Noncommutative geometry and applications to fractals

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A quotation

“Вселенную нельзя понять, не научившись сначала понимать её язык, и не изучив буквы, которыми она написана. А написана она на математическом языке, и её буквы это треугольники, дуги и другие геометрические фигуры, без каковых невозможно понять по-человечески её слова, без них это тщетное кружение в тёмном лабиринте."

- Galileo Galilei, Opere Il Saggiatore, 1623.

What this talk is about

This talk will cover the details of the statement of the conformal trace theorem for Julia sets of quadratic polynomials. I will discuss:

1. A brief introduction to Julia sets
2. The Conformal Trace Theorem

This talk is based on the recent papers:
Trace theorem for quasi-Fuchsian groups, Connes, Sukochev, Zanin, *Math. Sb.*
Conformal trace theorem for Julia sets of quadratic polynomials, Connes, McDonald, Sukochev, Zanin, *ETDS*.

Complex polynomial dynamics

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial with complex coefficients. Let $z_0 \in \mathbb{C}$. Consider the recursive sequence:

$$z_{n+1} := f(z_n) \quad n \geq 0.$$

We are especially interested in studying the asymptotic behaviour of $\{z_n\}_{n\geq 0}$ for different $z_0 \in \mathbb{C}$. In particular, given any complex number, it can be shown that exactly one of the following happens:

1. Either $|z_n| \to \infty$.
2. $\{z_n\}_{n\geq 0}$ remains bounded.

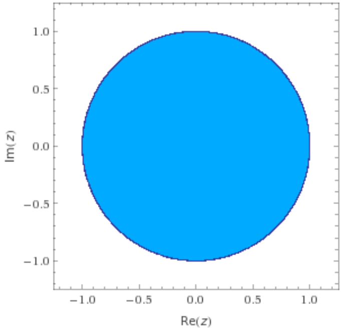
Complex polynomial dynamics

The simplest nontrivial example is $f(z) = z^2$. Then $z_k = f^k(z_0) = {z_0}^{2^k}$, and the behaviour of $f^k(z_0)$ neatly splits into three separate cases:

1. If $|z_0| < 1$, then $f^k(z_0) \to 0$ as $k \to \infty$.
2. If $|z_0| = 1$, then $|f^k(z_0)| = 1$ for all $k \geq 0$.
3. If $|z_0| > 1$, then $|f^k(z_0)| \to \infty$ as $k \to \infty$.

Complex polynomial dynamics

Pictorially,



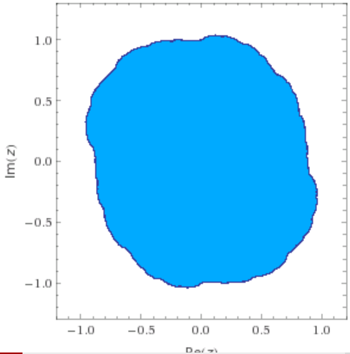
Here, the set of z_0 such that z_n remains bounded is coloured in blue. The set of z_0 such that z_n is unbounded is white. The boundary of the blue set is highlighted to make it easier to see.

Complex polynomial dynamics

What if we perturb the polynomial $f(z) = z^2$ slightly? Consider $f(z) = z^2 + 0.1 + 0.1i$.

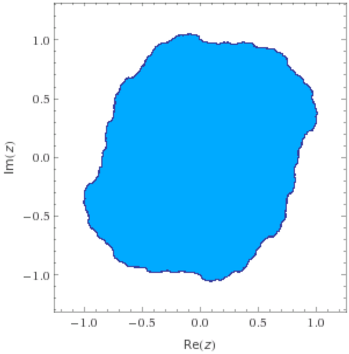
It is not feasible to determine analytically the behaviour of $\{f^n(z)\}_{n\geq 0}$. On a large grid of complex numbers, colour each point z blue if $|f^N(z)| < 10$ for some suitably large number N.

The result looks like this:



