Noncommutative geometry and applications to fractals

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14 февраля 2018 г.

A quotation

“Вселенную нельзя понять, не научившись сначала понимать её язык, и не изучив буквы, которыми она написана. А написана она на математическом языке, и её буквы это треугольники, дуги и другие геометрические фигуры, без каковых невозможно понять по-человечески её слова, без них это тщетное кружение в тёмном лабиринте."

- Galileo Galilei, Opere Il Saggiatore, 1623.

Quantum mechanics

" In my paper the fact the XY was not equal to YX was very disagreeable to me. I felt this was the only point of difficulty in the whole scheme...and I was not able to solve it."

- physicist Werner Heisenberg

Quantum mechanical observables

Early on in the history of quantum mechanics, physicists realised that noncommutativity was essential to understand the structure of observables:
... rewriting Heisenberg's form of Bohr's quantum condition, I recognized at once its formal significance. It meant that the two matrix products pq **and** qp **are not identical.**
- physicist Max Born, describing his realisation c.1925 that observables in quantum mechanics do not commute.

What this talk is about

This talk will cover the details of the statement of the conformal trace theorem for Julia sets of quadratic polynomials. I will discuss:

1. What is noncommutative geometry about?
2. What is a Julia set?
3. What is the Hausdorff measure, and how does it relate to singular traces?

This talk is based on the recent papers:
Trace theorem for quasi-Fuchsian groups, Connes, Sukochev, Zanin, *Math. Sb.*
Conformal trace theorem for Julia sets of quadratic polynomials, Connes, McDonald, Sukochev, Zanin, *ETDS*.

What is noncommutative geometry?

Noncommutative geometry is the field mathematics which attempts to understand noncommutative algebras geometrically.

Main question:

In what way is a noncommutative algebra like an algebra of functions on some kind of “space"?

Noncommutative Topological spaces

Let X be a compact Hausdorff topological space. The algebra $C(X)$ of continuous complex valued functions on X is a C^*-algebra.
Does it make sense to think of a noncommutative C^*-algebra A as being like a ring of functions on a (non-existent) “space"?

Examples

"A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one."

- Paul Halmos

Example: The noncommutative torus

An arbitrary polynomial function f on a torus $\mathbb{T}^2$ can be written in terms of the coordinate variables z_1, z_2 as:

$$f(z_1, z_2) = \sum_{j,k} a_{j,k} z_1^j z_2^k$$

where the sum runs over a finite set of pairs of integers j, k, the coefficients $a_{j,k}$ are complex numbers and $|z_1| = |z_2| = 1$.
Now let θ be a real number. Consider the abstract algebra $\mathcal{A}_\theta$ on two generators U and V of polynomials of the form:

$$\sum_{j,k} a_{j,k} U^j V^k$$

where $U^{-1} = U^*$, $V^{-1} = V^*$, and $UV = e^{2\pi i\theta} VU$.

Example: The noncommutative torus

The algebra $\mathcal{A}_\theta$ is very much like an algebra of functions on a space, and $\mathcal{A}_0$ is precisely the algebra of polynomial functions on a torus.
One can develop a theory of harmonic analysis associated to $\mathcal{A}_\theta$ in parallel to the theory of harmonic analysis on a torus.

Example: The noncommutative plane

A polynomial function f on the plane $\mathbb{R}^2$ can be written in terms of the coordinate variables x_1, x_2 by:

$$f(x_1, x_2) = \sum_{j,k} a_{j,k} x_1^j x_2^k$$

where the sum runs over a finite set of non-negative integers j, k.
Now let θ be a real number. Consider the algebra on two generators X and Y satisfying $[X, Y] = i\theta$. This is the "noncommutative plane".

Example: The noncommutative plane

People familiar with physics will notice that the position and momentum operators x and $-i\hbar\frac{d}{dx}$ satisfy

$$\left[x, -i\hbar\frac{d}{dx}\right] = i\hbar.$$

One can develop a theory of analysis (including harmonic analysis, function spaces, measure theory etc.) over the noncommutative plane in parallel to the usual Euclidean plane $\mathbb{R}^2$.

