Trace Theorem for Quasi-Fuchsian groups

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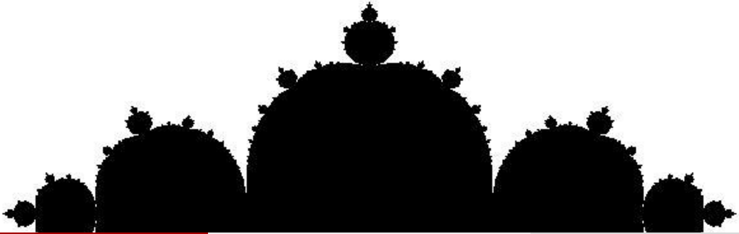


Aims of this talk

1. Overview of the theory of Kleinian groups
2. Brief review of the geometric measure theory of limit sets of Kleinian groups
3. Review of the theory of extended limits and Dixmier traces
4. Computing the geometric measure on limit sets

This talk is based on a forthcoming paper of Connes, Sukochev and Zanin, originally based on work by Connes detailed in [NCG].

Action of the group $\mathrm{SL}(2,\mathbb{C})$

Let

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in M_2(\mathbb{C}) : \ \det(g) = 1 \right\}.$$

Also, let $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm 1\}$.
We may identify the group $\mathrm{SL}(2,\mathbb{C})$ with its action on the Riemann sphere $\bar{\mathbb{C}}$ by the formula

$$g : z \to \frac{g_{11}z + g_{12}}{g_{21}z + g_{22}}, \quad z \in \bar{\mathbb{C}}.$$

We are interested in discrete subgroups of $\mathrm{SL}(2,\mathbb{C})$ such as, for example,

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in M_2(\mathbb{Z}) : \ \det(g) = 1 \right\}.$$

Kleinian groups

Let $G \subset \mathrm{SL}(2,\mathbb{C})$ (or $G \subset \mathrm{PSL}(2,\mathbb{C})$) be a discrete subgroup. We say that action of G at a point $z_0 \in \bar{\mathbb{C}}$ is *freely discontinuous* if there exists a neighborhood U of z_0 such that $g(U) \cap U = \varnothing$ for every $1 \neq g \in G$. In particular, the sequence $\{g(z_0)\}_{1 \neq g \in G}$ does not accumulate to z_0.

Definition

A discrete subgroup $G \subset \mathrm{SL}(2,\mathbb{C})$ is called Kleinian if its action is freely discontinuous at some point $z_0 \in \bar{\mathbb{C}}$.

For a Kleinian group, the set of all points at which G acts freely discontinuously is called regular set of the group G. Its complement is called the limit set of the group G. We denote the limit set by $\Lambda(G)$.

Limit set of a Kleinian group G

Lemma

For a Kleinian group, $\mathbb{C} \setminus \Lambda(G)$ is open and so $\Lambda(G)$ is closed.

If $\Lambda(G)$ contains 2 points or less, then G is one of the elementary Kleinian groups.
If $\Lambda(G)$ contains more than 2 points, then it is *perfect* (i.e., closed with no isolated points). In particular, $\Lambda(G)$ is uncountable.
Regular set is dense in $\bar{\mathbb{C}}$. So, limit set is nowhere dense.
Both regular and limit sets are $G-$invariant.

Quasi-Fuchsian groups

Definition

A Kleinian group G is called

1. Fuchsian if its limit set is a circle
2. quasi-Fuchsian if its limit set is a Jordan curve (i.e., homeomorphic to a circle)

If G is quasi-Fuchsian then by the Jordan Theorem $\Lambda(G)$ splits $\bar{\mathbb{C}}$ in 2 parts $\mathrm{int}(\Lambda(G))$ and $\mathrm{ext}(\Lambda(G))$.

Examples of quasi-Fuchsian groups

Let $a, b \in \mathrm{SL}(2, \mathbb{C})$ satisfy the condition

$$(\mathrm{Tr}(a))^2 + (\mathrm{Tr}(b))^2 + (\mathrm{Tr}(ab))^2 = \mathrm{Tr}(a) \cdot \mathrm{Tr}(b) \cdot \mathrm{Tr}(ab).$$

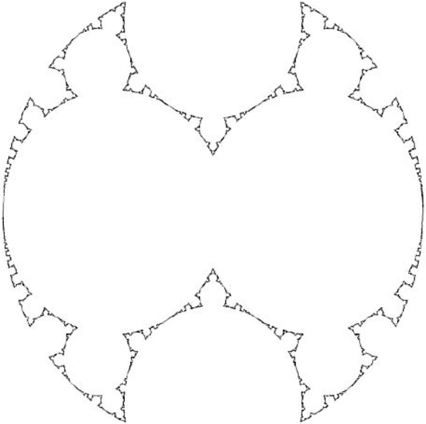
Let $G(a, b)$ be the group generated by a and b. It is showed in the book of Mumford, Series and Wright that $G(a, b)$ is quasi-Fuchsian.