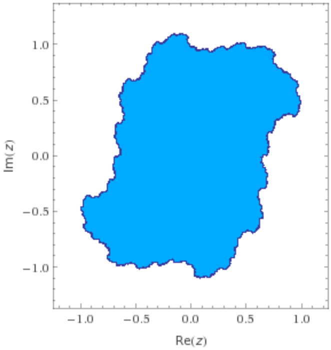
Complex polynomial dynamics

Try $f(z) = z^2 + 0.1 - 0.2i$,



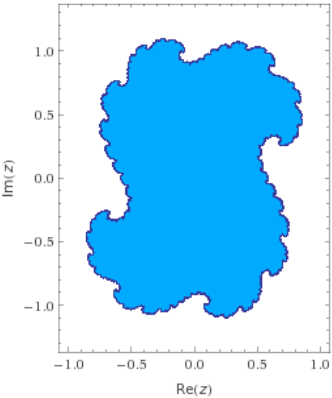
Complex polynomial dynamics

Try $f(z) = z^2 + 0.2 - 0.3i$,



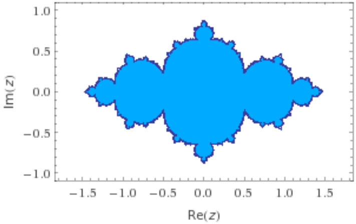
Complex polynomial dynamics

Try $f(z) = z^2 + 0.3 - 0.1i$,



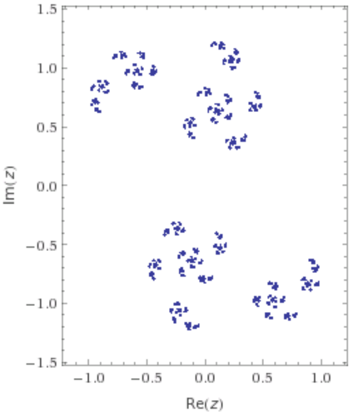
Complex polynomial dynamics

Try $f(z) = z^2 - 0.7 + 0.001i$,



Complex polynomial dynamics

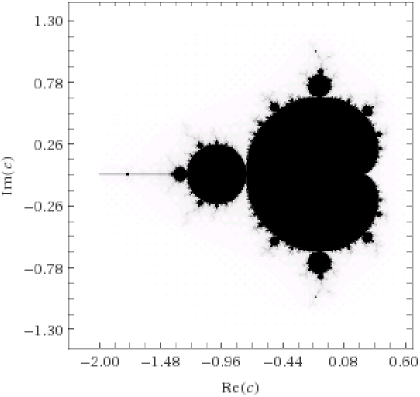
Let try a slightly bigger parameter. Consider $f(z) = z^2 + 0.5 + 0.5i$,



The Mandelbrot set

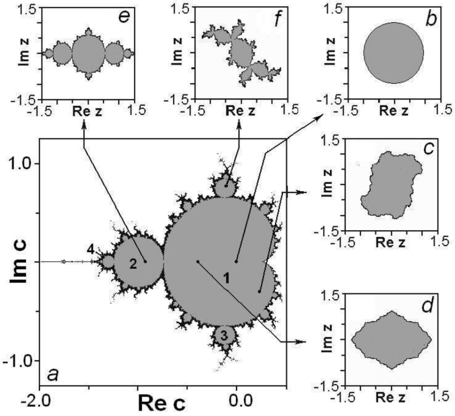
$$f_c(z) := z^2 + c.$$

Consider the case $z_0 = 0$. For which c is $\{f_c^k(0)\}_{k\geq 0}$ bounded? Define the Mandelbrot set $M := \{c \in \mathbb{C} \ : \ \{f_c^k(0)\}_{k\geq 0} \text{ is bounded}\}$.



The Mandelbrot set

A more informative image is this one:



The Julia set

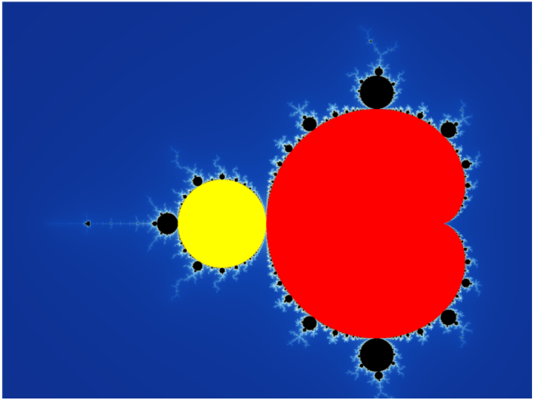
Let $c \in \mathbb{C}$, and consider $f_c(z) = z^2 + c$. The *Julia set* of f_c is the boundary of the set of points $z \in \mathbb{C}$ such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

Theorem

The Julia set $J(f_c)$ is connected if and only if $c \in M$ (the Mandelbrot set).

The main cardioid

Let M_0 be the set $\{\frac{z}{2}(1-\frac{z}{2}) \ : \ |z|<1\}$. M_0 is an open subset of the Mandelbrot set M called the *main cardioid*, shown below in red:



When Julia set is a Jordan curve?

The significance of the main cardioid is the following theorem:

Theorem

The Julia set $J(f_c)$ of f_c is a Jordan curve (i.e. homeomorphic to a circle) if and only if c is in the main cardioid M_0.