Noncommutative geometry

Noncommutative geometry is the systematic theory of noncommutative algebras viewed as representing geometric objects.
It lies at the intersection of differential geometry, operator algebras and index theory, and has applications in all of these areas.
Alain Connes has even promoted the viewpoint that the small-scale structure of space-time might be noncommutative.

Singular traces and noncommutative geometry

A fundamental tool in noncommutative geometry is the Dixmier trace – and more generally singular traces.
To explain the significance of this tool we need to go deeper into the theory.

Quantised Calculus: infinitesimals

Let H be a (complex, separable) Hilbert space. Let $B(H)$ denote the algebra of all bounded operators on H.
Following Connes' terminology, compact operators on H are called "infinitesimals".
For an operator $T \in B(H)$, the singular value sequence $\{\mu(n, T)\}_{n\geq 0}$ of T is defined by:

$$\mu(n, T) := \inf\{\|T - R\| \ : \ \mathrm{rank}(R) \leq n\}.$$

Quantised Calculus: infinitesimals

In general, $\mu(n,T)$ is a non-increasing sequence of non-negative numbers. By definition, an operator T is compact if and only if $\lim_{n\to\infty}\mu(n,T)=0$. The set $\mathcal{L}_{p,\infty}$ for $p\in(0,\infty)$ is defined to be:

$$\mathcal{L}_{p,\infty} := \{T\in B(H)\ :\ \mu(n,T)=O(n^{-1/p})\}.$$

Quantised Calculus: infinitesimals

It turns out that $\mathcal{L}_{p,\infty}$ is an ideal of $B(H)$, and equipped with the quasi-norm $\|T\|_{p,\infty} := \sup_{n\geq 0} n^{1/p}\mu(n,T)$ is a quasi-Banach space, and for all $T \in \mathcal{L}_{p,\infty}$ and $S, R \in B(H)$ we have:

$$\|RTS\|_{p,\infty} \leq \|R\|\|T\|_{p,\infty}\|S\|.$$

Traces on $\mathcal{L}_{1,\infty}$

A functional $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is called a trace if it is invariant under unitary conjugation. That is, for all unitary operators U and for all $T \in \mathcal{L}_{1,\infty}$, we have

$$\varphi(U^* TU) = \varphi(T).$$

Equivalently, for all $S \in B(H)$ and for all $T \in \mathcal{L}_{1,\infty}$, we have

$$\varphi(ST) = \varphi(TS).$$

We call a trace φ normalised if

$$\varphi\left(\mathrm{diag}\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}\right) = 1.$$

Traces on $\mathcal{L}_{1,\infty}$

The definition of a trace mimics the properties of the matrix trace $\mathrm{Tr} : M_n(\mathbb{C}) \to \mathbb{C}$.

There are traces on $\mathcal{L}_{1,\infty}$, but in great contrast to the matrix trace, they are all *singular*. That is, all traces on $\mathcal{L}_{1,\infty}$ vanish on finite rank operators.

However, traces can still be continuous with respect to the $\mathcal{L}_{1,\infty}$ topology.

We say that a trace φ is continuous if there is a constant $C > 0$ such that for all $T \in \mathcal{L}_{1,\infty}$ we have

$$|\varphi(T)| \leq C\|T\|_{1,\infty}.$$

Singular traces and fractals

Surprisingly, singular traces can be used to integrate functions on certain fractals.
To explain this we can focus attention on a certain very special class of fractals: Julia sets of quadratic polynomials.

Complex polynomial dynamics

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial with complex coefficients. Let $z_0 \in \mathbb{C}$. Consider the recursive sequence:

$$z_{n+1} := f(z_n) \quad n \geq 0.$$

We are especially interested in studying the asymptotic behaviour of $\{z_n\}_{n\geq 0}$ for different $z_0 \in \mathbb{C}$. In particular, given any complex number z_0, it can be shown that exactly one of the following happens:

1. $|z_n| \to \infty$ or,
2. $\{z_n\}_{n\geq 0}$ remains bounded.

