose the following conditions hold:*
1) *the right-hand sides of* $(*)$ *are real-analytic functions in a neighborhood of the point* $x = 0$, $z = 0$,
2) $\det B \neq 0$ *and* $c \neq 0$.
Then the system $(*)$ *admits two solutions with asymptotic expansions for* $t \to +\infty$ *and* $t \to -\infty$ *of the following form:*

$$x = -\frac{1}{c^2t^2}B^{-1}a + \sum_{k\geqslant 3}\frac{x^{(k)}(\ln|t|)}{t^k}, \qquad z = -\frac{1}{ct} + \sum_{l\geqslant 2}\frac{z^{(l)}(\ln|t|)}{t^l}.$$

The coefficients $x^{(k)}(\,\cdot\,)$ *and* $z^{(l)}(\,\cdot\,)$ *are polynomials with constant coefficients.*

Corollary. *Under conditions of Theorem 3 systems* $(*)$ *has an incoming and outgoing nontwisting asymptotic trajectories.*

Remarks.

1. $\dot{z} = -z^2 + az^3 + \cdots$ has a solution

$$z = \frac{1}{t} + \frac{a \ln t}{t^2} + \cdots .$$

2. $\dot{x} = -x - (x+z)^2$, $\dot{z} = x$ has a formal solution

$$z(t) = \sum_{k=1}^{\infty} \frac{(k-1)!}{t^k}, \qquad x(t) = \frac{1}{t} - z(t).$$

$z(t) = e^{-t} \int_{-\infty}^{t} \frac{e^{\tau}}{\tau} \, d\tau$ is the Borel sum of the diverging series.