Geometric properties of the Julia set

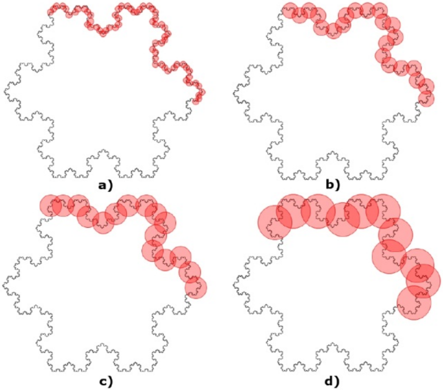
The study of Julia sets of quadratic polynomials is a classical one, and there are many detailed references (such as Carleson and Gamelin's *Complex Dynamics* and Milnor's *Dynamics in One Complex Variable*). Of particular interest is the Hausdorff dimension of $J(f_c)$.

Hausdorff dimension

Let (X, d) be a metric space. Let $\mathcal{C}$ be the set of $q \in [0, \infty)$ such that, for every $\varepsilon > 0$, there exists a covering of X by balls $\{B(a_j, r_j)\}_{j \geq 0}$ with $r_j < \varepsilon$ and $\sum_{j \geq 0} r_j^q < \infty$.
The Hausdorff dimension of X is defined to be the infimum of $\mathcal{C}$.

Hausdorff dimension

The Hausdorff dimension is a kind of "scaling dimension":



Roughly speaking if X has Hausdorff dimension p then, for all $\varepsilon > 0$ one needs at least $O(\varepsilon^{-p})$ balls of radius ε to cover X. (It is possible to make this heuristic precise for certain special X).

Hausdorff dimension of Julia sets

Fact: If c is in the main cardioid M_0 of the Mandelbrot set M, then the Julia set $J(f_c)$ is a Jordan curve with Hausdorff dimension $p \in [1, 2)$. In fact $p = 1$ if and only if $c = 0$.
Moreover: If $c \neq 0$, then $J(f_c)$ has infinite length!
(This is established in classical references such as Carleson and Gamelin, and Milnor).

Hausdorff measure

Associated to the Hausdorff dimension p there is a p-dimensional Hausdorff measure m_p.
If $S \subseteq X$, and $\delta > 0$, we define:

$$H^p_\delta(S) := \inf\{\sum_{k=1}^{\infty}(\mathrm{diam}(U_k))^p \;:\; S \subseteq \bigcup_{k=1}^{\infty} U_k,\, \mathrm{diam}(U_k) < \delta\}.$$

Define:

$$m_p(S) := \sup_{\delta>0} H^p_\delta(S).$$

It can be shown that m_p is a (countably additive) measure on the Borel σ-algebra on X.

In many cases, the Hausdorff measure of a ball of radius r scales like r^p,

$$cr^p \leq m_p(B(z,r)) \leq Cr^p$$

for some fixed c, C and all $r > 0$.
Fact: The p-dimensional Hausdorff measure on $J(f_c)$, where c is in the main cardioid satisfies the above inequality.

The Conformal trace theorem

Let $c \neq 0$ be in the main cardioid of M, and let $p \in (1,2)$ be the Hausdorff dimension of $J(f_c)$ with corresponding Hausdorff measure m_p. Let $g : J(f_c) \to \mathbb{C}$ be a continuous function. In his 1994 book *Noncommutative geometry*, Alain Connes announced a formula for $\int_J g\, dm_p$ given in terms of his "quantised calculus". We have now completed the proof of this formula. To state this formula we need to go into detail on Connes' quantised calculus.

Quantised Calculus: infinitesimals

Let H be a (complex, separable) Hilbert space. Let $B(H)$ denote the algebra of all bounded operators on H.

Following Connes' terminology, compact operators on H are called "infinitesimals".

For an operator $T \in B(H)$, the singular value sequence $\{\mu(n,T)\}_{n\geq 0}$ of T is defined by:

$$\mu(n,T) := \inf\{\|T-R\| \;:\; \mathrm{rank}(R) \leq n\}.$$

Quantised Calculus: infinitesimals

In general, $\mu(n, T)$ is a non-increasing sequence of non-negative numbers. By definiition, an operator T is compact if and only if $\lim_{n\to\infty} \mu(n, T) = 0$. The set $\mathcal{L}_{p,\infty}$ for $p \in (0, \infty)$ is defined to be:

$$\mathcal{L}_{p,\infty} := \{T \in B(H) \ : \ \mu(n, T) = O(n^{-1/p})\}.$$

Quantised Calculus: infinitesimals

It turns out that $\mathcal{L}_{p,\infty}$ is an ideal of $B(H)$, and equipped with the quasi-norm $\|T\|_{p,\infty} := \sup_{n\geq 0} n^{1/p}\mu(n,T)$ is a quasi-Banach space, and for all $T \in \mathcal{L}_{p,\infty}$ and $S, R \in B(H)$ we have:

$$\|RTS\|_{p,\infty} \leq \|R\|\|T\|_{p,\infty}\|S\|.$$

Traces on $\mathcal{L}_{1,\infty}$

A functional $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is called a trace if it is invariant under unitary conjugation. That is, for all unitary operators U and for all $T \in \mathcal{L}_{1,\infty}$, we have

$$\varphi(U^* TU) = \varphi(T).$$

Equivalently, for all $S \in B(H)$ and for all $T \in \mathcal{L}_{1,\infty}$, we have

$$\varphi(ST) = \varphi(TS).$$

We call a trace φ normalised if

$$\varphi\left(\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}\right) = 1.$$

Traces on $\mathcal{L}_{1,\infty}$

The definition of a trace mimics the properties of the matrix trace $\mathrm{Tr} : M_n(\mathbb{C}) \to \mathbb{C}$.

There are traces on $\mathcal{L}_{1,\infty}$, but in great contrast to the matrix trace, they are all *singular*. That is, all traces on $\mathcal{L}_{1,\infty}$ vanish on finite rank operators. However, traces can still be continuous with respect to the $\mathcal{L}_{1,\infty}$ topology. We say that a trace φ is continuous if there is a constant $C > 0$ such that for all $T \in \mathcal{L}_{1,\infty}$ we have

$$|\varphi(T)| \leq C\|T\|_{1,\infty}.$$

The Hilbert transform

Let $\mathbb{T}$ be the unit circle (in the complex plane). The Hilbert space $L_2(\mathbb{T})$ is defined with respect to the arc-length measure (the Haar measure). There is the trigonometric orthonormal basis for $L_2(\mathbb{T})$,

$$e_n(z) = z^n, \quad n \in \mathbb{Z}, z \in \mathbb{T}.$$

The Hilbert transform F is defined on the basis e_n by $Fe_n = \mathrm{sgn}(n)e_n$.

Quantised differentials

If f is a bounded function on $\mathbb{T}$, then pointwise multiplication by f defines a bounded linear operator M_f on $L_2(\mathbb{T})$. Connes calls the commutator $i[F, M_f] = i(FM_f - M_f F)$ the "quantised differential" of f.
The name is intended to imply that this is something like a differential df.
So we use the symbol $đf$,

$$đf := i[F, M_f].$$

The advantage of quantised differentials is that they work for arbitrary bounded measurable functions, not just differentiable functions.

Quantised differentials

It is not easy to see why this definition is the right one. One motivation comes from the higher dimensional situation:
The Hilbert transform on $\mathbb{R}^d$, $d \geq 2$ is an operator on $L_2(\mathbb{R}^d, \mathbb{C}^N)$, where $N = 2^{\lfloor d/2 \rfloor}$, given by:

$$F\xi(s) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \left(\sum_{j=1}^{d} \gamma_j \frac{t_j}{|t|} \right) \widehat{\xi}(t) e^{i(t,s)}\, dt.$$

where $\xi \in C_c^\infty(\mathbb{R}^d)$ and $\widehat{\xi}$ is the Fourier transform, and $\{\gamma_j\}_{j=1}^d$ are a set of $N \times N$ matrices satisfying $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{j,k}1$, $1 \leq j, k \leq d$.

Quantised differentials in higher dimensions

The following result is from *Quantum differentiability of essentially bounded functions on Euclidean spaces*, Lord S., McDonald E., Sukochev F. and Zanin D.: For a bounded function f on $\mathbb{R}^d$, we define $đf := i[F, 1 \otimes M_f]$.

Theorem

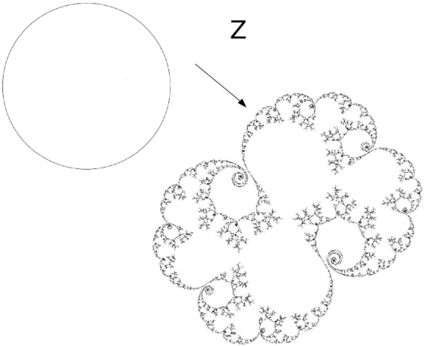
If $g, f \in L_\infty(\mathbb{R}^d)$, then $đf \in \mathcal{L}_{d,\infty}(L_2(\mathbb{R}^d, \mathbb{C}^N))$ if and only if $\left(\sum_{j=1}^d |\partial_j f|^2\right)^{d/2} \in L_1(\mathbb{R}^d)$, and:

$$\varphi(1 \otimes M_g |đf|^d) = c_d \int_{\mathbb{R}^d} g(x) \left(\sum_{j=1}^d |\partial_j f(x)|^2\right)^{d/2} dx.$$

for any continuous normalised trace φ on $\mathcal{L}_{1,\infty}$, and where c_d is a known constant.