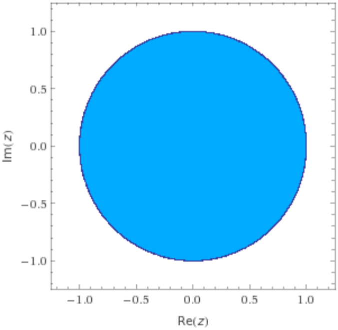
Complex polynomial dynamics

The simplest nontrivial example is $f(z) = z^2$. Then $z_k = f^k(z_0) = {z_0}^{2^k}$, and the behaviour of $f^k(z_0)$ neatly splits into three separate cases:

1. If $|z_0| < 1$, then $f^k(z_0) \to 0$ as $k \to \infty$.
2. If $|z_0| = 1$, then $|f^k(z_0)| = 1$ for all $k \geq 0$.
3. If $|z_0| > 1$, then $|f^k(z_0)| \to \infty$ as $k \to \infty$.

Complex polynomial dynamics

Pictorially,



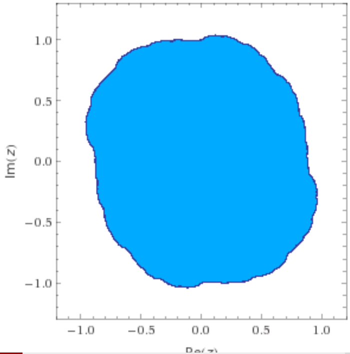
Here, the set of z_0 such that z_n remains bounded is coloured in blue. The set of z_0 such that z_n is unbounded is white. The boundary of the blue set is highlighted to make it easier to see.

Complex polynomial dynamics

What if we perturb the polynomial $f(z) = z^2$ slightly? Consider $f(z) = z^2 + 0.1 + 0.1i$.

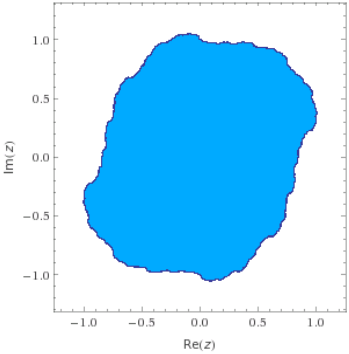
It is not feasible to determine analytically the behaviour of $\{f^n(z)\}_{n\geq 0}$. On a large grid of complex numbers, colour each point z blue if $|f^N(z)| < 10$ for some suitably large number N.

The result looks like this:



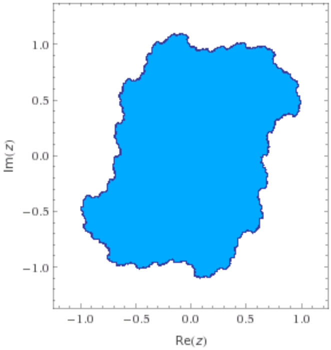
Complex polynomial dynamics

Try $f(z) = z^2 + 0.1 - 0.2i$,



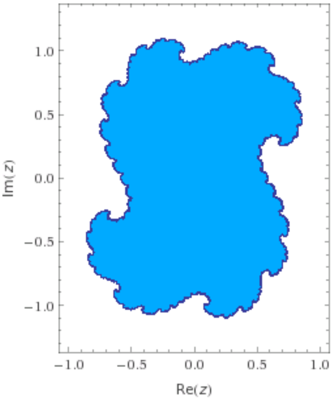
Complex polynomial dynamics

Try $f(z) = z^2 + 0.2 - 0.3i$,



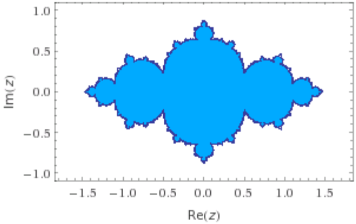
Complex polynomial dynamics

Try $f(z) = z^2 + 0.3 - 0.1i$,



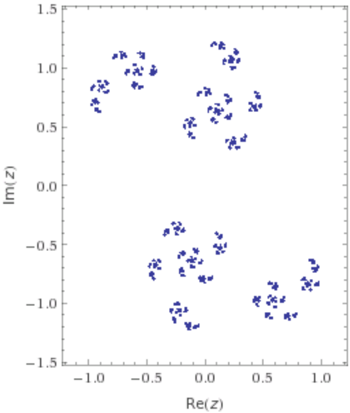
Complex polynomial dynamics

Try $f(z) = z^2 - 0.7 + 0.001i$,



Complex polynomial dynamics

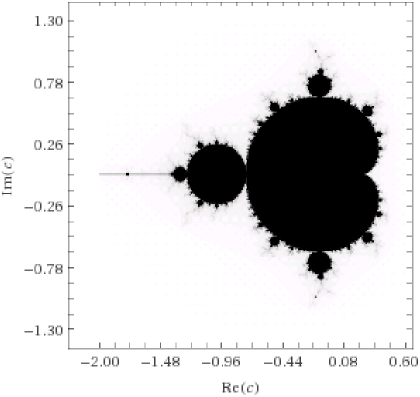
Let try a slightly bigger parameter. Consider $f(z) = z^2 + 0.5 + 0.5i$,



The Mandelbrot set

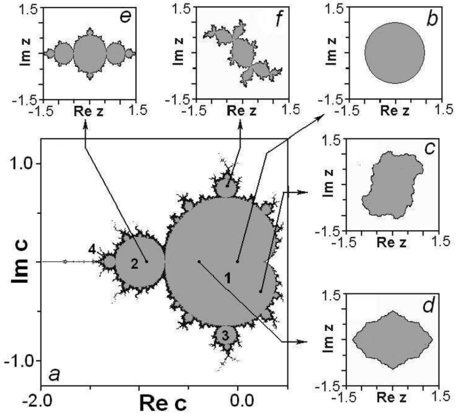
$$f_c(z) := z^2 + c.$$

Consider the case $z_0 = 0$. For which c is $\{f_c^k(0)\}_{k\geq 0}$ bounded? Define the Mandelbrot set $M := \{c \in \mathbb{C} \ : \ \{f_c^k(0)\}_{k\geq 0} \text{ is bounded}\}$.



The Mandelbrot set

A more informative image is this one:



The Julia set

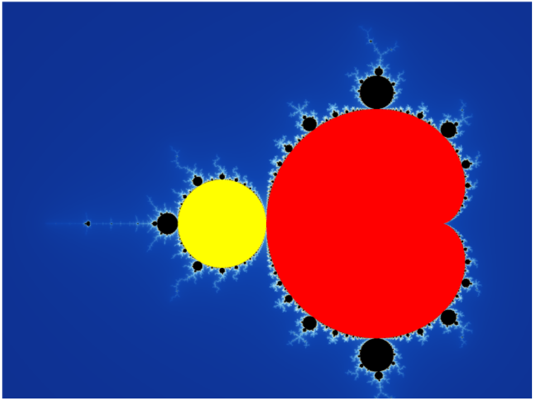
Let $c \in \mathbb{C}$, and consider $f_c(z) = z^2 + c$. The *Julia set* of f_c is the boundary of the set of points $z \in \mathbb{C}$ such that $\{f_c^n(z)\}_{n \geq 0}$ is bounded.

Theorem

The Julia set $J(f_c)$ is connected if and only if $c \in M$ (the Mandelbrot set).

The main cardioid

Let M_0 be the set $\{\frac{z}{2}(1-\frac{z}{2}) \ : \ |z|<1\}$. M_0 is an open subset of the Mandelbrot set M called the *main cardioid*, shown below in red:



When is the Julia set a Jordan curve?