In the same book, there is an algorithm for approximating the limit set of $G(a, b)$.

Note that the couple (a, b) is determined modulo conjugation by the numerical couple $(\mathrm{Tr}(a), \mathrm{Tr}(b))$. The following examples and images are from the honours thesis of Jacob Geerlings.

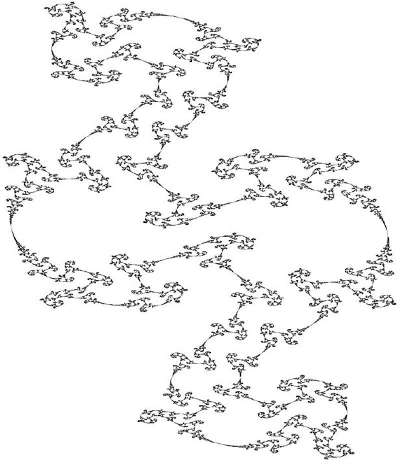
Pictures of $\Lambda(G(a,b))$

$\mathrm{Tr}(a) = \mathrm{Tr}(b) = 2.2$



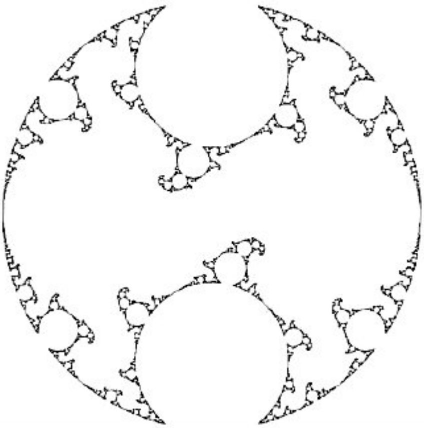
Pictures of $\Lambda(G(a,b))$

$\mathrm{Tr}(a) = 1.8 + 0.251i$, $\mathrm{Tr}(b) = 1.8 - 0.251i$



Pictures of $\Lambda(G(a, b))$

$\mathrm{Tr}(a) = 1.9 + 0.3i$, $\mathrm{Tr}(b) = 2.05$



Hausdorff dimension

Definition

Given a metric space (X, d), let $\mathcal{C}(X, d)$ be the set of all $q > 0$ such that there exists a covering of X by open balls $\{B(x_n, r_n)\}_{n=0}^{\infty}$ such that

$$\sum_{n=0}^{\infty} r_n^q < \infty.$$

The Hausdorff dimension of X is defined to be the infimum of $\mathcal{C}(X, d)$.

It is known that a limit set of a finitely generated quasi-Fuchsian group (which is not Fuchsian) has Hausdorff dimension $\dim(\Lambda(G))$ strictly greater than 1. In particular, $\Lambda(G)$ is never a smooth curve (unless G is Fuchsian).

On the other hand, $\Lambda(G)$ is a quasi-conformal image of a circle and, so (see e.g. Gehring-Vaisala-1973), its Hausdorff dimension is strictly less than 2.

Geometric measure of the group G.

Definition

A measure ν on $\bar{\mathbb{C}}$ is called a $p-$dimensional geometric measure (relative to G) if $d(\nu \circ g)(z) = |g'(z)|^p d\nu(z)$ for every $g \in G$.

Theorem (Sullivan-1984-Acta)

If G is a quasi-Fuchsian group where $\Lambda(G)$ has Hausdorff dimension p, then there exists a unique p-dimensional geometric measure on $\bar{\mathbb{C}}$ (relative to G). This measure is supported on $\Lambda(G)$, and is Radon on $\Lambda(G)$.

The main aim of this talk is to give a means of computing the geometric measure by means of a "Conformal Trace Formula".

General notations

Fix throughout a separable infinite dimensional Hilbert space H. We let $B(H)$ denote the algebra of all bounded operators on H. For a compact operator T on H, let $\mu(k, T)$ denote $k-$th largest singular value (these are the eigenvalues of $|T|$). The sequence $\mu(T) = \{\mu(k, T)\}_{k\geq 0}$ is referred to as to the singular value sequence of the operator T. The standard trace on $B(H)$ is denoted by Tr.
Fix an orthonormal basis in H (the particular choice of a basis is inessential). We identify the algebra l_∞ of bounded sequences with the subalgebra of all diagonal operators with respect to the chosen basis. For a given sequence $\alpha \in l_\infty$, we denote the corresponding diagonal operator by $\mathrm{diag}(\alpha)$.