Parametrising the Julia set

Let $c \in M_0$ (the main cardioid of the Mandelbrot set), and let $c \neq 0$ so that $J(f_c)$ is a Jordan curve with Hausdorff dimension $1 < p < 2$. Since J is a Jordan curve, there is a conformal mapping Z from the exterior of the unit disc to the exterior of $J(f_c)$,



Parametrising the Julia set

There is a theorem of complex analysis (Caratheodory's theorem) which implies that Z extends to a continuous function on the unit circle $\mathbb{T} := \{z \in \mathbb{C} \,:\, |z| = 1\}$ to $J(f_c)$. We can choose Z so that it satisfies the functional equation

$$Z(z)^2 + c = Z(z^2), \quad z \in \mathbb{T}.$$

(this is called the Böttcher equation, and Z is sometimes called a Böttcher coordinate)

Differentiating non-differentiable functions

The function Z is not differentiable, so we cannot define its derivative in the usual sense, or its differential dZ.

It is possible to rigourously define differentials of nondifferentiable functions using the theory of Schwartz distributions (also called "generalised functions").

Impossibility of multiplying distributions

The space of Schwartz distributions on the circle is denoted $\mathcal{D}'(\mathbb{T})$, and defined as the dual of the space of smooth functions on $\mathbb{T}$. L. Schwartz proved in 1954 that there does not exist an associative algebra A such that:

- $\mathcal{D}'(\mathbb{T})$ embeds linearly into A, and the constant function $1 \in \mathcal{D}'(\mathbb{T})$ becomes the unit of A.
- The multiplication of A extends the pointwise multiplication of continuous functions.
- There is a derivation ∂ on A which extends the differentiation of elements of $\mathcal{D}'(\mathbb{T})$.

A disadvantage of distributions

The outcome is that distributions cannot be multiplied in a reasonable way. For a general statement of the impossibility result and a full bibliography, see Section 1.1 and Theorem 1.1.4 of

Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R. **Geometric theory of generalized functions with applications to general relativity**, Mathematics and its Applications, 537. *K*luwer Academic Publishers, Dordrecht, 2001.

Fractional powers of quantised differentials

We can, however, define $|\bar{d}Z|^p$ for a non-integer p.
Since one cannot multiply distributions, it is impossible to raise the distributional differential dZ to a fractional power.

Description of the Conformal Trace Formula

Lemma

Let p be the Hausdorff dimension of the Julia set. The $p-$th power $|\bar{d}Z|^p$ of the absolute value of the quantised differential $\bar{d}Z$ is in the ideal $\mathcal{L}_{1,\infty}$.

Hence, If f is a bounded function on the Julia set $J(f_c)$, then the operator $M_{f\circ Z}|\bar{d}Z|^p$ is also in $\mathcal{L}_{1,\infty}$.
Motivated by noncommutative geometry, one might guess that the correct way of “integrating"this infinitesimal is to take a trace.

Description of the Conformal Trace Formula

Theorem

Let φ be a continuous trace on $\mathcal{L}_{1,\infty}$. Then there is a constant $K(\varphi, c)$ such that for all $f \in C(J(f_c))$,

$$\varphi(M_{f\circ Z}|\bar{d}Z|^p) = K(\varphi, c)\int f\, dm_p$$

where m_p is the p-dimensional Hausdorff measure on $J(f_c)$. Also, there exist traces φ such that $K(\varphi, c) > 0$.

Thank you for listening!

The paper discussed in this talk is:

Connes A., McDonald E., Sukochev F., Zanin D. **Conformal trace formula for Julia sets.** to appear at Ergodic Theory Dynam. Systems.

More information on operator ideals and their traces can be found in:

Lord S., Sukochev F., Zanin D. **Singular traces. Theory and applications.** De Gruyter Studies in Mathematics, 46. De Gruyter, Berlin, 2013.

More information on the quantised calculus may be found in:

Connes A. **Noncommutative geometry**. Academic Press, Inc., San Diego, CA, 1994.

For details about complex dynamical systems, I recommend:

Carleson L. and Gamelin T. **Complex dynamics**. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

Milnor J.**Dynamics in one complex variable**, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.