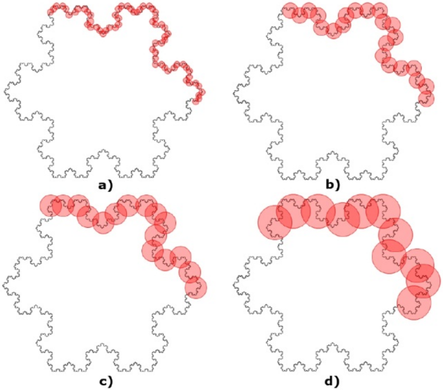
The significance of the main cardioid is the following theorem:

Theorem

The Julia set $J(f_c)$ of f_c is a Jordan curve (i.e. homeomorphic to a circle) if and only if c is in the main cardioid M_0.

Hausdorff dimension

The Hausdorff dimension is a kind of "scaling dimension":



Roughly speaking if X has Hausdorff dimension p then, for all $\varepsilon > 0$ one needs at least $O(\varepsilon^{-p})$ balls of radius ε to cover X. (It is possible to make this heuristic precise for certain special X).

Hausdorff dimension of Julia sets

Fact: If c is in the main cardioid M_0 of the Mandelbrot set M, then the Julia set $J(f_c)$ is a Jordan curve with Hausdorff dimension $p \in [1, 2)$. In fact $p = 1$ if and only if $c = 0$.
(This is established in classical references such as Carleson and Gamelin, and Milnor).
Moreover: for $c \neq 0$, the Julia set has infinite length.

Hausdorff measure

Informally, Hausdorff measure, the p-dimensional Hausdorff measure satisfies:

$$cr^p \leq m_p(B(z,r)) \leq Cr^p$$

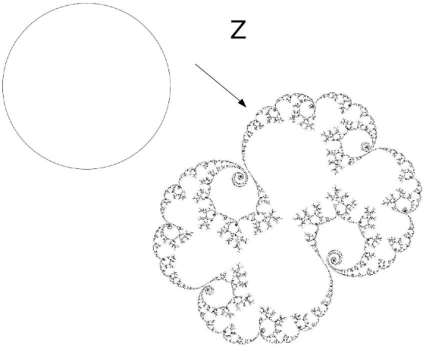
for some fixed c, C and all $r > 0$.
Fact: The p-dimensional Hausdorff measure on $J(f_c)$, where c is in the main cardioid satisfies the above inequality.

The Conformal trace theorem

Let $c \neq 0$ be in the main cardioid of M, and let $p \in (1,2)$ be the Hausdorff dimension of $J(f_c)$ with corresponding Hausdorff measure m_p. Let $g : J(f_c) \to \mathbb{C}$ be a continuous function. In his 1994 book *Noncommutative geometry*, Alain Connes announced a formula for $\int_J g\, dm_p$ given in terms of his "quantised calculus". We have now completed the proof of this formula. To state this formula we need to go into detail on Connes' quantised calculus.

Parametrising the Julia set

Let $c \in M_0$ (the main cardioid of the Mandelbrot set), and let $c \neq 0$ so that $J(f_c)$ is a Jordan curve with Hausdorff dimension $1 < p < 2$. Since J is a Jordan curve, there is a conformal mapping Z from the exterior of the unit disc to the exterior of $J(f_c)$,



The Hilbert transform

Let $\mathbb{T}$ be the unit circle (in the complex plane). The Hilbert space $L_2(\mathbb{T})$ is defined with respect to the arc-length measure (the Haar measure). There is the trigonometric orthonormal basis for $L_2(\mathbb{T})$,

$$e_n(z) = z^n, \quad n \in \mathbb{Z}, z \in \mathbb{T}.$$

The Hilbert transform F is defined on the basis e_n by $Fe_n = \mathrm{sgn}(n)e_n$.

Quantised Differentials

Let f be a bounded measurable function on the circle. The operator M_f defined (pointwise almost everywhere) by:

$$(M_f \xi)(z), \quad \xi \in L_2(\mathbb{T}), \quad z \in \mathbb{T}.$$

Connes introduces what he calls the "quantised differential" $đf$ of f defined by:

$$đf = i[F, M_f].$$

$đf$ is intended to be something like an infinitesimal "differential" df of f. The advantage of quantised differentials is that they work for arbitrary bounded measurable functions, not just differentiable functions.