Principal ideals $\mathcal{L}_{p,\infty}$

Let $\mathcal{L}_{p,\infty}$ be the principal ideal in $B(l_2)$ generated by the element $A_0 = \mathrm{diag}(\{(k+1)^{-\frac{1}{p}}\}_{k\geq 0})$. Equivalently,

$$\mathcal{L}_{p,\infty} = \{A \in B(l_2) : \sup_{k\geq 0}(k+1)^{\frac{1}{p}}\mu(k,A) < \infty\}.$$

In Noncommutative Geometry, a compact operator A is called an infinitesimal of order $\frac{1}{p}$ if

$$\mu(k,A) = O((k+1)^{-\frac{1}{p}}), \quad k \in \mathbb{Z}_+.$$

In other words, $\mathcal{L}_{p,\infty}$ is the set of all infinitesimals of order $\frac{1}{p}$.

Traces on $\mathcal{L}_{1,\infty}$

Definition

A linear functional $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is called a trace if $\varphi(AB) = \varphi(BA)$ for every $A \in \mathcal{L}_{1,\infty}$ and for every $B \in B(H)$.

There exists a plethora of traces on $\mathcal{L}_{1,\infty}$. The most famous ones are Dixmier traces.

Definition (Dixmier)

If ω is a free ultrafilter on $\mathbb{Z}_+$, then the functional

$$A \to \lim_{n\to\omega} \frac{1}{\log(n+2)} \sum_{k=0}^{n} \mu(k, A), \quad 0 \le A \in \mathcal{L}_{1,\infty}$$

is finite and additive on the positive cone of $\mathcal{L}_{1,\infty}$. Thus, it uniquely extends to a unitarily invariant linear functional on $\mathcal{L}_{1,\infty}$.

Basic properties of traces

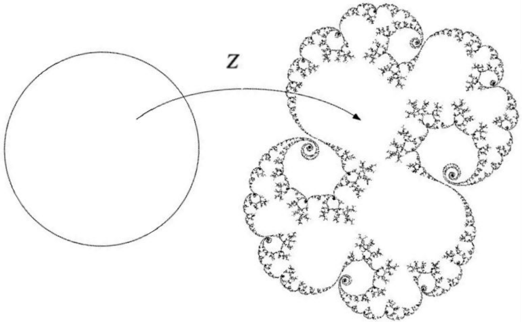
1. Every Dixmier trace is positive
2. Every positive trace is continuous with respect to the natural quasi-norm on $\mathcal{L}_{1,\infty}$
3. Every continuous trace is a linear combination of 4 positive traces.
4. There are positive traces which are not Dixmier traces

More information on the traces is available in [LSZ].

Statement of the main result

Let G be quasi-Fuchsian. Then by the Riemann mapping theorem, there is a conformal (i.e., biholomorphic) function $Z : \{|z| < 1\} \to \mathrm{int}(\Lambda(G))$. Since $\Lambda(G)$ is a Jordan curve, the Caratheòdory extension theorem for conformal mappings tells us that Z extends to a continuous function

$$Z : \{|z| = 1\} \to \Lambda(G).$$



Statement of the main result

Let F be the Hilbert transform on the circle (the sign of the differentiation operator). All the operators below act on $L_2(\mathbb{S}^1)$.

Theorem

Let G be a finitely generated quasi-Fuchsian (but not Fuchsian) without parabolic elements. Let $p = \dim(\Lambda(G))$. We have $[F, Z] \in \mathcal{L}_{p,\infty}$ and

$$\varphi(f(Z)|[F, Z]|^p) = c(G, \varphi) \int_{\Lambda(G)} f(z) d\nu(z), \quad f \in C(\Lambda(G)),$$

for every continuous trace φ on $\mathcal{L}_{1,\infty}$. Here, ν is the $p-$dimensional geometric measure.

History and credits

The preceding theorem was stated in [NCG] for the groups which arise in Bers's theorem on simultaneous uniformization (for Dixmier traces only). Strategy of the proof is given in [NCG] and it is stated there that a full proof will be available in a subsequent paper by Connes and Sullivan. This paper never appeared and the strategy proposed in [NCG] is vulnerable to a number of heavy obstacles of technical nature. These obstacles were fully resolved by the authors in [C-S-Z] using Double Operator Integral theory and a number of important auxiliary results by other authors (most notable: Sullivan [Sul-79,Sul-84] and Bishop-Jones [B-J]).