Quantised differentials

It is not easy to see why this definition is the right one. One motivation comes from the higher dimensional situation:
The Hilbert transform on $\mathbb{R}^d$, $d \geq 2$ is an operator on $L_2(\mathbb{R}^d, \mathbb{C}^N)$, where $N = 2^{\lfloor d/2 \rfloor}$, given by:

$$F\xi(s) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \left(\sum_{j=1}^{d} \gamma_j \frac{t_j}{|t|} \right) \widehat{\xi}(t) e^{i(t,s)} \, dtds.$$

where $\xi \in C_c^\infty(\mathbb{R}^d)$ and $\widehat{\xi}$ is the Fourier transform, and $\{\gamma_j\}_{j=1}^d$ are a set of $N \times N$ matrices satisfying $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{j,k}1$, $1 \leq j, k \leq d$.

Quantised differentials in higher dimensions

The following result is from *Quantum differentiability of essentially bounded functions on Euclidean spaces*, Lord S., McDonald E., Sukochev F. and Zanin D.:

Theorem

If $g, f \in L_\infty(\mathbb{R}^d)$, then $[F, 1 \otimes M_f] \in \mathcal{L}_{d,\infty}(L_2(\mathbb{R}^d, \mathbb{C}^N))$ if and only if $\left(\sum_{j=1}^d |\partial_j f|^2\right)^{d/2} \in L_1(\mathbb{R}^d)$, and:

$$\varphi(1 \otimes M_g |[F, 1 \otimes M_f]|^d) = c_d \int_{\mathbb{R}^d} g(x) \left(\sum_{j=1}^{d} |\partial_j f(x)|^2\right)^{d/2} dx.$$

where c_d is a known constant.

So there is a direct link between the quantised differential $[F, 1 \otimes M_f]$ and the classical gradient $\nabla f = (\partial_1 f, \partial_2 f, \ldots, \partial_d f)$.

Description of the Conformal Trace Formula

Recall that $Z : \mathbb{T} \to J(f_c)$ is the parametrisation of the Julia set.

Lemma

Let p be the Hausdorff dimension of the Julia set. Then $[F, M_Z] \in \mathcal{L}_{p,\infty}$.

Hence, If f is a bounded function on the Julia set $J(f_c)$, then the operator $M_{f\circ Z}|[F, M_Z]|^p$ is also in $\mathcal{L}_{1,\infty}$.
Motivated by noncommutative geometry, one might guess that the correct way of "integrating"this infinitesimal is to take a trace.

Description of the Conformal Trace Formula

Theorem

Let φ be a continuous trace on $\mathcal{L}_{1,\infty}$. Then there is a constant $K(\varphi, c)$ such that for all $f \in C(J(f_c))$,

$$\varphi(M_{f\circ Z}|[F, M_Z]|^p) = K(\varphi, c) \int_{J(f_c)} f \, dm_p$$

where m_p is the p-dimensional Hausdorff measure on $J(f_c)$. Also, there exist traces φ such that $K(\varphi, c) > 0$.

Thank you for listening!

The paper discussed in this talk is:

Connes A., McDonald E., Sukochev F., Zanin D. **Conformal trace formula for Julia sets.** to appear at Ergodic Theory Dynam. Systems.

More information on operator ideals and their traces can be found in:

Lord S., Sukochev F., Zanin D. **Singular traces. Theory and applications.** De Gruyter Studies in Mathematics, 46. De Gruyter, Berlin, 2013.

More information on the quantised calculus may be found in:

Connes A. **Noncommutative geometry**. Academic Press, Inc., San Diego, CA, 1994.

For details about complex dynamical systems, I recommend:

Carleson L. and Gamelin T. **Complex dynamics**. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

Milnor J.**Dynamics in one complex variable**, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.