Proof strategy

The first step in the proof is to show that $[F, Z] \in \mathcal{L}_{p,\infty}$.
Hence from the Riesz theorem,

$$\varphi(f(Z)|[F,Z]|^p) = \int_{\Lambda(G)} f\, d\mu, \quad f \in C(\Lambda(G)),$$

for some measure μ. The second step in the proof is to show that

$$\varphi(f(g^{-1}(Z))|[F,Z]|^p) = \varphi(f(Z)|g'(Z)|^p|[F,Z]|^p), \quad f \in C(\Lambda(G)).$$

The p-dimensional geometric measure, satisfying $d(\nu \circ g) = |g'|^p d\nu$, is uniquely specified up to a scaling factor by the relation, for all $f \in C(\Lambda(G))$,

$$\int_{\Lambda(G)} f \circ g^{-1}\, d\nu = \int_{\Lambda(G)} f \cdot |g'|^p\, d\nu. \tag{1}$$

Since μ satisfies the covariance property in (1), it follows from the uniqueness part of Sullivan's theorem that $\mu = \nu$ (up to a constant factor).

Hausdorff dimension vs critical exponent

First we prove that $[F, Z] \in \mathcal{L}_{p,\infty}$.

Lemma

If G is a finitely generated quasi-Fuchsian group, then Hausdorff dimension of $\Lambda(G)$ equals to the critical exponent of G.

$$\dim(\Lambda(G)) = \inf\left\{q : \sum_{g\in G} |g'(z)|^q < \infty \text{ for almost every } z \in \bar{\mathbb{C}}\right\}.$$

Proof.

[Thanks to C. Bishop] We have $\dim(\Lambda(G)) < 2$. By Ahlfors Finiteness Theorem, G is analytically finite. By Bishop-Jones theorem [B-J], G is geometrically finite. By Theorem 1 in [Sul-84], Hausdorff dimension equals to the critical exponent. □

Key lemma

The proof of the following lemma combines Peller's characterization of Hankel operators in $\mathcal{L}_p$, Stein's description of the holomorphic part of Besov spaces, real interpolation of analytic Besov classes and fine properties of Fuchsian (!) groups.

Lemma

Let G be a finitely generated quasi-Fuchsian group without parabolic elements. If p is the critical exponent of G, then

$$\|[F,Z]\|_{p,\infty} \leq c(G)\Big\|\Big\{\frac{1}{|g_{21}|^2}\Big\}_{1\neq g\in G}\Big\|_{p,\infty}.$$

Thus, in order to conclude that $[F,Z]\in\mathcal{L}_{p,\infty}$, it suffices to obtain that $\Big\{\frac{1}{|g_{21}|^2}\Big\}_{1\neq g\in G}\in l_{p,\infty}(G)$.

Sullivan's estimates

A few hours of meditation over Corollary 5 in [Sul-79] yields.

Lemma

If G is a Kleinian group with critical exponent p, then $\{\|g\|_\infty^{-2}\}_{g\in G} \in l_{p,\infty}(G)$.

If $\infty \notin \Lambda(G)$, then $\|g\|_\infty = O(|g_{21}|)$, $1 \neq g \in G$. As a corollary, we get

Corollary

If G is a Kleinian group with critical exponent p and if $\infty \notin \Lambda(G)$, then $\{|g_{21}|^{-2}\}_{1\neq g\in G} \in l_{p,\infty}(G)$.

Using lemma from the preceding slide, we infer that $[F, Z] \in \mathcal{L}_{p,\infty}$.

Unitary transform

Set

$$g \circ Z = Z \circ \pi(g), \quad g \in G,$$

where $\pi(g)$ is a conformal automorphism of the unit ball.
For every conformal automorphism h of $D = \{|z| < 1\}$, we set

$$(U_h \xi)(z) = \xi(\frac{h_{11}z + h_{12}}{h_{21}z + h_{22}})\frac{1}{h_{21}z + h_{22}}, \quad |z| = 1.$$

We have that $U_h : L_2(\partial D) \to L_2(\partial D)$ is a unitary operator. It is of crucial importance that (Lemma 6.4 in [C-S-Z]) U_h commutes with F.
We have

$$(f \circ g^{-1})(Z) = U_{\pi(g)}^{-1} f(Z) U_{\pi(g)},$$

$$[F, Z] = U_{\pi(g)}^{-1} [F, g \circ Z] U_{\pi(g)}.$$

The transformation property

We wish to show that,

$$\varphi(f(g^{-1}(Z))|[F,Z]|^p) = \varphi(f(Z)|g'(Z)|^p|[F,Z]|^p).$$

Using the identities from the previous slide and unitary invariance of φ, we obtain

$$\begin{aligned}\varphi(f(g^{-1}(Z))|[F,Z]|^p) &= \varphi(U_{\pi(g)}^{-1}f(Z)|[F,g\circ Z]|^p U_{\pi(g)}) \\ &= \varphi(f(Z)|[F,g\circ Z]|^p).\end{aligned}$$

It suffices now to show that

$$\varphi(f(Z)|[F,g\circ Z]|^p) = \varphi(f(Z)|g'(Z)|^p|[F,Z]|^p).$$

Since f is bounded, it would be enough to know that

$$|[F,g\circ Z]|^p - |g'(Z)|^p|[F,Z]|^p \in (\mathcal{L}_{1,\infty})_0.$$

The transformation property

By Theorem 8.a in Section IV.3.β in Connes-NCG and by the first few lines of the proof of Lemma 6.3 in [C-S-Z], we have

$$[F, g \circ Z] - g'(Z)[F, Z] \in (\mathcal{L}_{p,\infty})_0.$$

This implies (see the proof of Lemma 6.3 in [C-S-Z])

$$|[F, g \circ Z]|^p - \left| |g'(Z)|^{\frac{1}{2}} |[F, Z]| |g'(Z)|^{\frac{1}{2}} \right|^p \in (\mathcal{L}_{1,\infty})_0.$$

We then wish to conclude (see the statement of Lemma 6.3 in [C-S-Z]),

$$|[F, g \circ Z]|^p - |[F, Z]|^p |g'(Z)|^p \in (\mathcal{L}_{1,\infty})_0.$$

This conclusion requires much more work (done in Lemma 5.3 in [C-S-Z]).

Operator estimates

To conclude the assertion on the preceding slide, we use the following theorem (it is adjusted Lemma 3.β.11 in [NCG]).

Theorem

Let $A, B \geq 0$ be bounded. If $B \in \mathcal{L}_{p,\infty}$ and if $[A^{\frac{1}{2}}, B] \in (\mathcal{L}_{p,\infty})_0$, then

$$B^p A^p - (A^{\frac{1}{2}} B A^{\frac{1}{2}})^p \in (\mathcal{L}_{1,\infty})_0.$$

We use this theorem for $A = |g'(Z)|$ and $B = |[F, Z]|$. Note that (done in the proof of Lemma 6.3 in [C-S-Z]) the commutator condition of the above theorem indeed holds for so-defined A and B.

Integral representation

If A and B are positive bounded operators, then [Thanks to D. Potapov]

$$B^p A^p - (A^{\frac{1}{2}} B A^{\frac{1}{2}})^p = T(0) - \int_{\mathbb{R}} T(s) h(s) ds.$$

Here, $s \to T(s)$ is some operator-valued function (in this talk, we do not provide a particular expression for it) and h is a fixed Schwartz function.

For this integral representation we prove that $T(s) \in (\mathcal{L}_{1,\infty})_0$ for every $s \in \mathbb{R}$. Then [Lying through our teeth]

$$\int_{\mathbb{R}} T(s) h(s) ds \in (\mathcal{L}_{1,\infty})_0.$$

Final result

Combining the above arguments, there is a constant $c(G,\varphi)$ such that

$$\varphi(f(Z)|[F,Z]|^p) = c(G,\varphi)\int_{\Lambda(G)} f\,d\nu.$$

This formula is not interesting if $c(G,\varphi)=0$, so we also show that $c(G,\varphi)>0$ for at least some φ. In particular, $c(G,\mathrm{Tr}_\omega)>0$ when ω is a power-invariant extended limit.

References

- [B-J] Bishop C., Jones P. *Hausdorff dimension and Kleinian groups.*
- [NCG] Connes A. *Noncommutative Geometry.*
- [CSZ] Connes A., Sukochev F., Zanin D. *Trace Theorem for quasi-Fuchsian groups.*
- [LSZ] Lord S., Sukochev F., Zanin D. *Singular traces. Theory and applications.*
- [Sul-79] Sullivan D. *The density at infinity of a discrete group of hyperbolic motions.*
- [Sul-84] Sullivan D. *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups.*